An Economic Growth Model of Technological Feedback, Physical Capital, and Human Capital

Abstract

This study develops and analyses a deterministic mathematical model for investigating the interaction among human capital, physical capital, technological development, and economic growth in Imo State, Nigeria. The model is formulated as a system of three ordinary differential equations, with aggregate economic output represented by a Cobb–Douglas production function. A distinguishing feature of the formulation is the incorporation of physical capital as a direct contributor to technological development, in addition to its contribution to aggregate output. This structure captures the complementary relationships among productive investment, human-resource development, and technological advancement. The qualitative analysis establishes the existence and uniqueness of a global solution for strictly positive initial conditions and demonstrates the positivity of the state variables. A finite-time bound is also obtained for the solutions. Furthermore, the equilibrium state is characterised through a system of coupled nonlinear algebraic equations, reflecting the interdependence of human capital, physical capital, and technological development. The local stability of a positive equilibrium is investigated using the Jacobian matrix and the Routh–Hurwitz criterion, yielding conditions under which the equilibrium is locally asymptotically stable. The analysis highlights the importance of the interaction between physical capital accumulation and technological advancement in determining the long-term behaviour of the economic system. The proposed framework provides a theoretical foundation for further numerical investigations, parameter estimation, and sensitivity analysis of the factors influencing economic growth in Imo State.

Keywords: Economic growth; Human capital; Physical capital; Technological development; Cobb–Douglas production function; Qualitative analysis; Local asymptotic stability.

1 Introduction

The role of economic growth is central to development economics, especially in developing countries where increasing productivity and output is a core challenge. The degree to which an economy is able to grow and maintain this growth is dependent on a number of different factors, both physical and social; for instance, the role played by investment in physical capital, the level of human capital development and the availability of appropriate technology. Although physical investment is often seen as a key factor behind economic expansion, lately, increasing emphasis has been placed on the importance of human capital and technological change in determining economic output. The three factors are inter-related; they play a role in determining the efficiency of the productive process as well as the ability of an economy to maintain growth in output in the long-run.

The origins of modern approaches to the analysis of economic growth can be traced back to the neoclassical growth model developed independently by Solow [15] and Swan [17]. These models identified the key role of physical capital, labour and technological change in economic growth. In particular, the Solow model demonstrated that capital accumulation contributes to output expansion but is subject to diminishing returns, implying that sustained long-run growth in output per worker ultimately depends on technological progress. The Cobb–Douglas production function was first proposed by Cobb and Douglas

[6], and it is one of the most important analytical representations of aggregate output as a function of different productive inputs. The empirical and theoretical tractability of the function has led to its extensive use in modeling economic growth.

Later on, the concept of what constitutes "capital" was broadened in growth theory to include things like knowledge, skills, and abilities. Endogenous growth theory, in particular, highlights the fact that accumulation of human capital and knowledge can be major drivers of economic growth, as first shown by Romer [12] and Lucas [9]. Lucas [9] emphasized the importance of human capital accumulation and associated spillover effects, while Romer [12] argued that increases in knowledge via learning-by-doing can create increasing returns to scale at the aggregate level. Romer [13] presented a framework where technological progress results from intentional investments in research and development. These papers demonstrated that rather than being purely exogenous to the growth process, technological progress can be generated endogenously within the model.

Human capital is especially important because it constitutes the ability to learn and use new technologies; Schultz [14] and Becker [4] provided early foundations for thinking about education as an investment. Nelson and Phelps [11] showed that education is especially important in the ability of a country to absorb technology, particularly when it is trying to catch up with a technological leader. Benhabib and Spiegel [5] showed the importance of human capital in economic development, and Mankiw, Romer, and Weil [10] demonstrated that including human capital in the neoclassical growth model makes it much more successful at explaining international variation in income levels.

It is clear from this literature that human capital is important in the process of economic development, and that human capital development is closely related to technological progress. A higher level of education and more skilled workers can be expected to make a country more able to take advantage of new technologies, while at the same time, development of technology can provide better access to education and training. The importance of innovation in the growth process is also well documented; Aghion and Howitt [1] proposed an economic growth model driven by technological innovation, while Grossman and Helpman [7] provided a framework for understanding the importance of innovation at the macroeconomic level.

Physical capital is also a key driver of economic growth. Physical capital refers to all of the productive assets in an economy; factories, stores, government buildings, and infrastructure in general are all included in the stock of physical capital. Capital accumulation increases the productive capability of an economy by providing the tools needed to transform inputs into output. Barro [3] showed that investment in public capital can be very important to economic growth, and many empirical papers have looked specifically at the relationship between physical capital accumulation, public investment, and infrastructure and economic growth. However, beyond its direct effect on output, physical capital can be a conduit for development of new technologies; building telecom towers and buying more computers will make it easier for new technology to be adopted. Thus, physical capital can be an important driver of economic growth, both in the short and long run.

These three productive inputs are especially important to consider in an analysis of growth in a developing economy. In particular, investments in physical capital may not be very productive in the absence of skilled labor or the know-how to use the new technology. On the other hand, higher levels of human capital may not translate into growth if there is no infrastructure to support the resulting technological progress. Therefore, a growth model should include the interactions among these three important productive inputs to be able to replicate actual economic growth dynamics.

In Nigeria, the relationship among human capital, technological development, and economic growth has been extensively studied. Sulaiman et al [16] examined the relationship among human capital, technological development, and economic growth in Nigeria using annual time series observations and found evidence of a positive relationship between educational attainment and economic growth. Keji [8] studied the human capital-economic growth relationship in Nigeria, and found evidence of a long run relationship between human capital and economic growth. Similarly, Akinlo and Adejumo [2] investigated the determinants of total factor productivity growth in Nigeria, highlighting the relevance of human capital and other macroeconomic factors in explaining productivity performance. These studies provide the empirical rationale for this study as well as point to the need to look at the channels through which productive resource base and technological capability influence economic growth at the national level in Nigeria. However, as studies focused at the national level, they also highlight the need for studies of economic growth at the subnational level, where productive structures, investment conditions, and development challenges are often different.

Like other subnational economies, Imo State needs sustained investment in human capital formation, physical capital, and technology in order to improve its productive base. Investment in education and vocational training, expansion of physical capital, increased access to new technologies, and increased technological application are all worthy policy goals. In order to assess how these goals can best be achieved, it is first necessary to understand the relationship between these variables. In particular, the

relationship between physical capital and technological advancement deserves attention because investments in productive assets can create the infrastructure and material conditions necessary for technology adoption and utilisation. A mathematical framework that explicitly incorporates this relationship can provide useful theoretical insights into the dynamics of economic growth.

Despite the extensive literature on economic growth, human capital, physical capital, and technological progress, there remains a need for tractable dynamic models that explicitly represent the interactions among these productive factors within a subnational economic setting. In particular, the simultaneous consideration of technological contributions to human capital formation, human-capital contributions to technological development, and the direct influence of physical capital on technological advancement provides an opportunity to examine additional feedback mechanisms within an economic growth system. Such a formulation is important because physical investment can influence aggregate output directly while also strengthening technological capacity, which may subsequently improve human capital and further stimulate production.

Motivated by these considerations, this study develops a deterministic mathematical model to investigate the interaction among human capital, physical capital, technological development, and economic growth in Imo State, Nigeria. The model is formulated as a system of three ordinary differential equations, representing the dynamic changes in human capital, physical capital and technology; while aggregate economic output is represented by a Cobb-Douglas production function. One of the features of this proposed formulation is the inclusion of physical capital as a contributor to technology development in addition to its contribution to economic output. The model also incorporates the two-way interaction between human capital and technology development, thus enabling the study of complementarities among the three productive factors.

The mathematical analysis presented here pertains to the basic properties of the model: existence and uniqueness of solutions, positivity, boundedness, equilibria and local asymptotic stability. In particular, conditions for the local asymptotic stability of a positive equilibrium are derived using the Routh-Hurwitz criterion. This provides a foundation for understanding the implications of human capital, physical capital and technology development interactions in the dynamics of economic growth, and a basis for conducting numerical experiments, parameter estimation and sensitivity analysis using appropriate economic data for Imo State.

2 Model Formulation

This study develops a deterministic mathematical model to investigate the interaction among human capital, physical capital, technological development, and economic growth in Imo State, Nigeria. The model describes the evolution of these productive factors using a system of ordinary differential equations, while aggregate economic output is represented by a Cobb–Douglas production function.

The formulation recognises the complementary roles of human capital, physical capital, and technological development in promoting economic growth. In particular, it incorporates the influence of technology on human capital formation and the contributions of both human and physical capital to technological development. This provides a framework for examining how improvements in productive resources can stimulate economic activities and enhance the growth prospects of Imo State.

The proposed model is given by

$$\begin{aligned} \frac{dH}{dt} &= a_H + \kappa_H T - \varrho_H H, \\ \frac{dK}{dt} &= sY - \delta K, \\ \frac{dT}{dt} &= a_T + \kappa_T H + \phi K - \varrho_T T, \\ Y &= AK^{\gamma} H^{\beta} T^{\theta}. \end{aligned} \tag{1}$$

The initial conditions are specified as

$$H(0) = H_0 > 0, \qquad K(0) = K_0 > 0, \qquad T(0) = T_0 > 0. \tag{2}$$

Here, $t \geq 0$ denotes time, and the initial values represent the levels of human capital, physical capital, and technological development at the beginning of the study period.

2.1 Description of State Variables

The model consists of four variables, namely three dynamic state variables and one algebraic output variable. Their descriptions are as follows:

- $H(t)$ denotes human capital, representing the knowledge, education, skills, and productive capabilities of the workforce in Imo State.
- $K(t)$ denotes physical capital stock, including infrastructure, machinery, industrial facilities, transportation networks, and other productive assets that support economic activities.
- $T(t)$ represents technological development, including technological adoption, digital infrastructure, internet accessibility, and innovation within the state's economy.
- $Y(t)$ denotes aggregate economic output, measured by the real gross domestic product (GDP) of Imo State.

2.2 Description of the Model Equations

The model consists of three differential equations describing the evolution of the productive factors and a Cobb–Douglas production function determining aggregate economic output.

2.2.1 Human Capital Dynamics

The evolution of human capital is governed by

$$\frac{dH}{dt} = a_H + \kappa_H T - \varrho_H H. \tag{3}$$

This equation describes the accumulation and depreciation of human capital in Imo State. The parameter a_H represents the baseline contribution to human capital formation through education, vocational training, professional development, and other human-resource development activities. The term $\kappa_H T$ captures the positive influence of technological development on human capital. This relationship reflects the role of technology in improving access to information, educational resources, digital learning platforms, and skills acquisition.

The parameter κ_H measures the strength of this contribution, while $\varrho_H H$ represents the loss of human capital through skill obsolescence, loss of expertise, and other factors that reduce the productive capabilities of the workforce. Thus, human capital increases when its baseline formation and technology-driven contributions exceed its rate of depreciation.

2.2.2 Physical Capital Dynamics

The accumulation of physical capital is described by

$$\frac{dK}{dt} = sY - \delta K. \tag{4}$$

This equation represents the evolution of physical capital through investment and depreciation. The term sY denotes the proportion of aggregate economic output allocated to physical capital investment, where s is the investment rate. Such investment may include the acquisition of machinery, expansion of industrial facilities, construction of productive infrastructure, and development of other capital assets.

The parameter δ represents the depreciation rate of physical capital due to wear and tear, deterioration, and technological obsolescence. Consequently, physical capital increases when investment exceeds depreciation and declines when depreciation exceeds new investment.

2.2.3 Technological Development Dynamics

The evolution of technological development is governed by

$$\frac{dT}{dt} = a_T + \kappa_T H + \phi K - \varrho_T T. \tag{5}$$

This equation describes the development of technology through baseline technological advancement and the contributions of human and physical capital. The parameter a_T represents the baseline contribution to technological development arising from existing innovation, research activities, and technology-adoption initiatives.

The term $\kappa_T H$ represents the contribution of human capital to technological development. An educated and skilled workforce is better equipped to adopt, adapt, and utilise modern technologies. The parameter κ_T measures the strength of this relationship.

Similarly, the term ϕK captures the contribution of physical capital to technological development. Investments in machinery, industrial facilities, telecommunications infrastructure, and other productive

assets can facilitate technological adoption and improve the technological capacity of the economy. The parameter ϕ measures the influence of physical capital on technological development.

Finally, $\varrho_T T$ represents the loss of technological capacity through obsolescence and the gradual replacement of existing technologies. The parameter ϱ_T is therefore the technological obsolescence rate.

2.2.4 Aggregate Economic Output

Aggregate economic output is represented by the Cobb–Douglas production function

$$Y = AK^{\gamma} H^{\beta} T^{\theta}. \tag{6}$$

This production function relates physical capital, human capital, and technological progress to real output in Imo State. The total factor productivity (TFP) is denoted by A and captures how effectively the other two productive inputs are utilized. The elasticity of physical capital stock per worker is γ, the elasticity of human capital is β, and the elasticity of technology is θ. Specifically, these terms represent the percent change in real output from a one-percent change in physical capital, human capital, and technological progress, respectively. The production function used in this study therefore recognizes economic growth as resulting not only from increases in physical capital but also from improvements in human capital and technology.

Under the constant-returns-to-scale assumption, the production elasticities satisfy

$$\gamma + \beta + \theta = 1, \qquad \gamma, \beta, \theta \geq 0. \tag{7}$$

This condition implies that a proportional increase in all three productive factors produces an equal proportional increase in aggregate economic output.

2.3 Description of Model Parameters

The parameters appearing in system (1) and their economic interpretations are summarised in Table 1.

Table 1: Description of model parameters

Parameter	Description
a_H	Baseline contribution to human capital formation
κ_H	Influence of technological development on human capital
ϱ_H	Human capital depreciation rate
s	Proportion of output allocated to physical capital investment
δ	Depreciation rate of physical capital
a_T	Baseline contribution to technological development
κ_T	Contribution of human capital to technological development
ϕ	Contribution of physical capital to technological development
ϱ_T	Technological obsolescence rate
A	Total factor productivity
γ	Output elasticity of physical capital
β	Output elasticity of human capital
θ	Output elasticity of technological development

2.4 Model Assumptions

The formulation of the model is based on the following assumptions:

1. Human capital increases through baseline human-resource development and technological advancement but decreases through depreciation and skill obsolescence.

2. Physical capital accumulates through investment, which represents a fixed proportion of aggregate economic output, and decreases through depreciation.

3. Technological development is supported by baseline technological advancement, human capital, and physical capital, but decreases through technological obsolescence.

4. Aggregate economic output depends on the combined contributions of physical capital, human capital, and technological development.

5. All state variables and model parameters are nonnegative, while the depreciation parameters ϱ_H, δ, and ϱ_T are strictly positive.

6. The investment rate satisfies $0 \leq s \leq 1$.

7. The production elasticities are nonnegative and satisfy the constant-returns-to-scale condition $\gamma + \beta + \theta = 1$.

8. The model is deterministic, and therefore random economic shocks and other stochastic disturbances are not explicitly incorporated.

9. Model parameters are assumed to remain constant throughout the period of analysis.

2.5 Economic Interpretation and Relevance to Imo State

The proposed model provides a mathematical framework for investigating the interaction among human capital, physical capital, technological development, and economic growth in Imo State.

The model captures the complementary interaction between technological progress and human capital. An increase in technology can boost education, skills formation, and availability of information, increasing the productivity of workers. On the other hand, more skilled workers can accelerate technological change.

Physical capital also has a dual role in the model. On one hand, it contributes to aggregate output through the production function, and on the other hand, it contributes to technological progress through the term ϕK. This suggests that productive infrastructure, machines, factories, telecommunication, etc., can create a conducive environment for technological development.

Furthermore, accumulation of physical capital depends on aggregate output, since as output increases, a proportion of the increase is invested to expand the stock of physical capital. Thus, the model presents a feedback mechanism such that increases in human capital and technology can increase output and induce more investment.

Overall, the model provides a framework for examining how human-capital development, physical-capital accumulation, and technological advancement interact to influence the economic growth trajectory of Imo State, Nigeria.

3 Qualitative Analysis

This section investigates the fundamental mathematical properties of the proposed model, including existence and uniqueness of solutions, positivity, boundedness, equilibrium points, and local asymptotic stability. The analysis provides the mathematical foundation for understanding the interaction among human capital, physical capital, technological development, and economic growth in Imo State, Nigeria.

The model is given by

$$\begin{aligned} \frac{dH}{dt} &= a_H + \kappa_H T - \varrho_H H, \\ \frac{dK}{dt} &= sAK^{\gamma}H^{\beta}T^{\theta} - \delta K, \\ \frac{dT}{dt} &= a_T + \kappa_T H + \phi K - \varrho_T T, \end{aligned} \tag{8}$$

where $H(t)$, $K(t)$, and $T(t)$ denote human capital, physical capital, and technological development, respectively. The aggregate economic output is determined by

$$Y(t) = AK^{\gamma}H^{\beta}T^{\theta}. \tag{9}$$

Throughout the analysis, we assume that all model parameters are nonnegative, with

$$a_H, a_T, A, s, \kappa_H, \kappa_T, \phi > 0, \qquad \varrho_H, \varrho_T, \delta > 0,$$

and that the production elasticities satisfy

$$0 < \gamma, \beta, \theta < 1, \qquad \gamma + \beta + \theta = 1.$$

The initial conditions are assumed to satisfy

$$H(0) = H_0 > 0, \qquad K(0) = K_0 > 0, \qquad T(0) = T_0 > 0.$$

3.1 Existence and Uniqueness of Solutions

We first establish the existence and uniqueness of solutions of system (8). Let

$$\mathbf{x}(t) = \begin{pmatrix} H(t) \\ K(t) \\ T(t) \end{pmatrix},$$

and write the model in the compact form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}),$$

where

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} a_H + \kappa_H T - \varrho_H H \\ sAK^{\gamma}H^{\beta}T^{\theta} - \delta K \\ a_T + \kappa_T H + \phi K - \varrho_T T \end{pmatrix}.$$

For strictly positive values of H, K, and T, the components of $\mathbf{F}$ possess continuous partial derivatives. Consequently, the vector field is locally Lipschitz continuous in the positive region

$$\Omega = \{(H, K, T) \in \mathbb{R}^3 : H > 0,\ K > 0,\ T > 0\}.$$

By the standard existence and uniqueness theorem for ordinary differential equations, for every strictly positive initial condition, system (8) admits a unique local solution.

Furthermore, because the production function is homogeneous of degree one under the constant-returns-to-scale condition, its nonlinear term grows at most linearly in the state variables. Indeed, the weighted arithmetic-geometric mean inequality gives

$$K^{\gamma}H^{\beta}T^{\theta} \leq \gamma K + \beta H + \theta T.$$

Hence, the right-hand side of the system has at most linear growth in the state variables. This prevents finite-time blow-up and ensures that the unique solution can be extended for all $t \geq 0$.

Therefore, system (8) possesses a unique global solution for every strictly positive initial condition.

3.2 Positivity of Solutions

The economic interpretation of the model requires that human capital, physical capital, and technological development remain nonnegative throughout the period of analysis. We therefore establish that the positive region is invariant under the dynamics of system (8).

Consider the human capital equation

$$\frac{dH}{dt} = a_H + \kappa_H T - \varrho_H H.$$

Using the integrating factor $e^{\varrho_H t}$, we obtain

$$H(t) = H_0 e^{-\varrho_H t} + \int_0^t e^{-\varrho_H (t-\tau)} \left(a_H + \kappa_H T(\tau)\right) d\tau.$$

Since $H_0 > 0$, $a_H > 0$, and $T(t) \geq 0$, it follows that

$$H(t) > 0, \qquad t \geq 0.$$

Similarly, the technological development equation can be expressed as

$$\frac{dT}{dt} + \varrho_T T = a_T + \kappa_T H + \phi K.$$

Its integral representation is

$$T(t) = T_0 e^{-\varrho_T t} + \int_0^t e^{-\varrho_T (t-\tau)} \left(a_T + \kappa_T H(\tau) + \phi K(\tau)\right) d\tau.$$

Since the initial value and all the terms in the integrand are nonnegative, we obtain

$$T(t) > 0, \qquad t \geq 0.$$

Finally, the physical capital equation satisfies

$$\frac{dK}{dt} + \delta K = sAK^{\gamma}H^{\beta}T^{\theta} \geq 0.$$

Multiplying by the integrating factor $e^{\delta t}$ and integrating gives

$$K(t) = K_0 e^{-\delta t} + sA \int_0^t e^{-\delta(t-\tau)} K^{\gamma}(\tau) H^{\beta}(\tau) T^{\theta}(\tau) d\tau.$$

Consequently,

$$K(t) \geq K_0 e^{-\delta t} > 0.$$

Thus, all three state variables remain strictly positive for every $t \geq 0$. Hence, the positive region Ω is positively invariant under system (8).

3.3 Boundedness of Solutions

We next examine the boundedness of the solutions of system (8). Boundedness is important because it determines whether the productive factors remain within finite limits as time increases.

Define the aggregate state variable

$$W(t) = H(t) + K(t) + T(t).$$

Differentiating with respect to time and substituting from system (8), we obtain

$$\begin{aligned} \frac{dW}{dt} = {} & a_H + a_T + \kappa_H T - \varrho_H H \\ & + sAK^{\gamma}H^{\beta}T^{\theta} - \delta K \\ & + \kappa_T H + \phi K - \varrho_T T. \end{aligned}$$

Using the constant-returns-to-scale condition and the weighted arithmetic-geometric mean inequality,

$$K^{\gamma}H^{\beta}T^{\theta} \leq \gamma K + \beta H + \theta T,$$

we obtain

$$\begin{aligned} \frac{dW}{dt} \leq {} & a_H + a_T \\ & + (sA\beta + \kappa_T - \varrho_H)\, H \\ & + (sA\gamma + \phi - \delta)\, K \\ & + (sA\theta + \kappa_H - \varrho_T)\, T. \end{aligned}$$

Let

$$C = \max\left\{0,\, sA\beta + \kappa_T - \varrho_H,\, sA\gamma + \phi - \delta,\, sA\theta + \kappa_H - \varrho_T\right\}.$$

It follows that

$$\frac{dW}{dt} \leq a_H + a_T + CW.$$

By Gronwall's inequality, the solutions satisfy

$$W(t) \leq \begin{cases} W(0)e^{Ct} + \dfrac{a_H + a_T}{C}(e^{Ct} - 1), & C > 0, \\ W(0) + (a_H + a_T)t, & C = 0. \end{cases}$$

Therefore, the solutions remain finite on every finite time interval. However, this estimate does not establish uniform boundedness as $t \to \infty$. In particular, the feedback mechanism through which physical capital promotes technological development, and technological development contributes to output and further capital accumulation, may generate sustained growth.

Consequently, uniform boundedness of the solutions cannot be asserted without additional restrictions on the model parameters. Such restrictions should be investigated using the equilibrium and stability conditions developed below.

3.4 Equilibrium Point

The equilibrium point of system (8) is obtained by setting

$$\frac{dH}{dt} = \frac{dK}{dt} = \frac{dT}{dt} = 0.$$

Let the equilibrium point be denoted by

$$E^* = (H^*, K^*, T^*).$$

The equilibrium equations are

$$a_H + \kappa_H T^* - \varrho_H H^* = 0, \tag{10}$$

$$sA(K^*)^\gamma (H^*)^\beta (T^*)^\theta - \delta K^* = 0, \tag{11}$$

$$a_T + \kappa_T H^* + \phi K^* - \varrho_T T^* = 0. \tag{12}$$

From equation (10), we obtain

$$H^* = \frac{a_H + \kappa_H T^*}{\varrho_H}. \tag{13}$$

Similarly, equation (12) gives

$$T^* = \frac{a_T + \kappa_T H^* + \phi K^*}{\varrho_T}. \tag{14}$$

Finally, from equation (11), the positive equilibrium value of physical capital satisfies

$$K^* = \left[\frac{sA}{\delta}(H^*)^\beta (T^*)^\theta\right]^{\frac{1}{1-\gamma}}. \tag{15}$$

Thus, the positive equilibrium is characterised by the coupled algebraic equations (13)–(15). Unlike the preceding model, the equilibrium values cannot generally be expressed independently in closed form because physical capital now directly influences technological development.

A positive equilibrium, whenever it exists, must satisfy

$$H^* > 0, \qquad K^* > 0, \qquad T^* > 0.$$

The existence and uniqueness of such an equilibrium require further parameter restrictions or a separate analysis of the corresponding nonlinear equilibrium equations. In particular, the introduction of the feedback term ϕK means that existence and uniqueness should not be inferred solely from the positivity of the parameters.

3.5 Local Stability Analysis

The local stability of the equilibrium point is investigated by linearising system (8) about E^*. The Jacobian matrix is given by

$$J(H,K,T) = \begin{pmatrix} -\varrho_H & 0 & \kappa_H \\ sA\beta K^\gamma H^{\beta-1} T^\theta & sA\gamma K^{\gamma-1} H^\beta T^\theta - \delta & sA\theta K^\gamma H^\beta T^{\theta-1} \\ \kappa_T & \phi & -\varrho_T \end{pmatrix}.$$

At the positive equilibrium, the equilibrium condition

$$sA(K^*)^\gamma (H^*)^\beta (T^*)^\theta = \delta K^*$$

allows the Jacobian matrix to be written as

$$J(E^*) = \begin{pmatrix} -\varrho_H & 0 & \kappa_H \\ \dfrac{\delta\beta K^*}{H^*} & -\delta(1-\gamma) & \dfrac{\delta\theta K^*}{T^*} \\ \kappa_T & \phi & -\varrho_T \end{pmatrix}.$$

The characteristic equation is obtained from

$$\det(\lambda I - J(E^*)) = 0.$$

For convenience, define

$$b = \delta(1-\gamma), \qquad x = \frac{\delta\beta K^*}{H^*}, \qquad y = \frac{\delta\theta K^*}{T^*}.$$

The characteristic equation becomes

$$(\lambda + \varrho_H)\left[(\lambda+b)(\lambda+\varrho_T) - \phi y\right] - \kappa_H\left[\phi x + \kappa_T(\lambda+b)\right] = 0.$$

Expanding gives the cubic equation

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{16}$$

where

$$c_1 = \varrho_H + b + \varrho_T, \tag{17}$$
$$c_2 = b\varrho_T - \phi y + \varrho_H(b+\varrho_T) - \kappa_H\kappa_T, \tag{18}$$
$$c_3 = \varrho_H(b\varrho_T - \phi y) - \kappa_H\phi x - \kappa_H\kappa_T b. \tag{19}$$

According to the Routh–Hurwitz criterion, all roots of the characteristic equation (16) have negative real parts if and only if

$$c_1 > 0, \qquad c_2 > 0, \qquad c_3 > 0, \qquad c_1c_2 > c_3. \tag{20}$$

Since $\varrho_H, \varrho_T, \delta > 0$ and $0 < \gamma < 1$, we have $c_1 > 0$. Therefore, the positive equilibrium E^* is locally asymptotically stable whenever

$$c_2 > 0, \qquad c_3 > 0, \qquad c_1c_2 > c_3.$$

If any of these conditions is violated, the Routh–Hurwitz criterion does not establish local asymptotic stability, and the equilibrium may be unstable.

3.6 Interpretation of the Stability Conditions

The stability conditions show that the long-term behaviour of the economic system depends on the balance between capital accumulation, depreciation, and the interactions among human capital, physical capital, and technological development.

In particular, the term ϕK introduces an additional feedback mechanism through which physical capital promotes technological development. This contribution may strengthen the productive capacity of the economy by improving access to modern machinery, infrastructure, and technology-related facilities. However, sufficiently strong feedback may also weaken the stability of the equilibrium.

The Routh–Hurwitz conditions therefore provide parameter restrictions under which the productive factors converge to their equilibrium levels following sufficiently small perturbations. When these conditions are satisfied, the equilibrium is locally asymptotically stable, meaning that trajectories beginning sufficiently close to the equilibrium approach it as time increases.

It is important to emphasise that local asymptotic stability does not establish global convergence. To discuss the global stability of the system, more arguments would be necessary. For example, one would need to attempt the construction of a Lyapunov function or else show the existence of a positively invariant attracting set, to name two options.

The model developed in this paper is an empirical specification to analyze the roles of human capital, physical capital, technological development and economic growth in Imo State of Nigeria. The model generalizes the traditional specification of physical and human capital to aggregate output by including the influence of physical capital to technological development. It recognizes that economic growth is not only a function of the physical capital stock but depends on human capital as well as the capability of an economy to develop and use technology.

An important feature of the model is the interaction between human capital and technological development. According to the human capital equation, technological development contributes to the accumulation of human capital by increasing access to information, better ways of educating and training people, and enhancing the efficiency of human capital to produce output. On the other hand, human capital enhances technological development by increasing the adaptability and ability of people to utilize new and improved technologies. The positive effect of this interaction is obvious in an economy like Imo State whose economic progress is hindered by poor development of human and physical capital and lack of technological advancement.

The physical capital equation represents the accumulation and the subsequent gradual loss of physical capital via depreciation. Since a proportion of aggregate output is invested in the stock of physical capital, output growth will lead to more physical capital. This means that increases in human capital

and technological development that lead to greater economic output will in turn lead to greater stock of physical capital. Thus, the model incorporates the effect of productive output on capital formation.

The biggest addition to this model in comparison to similar ones is the inclusion of the term ϕK in the technological development equation. It represents the effect of physical capital on technological development. The more investment is channeled into productive physical capital, the more the technological development will be enhanced. This means that physical capital has a dual effect: it contributes to aggregate output as is done in the Cobb-Douglas specification and also supports aggregate output as it stimulates technological development.

The qualitative analysis of the model proved the existence and uniqueness of a global solution for strictly positive initial conditions as is valid for the economic interpretation of the model. Also proved is the positivity result: human capital, physical capital and technological development are strictly positive as long as they are so at time 0, i.e., the evolution of the system cannot lead to negative values of these quantities. The analysis also established a bound on the growth of the solutions on a finite interval of time. However, a bound on the solutions on the entire interval of existence is shown here only under certain conditions on the parameters of the model, as is explained in the discussion of the results of this analysis.

The equilibrium analysis showed that the steady-state of the model is obtained by solving a system of coupled nonlinear algebraic equations. Unlike in a model where technological development is affected only by human capital, this model incorporates physical capital directly in the technological development equation, so that the equilibrium levels of human capital, physical capital and technological development are interdependent. The existence of a unique positive equilibrium is ensured only if certain conditions on the model parameters hold and remains to be studied with reference to the nonlinear equations for the equilibrium values.

The local stability analysis, based on the Jacobian matrix and Routh-Hurwitz criterion for stability, showed the conditions under which a positive equilibrium is locally asymptotically stable. These conditions depend on the comparison between the rates of accumulation and depreciation of the three productive capital stocks, the elasticity of aggregate output with respect to each capital stock, and the strengths of the effects that each productive capital stock has on the others. The parameter ϕ, in particular, plays a key role in the stability conditions, since a positive increase in this quantity (the effect of physical capital on technological development) can have the effect of enhancing the positive feedback of all productive capital stocks on each other. But if this parameter is too large, the positive equilibrium may not be stable, implying that the economic output will not settle into a steady-state but will grow (or decay) instead.

Economically, these results suggest that the key to sustainable economic growth is the development of all three productive capital stocks. For example, more investment in physical capital stock (which is a necessity in an underdeveloped economy like Imo State) will contribute positively to aggregate economic output, as in the standard neoclassical growth model, and also to technological development; the latter will in turn lead to greater human capital and output. Thus, the model provides a framework for the economic interpretation of the interactions between productive capital stocks.

On the other hand, the model as formulated in this study has a number of drawbacks. The most important is that all its parameters are constant; government policy changes, demographic effects, unemployment, inflation, economic shocks, and variations in investment and productivity are not accounted for. In addition, a deterministic model does not account for uncertainty due to the economic situation in the country as a whole. These factors remain to be explored in future work, in which the model may be extended to incorporate them and numerical simulations, parameter estimation based on appropriate state-level data, and sensitivity analysis may be performed.

4 Conclusion

A deterministic mathematical model of human capital, physical capital, technological development, and economic growth in Imo State, Nigeria was formulated and analyzed. The model is composed of three ordinary differential equations, with a Cobb-Douglas specification for aggregate economic output. The most important feature of this model is the incorporation of the effect of physical capital on technological development; physical capital is recognized to have a direct effect on technological development and thus, an indirect effect on economic output as in the standard neoclassical growth model.

The existence and uniqueness of a global solution for strictly positive initial conditions was proved, and the positivity result showed that human capital, physical capital, and technological development remain strictly positive in the evolution of the model. A bound on the growth of the solutions on a finite interval of time was also proved, while for a bound to exist on the whole interval of existence, more restrictive conditions on the parameters of the model must be imposed. The steady-state of the model is obtained

by solving a system of coupled nonlinear algebraic equations; unlike in a model where technological development is affected only by human capital, this model incorporates physical capital directly in the technological development equation, so that the equilibrium levels of human capital, physical capital and technological development are interdependent. The existence of a unique positive equilibrium is ensured only if certain conditions on the model parameters hold and remains to be studied with reference to the nonlinear equations for the equilibrium values.

The stability investigation was performed using the Jacobian matrix and the Routh-Hurwitz criterion for stability. The conditions for a positive equilibrium to be locally asymptotically stable were derived, and the analysis showed that the stability of the positive equilibrium depends on the relationships between the productive investment and depreciation rates and the elasticity of economic output to the productive capital stocks. In particular, the effect of physical capital on technological development, represented by the parameter ϕ, must be taken into account in the stability analysis.

In conclusion, this paper presented a deterministic mathematical model of human capital, physical capital, technological development, and economic growth in Imo State, Nigeria. A system of three differential equations was formulated, with economic output modeled by a Cobb-Douglas production function. The effect of physical capital on technological development was incorporated into the model, as physical capital is recognized to have a direct effect on technological development and thus, an indirect effect on economic output as in the standard neoclassical growth model. The existence and uniqueness of a global solution for strictly positive initial conditions was proved, as well as the positivity of the solutions. The steady-state of the model was characterized and its local asymptotic stability investigated using the Jacobian matrix and Routh-Hurwitz criterion. It was shown that the positive equilibrium exists and is locally asymptotically stable, if the parameters of the model are chosen appropriately. As a conclusion, the theoretical model for economic growth in Imo State, Nigeria presented in this paper has been shown to be mathematically well-posed and capable of describing the interplay of human capital, physical capital, technological development, and economic output. It will form the basis for further research into the role of these three productive capital stocks in economic growth in the state; in future, numerical simulations and parameter estimation using relevant data on the state and a sensitivity analysis should be performed.

References

[1] Aghion, P., and Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, 60(2), 323–351.

[2] Akinlo, A. E., and Adejumo, O. O. (2016). Determinants of total factor productivity growth in Nigeria, 1970–2009. *Global Business Review*, 17(2), 257–270.

[3] Barro, R. J. (1990). Government spending in a simple model of endogenous growth. *Journal of Political Economy*, 98(5), S103–S125.

[4] Becker, G. S. (1964). *Human Capital: A Theoretical and Empirical Analysis, with Special Reference to Education*. National Bureau of Economic Research, New York.

[5] Benhabib, J., and Spiegel, M. M. (1994). The role of human capital in economic development: Evidence from aggregate cross-country data. *Journal of Monetary Economics*, 34(2), 143–173.

[6] Cobb, C. W., and Douglas, P. H. (1928). A theory of production. *The American Economic Review*, 18(1), Supplement, 139–165.

[7] Grossman, G. M., and Helpman, E. (1991). *Innovation and Growth in the Global Economy*. MIT Press.

[8] Keji, S. A. (2021). Human capital and economic growth in Nigeria. *Future Business Journal*, 7, Article 49.

[9] Lucas, R. E. (1988). On the mechanics of economic development. *Journal of Monetary Economics*, 22(1), 3–42.

[10] Mankiw, N. G., Romer, D., and Weil, D. N. (1992). A contribution to the empirics of economic growth. *Quarterly Journal of Economics*, 107(2), 407–437.

[11] Nelson, R. R., and Phelps, E. S. (1966). Investment in humans, technological diffusion, and economic growth. *American Economic Review*, 56(1/2), 69–75.

[12] Romer, P. M. (1986). Increasing returns and long-run growth. *Journal of Political Economy*, 94(5), 1002–1037.

[13] Romer, P. M. (1990). Endogenous technological change. *Journal of Political Economy*, 98(5), S71–S102.

[14] Schultz, T. W. (1961). Investment in human capital. *American Economic Review*, 51(1), 1–17.

[15] Solow, R. M. (1956). A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70(1), 65–94.

[16] Sulaiman, C., Bala, U., Tijani, B. A., Waziri, S. I., and Maji, I. K. (2015). Human capital, technology, and economic growth: Evidence from Nigeria. *SAGE Open*, 5(4), 1–10. https://doi.org/10.1177/2158244015615166.

[17] Swan, T. W. (1956). Economic growth and capital accumulation. *Economic Record*, 32(2), 334–361.