UNDER PEER REVIEW

Original Research Article

Simple-Averaging Gradient Recovery and Error Estimation for the Steklov Eigenvalue Problem

Abstract

The Steklov eigenvalue problem is an elliptic eigenvalue problem in which the spectral parameter appears in the boundary condition. This paper investigates its linear finite element approximation and a simple-averaging gradient recovery method. We formulate the problem variationally, derive the corresponding finite element discretization, and establish a priori eigenvalue error estimates. We then define a nodal equal-weight averaging operator and analyze the superconvergence of the recovered gradient under suitable regularity and local mesh-structure assumptions, using a finite element superapproximation estimate. Based on the difference between the recovered and finite element gradients, we construct a computable error estimator. Using an eigenvalue error identity, we derive $\lambda_h - \lambda = \eta_h^2 + O(h^{2+\alpha_0})$ and establish its asymptotic exactness for the first isolated simple eigenvalue under the additional nondegeneracy condition $\lambda_h - \lambda \geq c_0 h^2$. Finally, numerical experiments illustrate the convergence behavior of the finite element eigenvalues and recovered gradients and assess the performance of the proposed estimator.

Keywords: Steklov eigenvalue problem; linear finite element method; simple averaging; gradient recovery; superconvergence; error estimation.

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1 Introduction

1.1 Research Background and Significance

Eigenvalue problems arise in many applications of partial differential equations. The Steklov eigenvalue problem is distinguished by the appearance of the spectral parameter in the boundary condition. In this paper, we consider

$$\begin{cases} -\Delta u + \kappa^2 u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda u, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, n is the unit outward normal, $\kappa > 0$, and (λ, u) is an eigenpair. The corresponding variational formulation contains a boundary bilinear form, so the approximation of the eigenfunction on $\partial\Omega$ also enters the error analysis.

For continuous piecewise linear finite elements on triangular meshes, u_h is continuous, whereas ∇u_h is piecewise constant and generally discontinuous across element interfaces. We employ the classical equal-weight simple-averaging gradient recovery operator, in which the element gradients in each nodal patch are averaged with identical weights (Hlaváček et al., 1996). The appeal of this recovery is its simplicity: it requires neither local polynomial fitting nor element-area weighting. Our focus is on its Steklov-specific superconvergence and on how the recovered gradient can be used for eigenvalue error estimation and postprocessing.

1.2 Related Work

Finite element approximation of symmetric elliptic eigenvalue problems has been extensively studied; Boffi provides a systematic account of spectral approximation and convergence theory (Boffi, 2010). For the Steklov problem, the spectral parameter occurs in the boundary condition, and error analysis must account for both domain approximation and boundary errors. Bi et al. established error estimates for linear triangular finite element approximations, including conforming and nonconforming methods (Bi et al., 2018). These results provide a basis for the finite element analysis in Section 2.

Gradient recovery is a postprocessing technique for improving finite element derivative approximations. The superconvergent patch recovery method of Zienkiewicz and Zhu uses element information around mesh nodes to construct a recovered gradient (Zienkiewicz and Zhu, 1992). Hlaváček, Křížek, and Pištora studied a simple averaging formula for recovering gradients of linear finite elements and established superconvergence properties on suitable triangulations (Hlaváček et al., 1996). Xu and Zhang analyzed recovery-type estimators on mildly structured grids and highlighted the role of local mesh geometry in superconvergence (Xu and Zhang, 2004).

Recovery techniques have also been applied to eigenvalue problems. Mao, Shen, and Zhou studied adaptive algorithms using local-averaging a posteriori estimators (Mao et al., 2006), while Naga, Zhang, and Zhou used polynomial-preserving recovery to improve eigenvalue approximations (Naga et al., 2006). For the Steklov problem, Xiong et al. developed residual-based a posteriori and superconvergence analysis with postprocessing through an auxiliary source problem (Xiong et al., 2023). Zhang and Han considered asymptotically exact recovery-based estimators for nonsymmetric elliptic eigenvalue problems (Zhang and Han, 2025).

The present work builds on the classical equal-weight simple-averaging recovery operator rather than introducing a new recovery formula. Its contribution is to develop, for the Steklov eigenvalue problem, a unified analysis that connects recovered-gradient superconvergence with a computable eigenvalue-error estimator, proves asymptotic exactness under a nondegeneracy condition, and derives a recovery-based postprocessed eigenvalue. The analysis explicitly accounts for the boundary spectral term characteristic of the Steklov problem and is carried out under specified regularity and local mesh-structure assumptions.

1.3 Main Contributions

Building on the classical simple-averaging recovery operator, this paper develops a Steklov-specific framework for gradient superconvergence, eigenvalue error estimation, asymptotic exactness, and eigenvalue postprocessing. The main contributions are as follows.

First, under suitable regularity and local mesh-structure assumptions, the recovered gradient is shown to satisfy the superconvergence estimate

$$\|G_h u_h - \nabla u\|_{0,\Omega} \le C h^{1+\alpha_0}, \qquad 0 < \alpha_0 \le \frac{1}{2}.$$

Second, a computable recovery-based error estimator is defined by

$$\eta_h = \|G_h u_h - \nabla u_h\|_{0,\Omega}.$$

Combining the superconvergence result with the eigenvalue error identity yields

$$\lambda_h - \lambda = \eta_h^2 + O(h^{2+\alpha_0}).$$

Moreover, for the first isolated simple Steklov eigenvalue, under the additional nondegeneracy condition

$$\lambda_h - \lambda \geq c_0 h^2,$$

the estimator is asymptotically exact:

$$\frac{\eta_h^2}{\lambda_h - \lambda} = 1 + O(h^{\alpha_0}).$$

Finally, a postprocessed eigenvalue is defined by

$$\widehat{\lambda}_h = \lambda_h - \eta_h^2,$$

and the following error estimate is established:

$$|\widehat{\lambda}_h - \lambda| \leq C h^{2+\alpha_0}.$$

1.4 Organization of the Paper

Section 2 presents the variational formulation, finite element discretization, a priori error estimates, and eigenvalue error identity. Section 3 defines the simple-averaging recovery operator and establishes its superconvergence, estimator exactness, and postprocessed eigenvalue estimate. Section 4 reports numerical experiments on convergence, estimator performance, and eigenvalue postprocessing. Section 5 concludes the paper and discusses possible extensions.

2 Steklov Eigenvalue Problem and Its Finite Element Approximation

Starting from the Steklov problem in Section 1, we formulate its weak form, define the linear finite element approximation, and derive the eigenfunction and eigenvalue error estimates needed for the recovery analysis in Section 3.

2.1 Basic Notation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. For a nonnegative integer m, define

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) \mid D^\alpha v \in L^2(\Omega),\ |\alpha| \leq m \right\}.$$

The corresponding norm and seminorm are

$$\|v\|_{m,\Omega} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{0,\Omega}^2 \right)^{1/2}, \qquad |v|_{m,\Omega} = \left(\sum_{|\alpha| = m} \|D^\alpha v\|_{0,\Omega}^2 \right)^{1/2}.$$

For $m = 0$, we use

$$(u, v)_\Omega = \int_\Omega uv \, dx, \qquad \|v\|_{0,\Omega} = (v, v)_\Omega^{1/2}.$$

On $\partial\Omega$, let

$$\langle u, v\rangle_{\partial\Omega} = \int_{\partial\Omega} uv\, ds, \qquad \|v\|_{0,\partial\Omega} = \langle v, v\rangle_{\partial\Omega}^{1/2}.$$

For vector fields $\boldsymbol{p}, \boldsymbol{q} \in [L^2(\Omega)]^2$, the L^2 inner product and norm are

$$(\boldsymbol{p}, \boldsymbol{q})_\Omega = \int_\Omega \boldsymbol{p}\cdot\boldsymbol{q}\, dx, \qquad \|\boldsymbol{p}\|_{0,\Omega} = (\boldsymbol{p}, \boldsymbol{p})_\Omega^{1/2}.$$

Set $V = H^1(\Omega)$ and define

$$\begin{aligned} a(u, v) &= \int_\Omega \left(\nabla u\cdot\nabla v + \kappa^2 uv\right)\, dx, \\ b(u, v) &= \int_{\partial\Omega} uv\, ds. \end{aligned} \tag{2.1}$$

Both forms are symmetric. Since $\kappa > 0$,

$$a(v, v) = \|\nabla v\|_{0,\Omega}^2 + \kappa^2\|v\|_{0,\Omega}^2 \geq \min\{1, \kappa^2\}\|v\|_{1,\Omega}^2,$$

so $a(\cdot,\cdot)$ is coercive and defines the energy norm $\|v\|_a = a(v, v)^{1/2}$. The trace theorem gives

$$\|v\|_{0,\partial\Omega} \leq C\|v\|_{1,\Omega}, \qquad v \in H^1(\Omega),$$

and hence $b(\cdot,\cdot)$ is continuous on $H^1(\Omega)$. Throughout, C denotes a positive constant independent of h, which may vary between occurrences.

2.2 The Steklov Eigenvalue Problem and Its Weak Formulation

For arbitrary $v \in H^1(\Omega)$, multiplying the interior equation in (1.1) by v, integrating over Ω, and applying Green's formula together with $\partial u/\partial n = \lambda u$ gives

$$\int_\Omega \left(\nabla u\cdot\nabla v + \kappa^2 uv\right)\, dx = \lambda\int_{\partial\Omega} uv\, ds.$$

Thus, the weak problem is to find $(\lambda, u) \in \mathbb{R}\times V$, with $u \neq 0$, such that

$$a(u, v) = \lambda b(u, v), \qquad \forall v \in V. \tag{2.2}$$

Since an eigenfunction is determined up to a nonzero multiplicative constant, we impose the boundary normalization

$$b(u, u) = \|u\|_{0,\partial\Omega}^2 = 1. \tag{2.3}$$

Taking $v = u$ in (2.2) yields

$$\lambda = a(u, u) = \|\nabla u\|_{0,\Omega}^2 + \kappa^2\|u\|_{0,\Omega}^2.$$

2.3 Finite Element Discretization

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω, with

$$h = \max_{K\in\mathcal{T}_h} h_K,$$

where h_K is the diameter of K. Define the continuous piecewise linear finite element space

$$V_h = \left\{v_h \in C(\overline{\Omega}) \mid v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\right\} \subset V, \tag{2.4}$$

where $P_1(K)$ is the space of linear polynomials on K. The discrete eigenvalue problem is to find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$, with $u_h \neq 0$, such that

$$a(u_h, v_h) = \lambda_h b(u_h, v_h), \qquad \forall v_h \in V_h. \tag{2.5}$$

We impose the boundary normalization

$$b(u_h, u_h) = \|u_h\|^2_{0,\partial\Omega} = 1. \tag{2.6}$$

Taking $v_h = u_h$ gives

$$\lambda_h = a(u_h, u_h) = \|\nabla u_h\|^2_{0,\Omega} + \kappa^2 \|u_h\|^2_{0,\Omega}.$$

Let $\{z_i\}_{i=1}^N$ and $\{\phi_i\}_{i=1}^N$ be the mesh nodes and their nodal basis functions. Writing

$$u_h = \sum_{i=1}^{N} U_i \phi_i, \qquad \boldsymbol{U} = (U_1, U_2, \ldots, U_N)^T,$$

and testing (2.5) with each ϕ_i yields

$$A\boldsymbol{U} = \lambda_h B\boldsymbol{U}, \tag{2.7}$$

where

$$A_{ij} = a(\phi_j, \phi_i), \qquad B_{ij} = b(\phi_j, \phi_i).$$

2.4 A Priori Error Estimates for the Finite Element Approximation

Let (λ, u) be an isolated simple eigenpair and (λ_h, u_h) its corresponding finite element eigenpair for sufficiently small h. Assume that both eigenfunctions are boundary-normalized, their signs are chosen consistently, $u \in H^2(\Omega)$, and the associated dual problem has H^2 regularity. Let $I_h u \in V_h$ be the nodal interpolant. Standard interpolation and spectral approximation estimates for isolated simple eigenvalues (see, e.g., Boffi, 2010; Bi et al., 2018) give

$$\|u - I_h u\|_{0,\Omega} + h\|u - I_h u\|_{1,\Omega} \leq Ch^2 \|u\|_{2,\Omega}. \tag{2.8}$$

The energy- and L^2-error estimates are

$$\|u - u_h\|_{1,\Omega} \leq Ch\|u\|_{2,\Omega}, \tag{2.9}$$

and, by the duality argument,

$$\|u - u_h\|_{0,\Omega} \leq Ch^2 \|u\|_{2,\Omega}. \tag{2.10}$$

The multiplicative trace inequality states that

$$\|v\|^2_{0,\partial\Omega} \leq C\left(\|v\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + \|v\|^2_{0,\Omega}\right), \qquad v \in H^1(\Omega). \tag{2.11}$$

Setting $e = u - u_h$ and applying (2.9)–(2.10), we obtain

$$\begin{aligned} \|e\|^2_{0,\partial\Omega} &\leq C\left(\|e\|_{0,\Omega}\|\nabla e\|_{0,\Omega} + \|e\|^2_{0,\Omega}\right) \\ &\leq C(h^3 + h^4)\|u\|^2_{2,\Omega} \leq Ch^3 \|u\|^2_{2,\Omega}. \end{aligned}$$

Consequently,

$$\|u - u_h\|_{0,\partial\Omega} \leq Ch^{3/2} \|u\|_{2,\Omega}. \tag{2.12}$$

2.5 An Eigenvalue Error Identity

The following identity relates the eigenvalue error to the eigenfunction error.

Lemma 2.1. *Let* (λ, u) *and* (λ_h, u_h) *be corresponding exact and discrete eigenpairs satisfying* $b(u,u) = b(u_h,u_h) = 1$*. With* $e = u - u_h$*, we have*

$$\lambda_h - \lambda = a(e,e) - \lambda b(e,e). \tag{2.13}$$

Proof. The eigenvalue equations and normalization give

$$\lambda = a(u,u), \qquad \lambda_h = a(u_h,u_h).$$

Since $u_h = u - e$, symmetry of $a(\cdot,\cdot)$ yields

$$\lambda_h - \lambda = a(e,e) - 2a(u,e).$$

Taking $v = e$ in the continuous eigenvalue equation gives $a(u,e) = \lambda b(u,e)$. Moreover, the normalization conditions imply

$$b(e,e) = b(u-u_h, u-u_h) = 2 - 2b(u,u_h) = 2b(u,e).$$

Hence $2a(u,e) = \lambda b(e,e)$, and substitution proves the identity. □

By the definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, (2.13) becomes

$$\lambda_h - \lambda = \|\nabla e\|_{0,\Omega}^2 + \kappa^2 \|e\|_{0,\Omega}^2 - \lambda \|e\|_{0,\partial\Omega}^2, \qquad e = u - u_h. \tag{2.14}$$

The estimates above imply

$$\|\nabla e\|_{0,\Omega}^2 = O(h^2), \qquad \|e\|_{0,\Omega}^2 = O(h^4), \qquad \|e\|_{0,\partial\Omega}^2 = O(h^3).$$

Thus, the last two terms in (2.14) are of higher order, and

$$\lambda_h - \lambda = \|\nabla(u - u_h)\|_{0,\Omega}^2 + O(h^3). \tag{2.15}$$

Taking absolute values in (2.14) and using the same estimates gives

$$\begin{aligned} |\lambda_h - \lambda| &\le \|\nabla e\|_{0,\Omega}^2 + \kappa^2 \|e\|_{0,\Omega}^2 + |\lambda| \|e\|_{0,\partial\Omega}^2 \\ &\le C(h^2 + h^4 + h^3) \le Ch^2. \end{aligned}$$

Therefore,

$$|\lambda_h - \lambda| \le Ch^2. \tag{2.16}$$

This is an upper bound and does not provide a second-order lower bound. For the first eigenvalue, the conforming min–max principle gives $\lambda_h \ge \lambda$, but does not ensure $\lambda_h - \lambda \ge c_0 h^2$. We therefore impose this nondegeneracy condition, with $c_0 > 0$ independent of h, as an additional assumption in the asymptotic exactness theorem in Section 3. Together with (2.16), it gives an eigenvalue error of order h^2. Relation (2.15) identifies the squared gradient error as the leading term and motivates the recovery-based estimator studied in Section 3.

3 Simple-Averaging Gradient Recovery and Error Estimation

Building on the finite element approximation theory established in Section 2, this section investigates a simple-averaging gradient recovery method for the Steklov eigenvalue problem and the associated error estimates. We first define an equal-weight simple-averaging gradient recovery operator and discuss its basic properties. We then analyze the superconvergence of the recovered gradient under suitable local mesh structures, in conjunction with a finite element interpolation superapproximation lemma. Finally, we construct a computable error estimator from the difference between the recovered gradient and the finite element gradient and, by combining it with the eigenvalue error identity derived in Section 2, investigate how accurately this estimator approximates the Steklov eigenvalue error.

3.1 The Simple-Averaging Gradient Recovery Operator

Let $\mathcal{T}_h$ be a triangulation of Ω, and let V_h be the continuous piecewise linear finite element space defined in Section 2. Since $v_h|_K \in P_1(K)$ for every $v_h \in V_h$, its gradient is constant on each element K, but is generally discontinuous across element interfaces.

For a mesh node z, define its nodal patch and the number of elements in the patch by

$$\omega_z = \{K \in \mathcal{T}_h : \ z \in \overline{K}\}, \qquad N_z = \#\omega_z.$$

The simple-averaging recovered gradient at z is

$$G_h v_h(z) = \frac{1}{N_z} \sum_{K \in \omega_z} \nabla v_h|_K, \qquad v_h \in V_h.$$

Thus, every element in the nodal patch receives the same weight $1/N_z$; no element-area weighting or local polynomial fitting is used. Here, $G_h v_h$ denotes the recovery of the elementwise gradient of v_h.

Let $\{\phi_z\}$ be the nodal basis of V_h. Extending the recovered nodal values by linear interpolation gives

$$G_h v_h(x) = \sum_z G_h v_h(z)\phi_z(x) \in [V_h]^2.$$

Hence, $G_h v_h$ is a continuous piecewise linear vector field obtained by element-gradient evaluation, equal-weight nodal averaging, and nodal interpolation.

3.2 Basic Properties of the Recovery Operator and Mesh Assumptions

We record the linearity and stability of G_h, and then state the local mesh conditions used in the superconvergence analysis.

3.2.1 Linearity

By the definition of G_h, for any $v_h, w_h \in V_h$ and $c_1, c_2 \in \mathbb{R}$,

$$G_h(c_1 v_h + c_2 w_h) = c_1 G_h v_h + c_2 G_h w_h.$$

In particular, $G_h(v_h - w_h) = G_h v_h - G_h w_h$.

3.2.2 Stability

Assume that $\mathcal{T}_h$ is shape-regular and quasi-uniform. Then the number of elements in each nodal patch is uniformly bounded. Jensen's inequality gives, for every node z,

$$|G_h v_h(z)|^2 \le \frac{1}{N_z} \sum_{K \in \omega_z} \left| \nabla v_h|_K \right|^2.$$

For an element K, let $\mathcal{N}(K)$ be its three vertices. Local norm equivalence for piecewise linear functions yields

$$\|G_h v_h\|_{0,K}^2 \le C|K| \sum_{z \in \mathcal{N}(K)} |G_h v_h(z)|^2.$$

Summing over $\mathcal{T}_h$, and using the comparability of element areas within each nodal patch and the uniformly bounded overlap of the patches, we obtain

$$\begin{aligned}\|G_h v_h\|_{0,\Omega}^2 &\le C \sum_{K\in\mathcal{T}_h} |K| \sum_{z\in\mathcal{N}(K)} \frac{1}{N_z} \sum_{K'\in\omega_z} \left|\nabla v_h|_{K'}\right|^2 \\ &\le C \sum_{K'\in\mathcal{T}_h} |K'| \left|\nabla v_h|_{K'}\right|^2 = C\|\nabla v_h\|_{0,\Omega}^2.\end{aligned}$$

Consequently,

$$\|G_h v_h\|_{0,\Omega} \le C\|\nabla v_h\|_{0,\Omega}, \qquad v_h \in V_h,$$

where C is independent of h. This stability estimate alone does not imply superconvergence: on a general unstructured mesh, simple averaging need not improve the convergence order. We therefore impose the following local mesh conditions.

3.2.3 Local Mesh Structure Assumptions

Let $\{\mathcal{T}_h\}$ be a family of shape-regular, quasi-uniform triangulations of the fixed polygonal domain Ω, and set

$$h = \max_{K\in\mathcal{T}_h} h_K.$$

Let $\mathcal{N}_h^\circ$ denote the set of interior nodes. For an interior node z, enumerate the triangles in ω_z counterclockwise as

$$K_i = \triangle(z, x_i, x_{i+1}), \qquad i = 1, \dots, N_z,$$

with cyclic indices. For the adjacent pair K_i, K_{i+1}, write the vertices of their union in boundary order as

$$A_i = x_i, \qquad B_i = x_{i+1}, \qquad C_i = x_{i+2}, \qquad D_i = z.$$

We call z a regular node if, for some fixed $0 < \alpha \le 1$ and a constant C independent of z and h,

$$|A_i + C_i - B_i - D_i| = |x_i + x_{i+2} - x_{i+1} - z| \le Ch^{1+\alpha}, \quad i = 1, \dots, N_z. \tag{3.1}$$

Let $\mathcal{Z}_{1,h}$ be the set of regular interior nodes and let

$$\mathcal{Z}_{2,h} = \mathcal{N}_h^\circ \setminus \mathcal{Z}_{1,h}.$$

Define the exceptional region

$$\Omega_{2,h} = \bigcup_{z\in\mathcal{Z}_{2,h}} \bigcup_{K\in\omega_z} \overline{K},$$

and assume

$$|\Omega_{2,h}| \le Ch^{\sigma_0}, \qquad \sigma_0 > 0. \tag{3.2}$$

Set

$$\alpha_0 = \min\left\{\alpha, \frac{\sigma_0}{2}, \frac{1}{2}\right\}. \tag{3.3}$$

Let $\Omega_{\partial,h}$ be the union of elements containing boundary nodes. Shape regularity and quasi-uniformity on the fixed polygonal domain imply

$$|\Omega_{\partial,h}| \le Ch. \tag{3.4}$$

Writing $\mathcal{Z}_{\partial,h}$ for the boundary nodes, partition the mesh into

$$\begin{aligned}\mathcal{T}_{2,h} &= \{K \in \mathcal{T}_h : \mathcal{N}(K) \cap (\mathcal{Z}_{2,h} \cup \mathcal{Z}_{\partial,h}) \neq \varnothing\}, \\ \mathcal{T}_{1,h} &= \mathcal{T}_h \setminus \mathcal{T}_{2,h}.\end{aligned}$$

Thus,

$$\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}, \qquad \mathcal{T}_{1,h} \cap \mathcal{T}_{2,h} = \varnothing.$$

If

$$\widetilde{\Omega}_{2,h} = \bigcup_{K \in \mathcal{T}_{2,h}} \overline{K} = \Omega_{2,h} \cup \Omega_{\partial,h}, \qquad \widetilde{\sigma}_0 = \min\{\sigma_0, 1\},$$

then (3.2) and (3.4) give

$$|\widetilde{\Omega}_{2,h}| \le C(h^{\sigma_0} + h) \le Ch^{\widetilde{\sigma}_0}.$$

Every node of an element in $\mathcal{T}_{1,h}$ is a regular interior node. Hence, for any two adjacent elements in $\mathcal{T}_{1,h}$ sharing an edge, their union satisfies the approximate parallelogram condition (3.1) at either endpoint of the common edge. The resulting partition therefore satisfies the $(\alpha, \widetilde{\sigma}_0)$ mesh condition of Xu and Zhang, with

$$\min\left\{\alpha, \frac{\widetilde{\sigma}_0}{2}, \frac{1}{2}\right\} = \min\left\{\alpha, \frac{\sigma_0}{2}, \frac{1}{2}\right\} = \alpha_0.$$

For a symmetric matrix $H \in \mathbb{R}^{2\times 2}$, define

$$q_{z,H}(x) = \frac{1}{2}(x - z)^T H (x - z),$$

and let $I_K q_{z,H}$ be its local linear Lagrange interpolant on K. We use the Frobenius norm

$$\|H\|_F = \left(\sum_{i,j=1}^{2} |H_{ij}|^2\right)^{1/2}.$$

The following local cancellation property is the key geometric estimate for the recovery analysis.

Lemma 3.1 (Local cancellation estimate under the approximate parallelogram condition)**.** *Let $\mathcal{T}_h$ be shape-regular and quasi-uniform, and let $z \in \mathcal{Z}_{1,h}$ be a regular interior node satisfying* (3.1) *throughout ω_z. For sufficiently small h, the patch ω_z consists of six triangles. Moreover, for every symmetric matrix $H \in \mathbb{R}^{2\times 2}$,*

$$\left| \frac{1}{N_z} \sum_{K \in \omega_z} \nabla I_K q_{z,H} \right| \le Ch^{1+\alpha} \|H\|_F, \tag{3.5}$$

where C is independent of h.

Proof. Write

$$r_i = x_i - z, \qquad e_i = r_i + r_{i+2} - r_{i+1}.$$

By (3.1),

$$|e_i| \le Ch^{1+\alpha}, \qquad i = 1, \dots, N_z, \tag{3.6}$$

and hence $r_{i+2} = r_{i+1} - r_i + e_i$. Shape regularity and quasi-uniformity imply that the number of elements in ω_z is uniformly bounded and

$$c \le \frac{|r_i|}{h} \le C, \qquad \det(r_i, r_{i+1}) \ge ch^2.$$

We first prove that $N_z = 6$ for sufficiently small h. Otherwise, there would exist a sequence $h_j \to 0$ and regular nodes z_j whose patches have a fixed number $N_* \ne 6$ of triangles along a subsequence. The normalized vectors

$$\xi_i^{(j)} = \frac{r_i^{(j)}}{h_j}$$

are uniformly bounded; after taking a subsequence, they converge to vectors ξ_i with $\det(\xi_i, \xi_{i+1}) \geq c > 0$. Dividing (3.6) by h_j and passing to the limit gives

$$\xi_{i+2} = \xi_{i+1} - \xi_i.$$

Thus,

$$\xi_3 = \xi_2 - \xi_1, \quad \xi_4 = -\xi_1, \quad \xi_5 = -\xi_2, \quad \xi_6 = \xi_1 - \xi_2, \quad \xi_7 = \xi_1, \quad \xi_8 = \xi_2.$$

The recurrence has minimal period six because ξ_1 and ξ_2 are linearly independent. Cyclic closure gives $\xi_{N_*+1} = \xi_1$ and $\xi_{N_*+2} = \xi_2$, so $N_* = 6m$ for some positive integer m.

Let $A = [\xi_1\ \xi_2]^{-1}$. Since $\det(\xi_1, \xi_2) > 0$, we have $\det A > 0$, and A is orientation-preserving. In particular, $A\xi_1 = (1,0)^T$ and $A\xi_2 = (0,1)^T$. The first six directions are mapped, in order, to

$$(1,0)^T,\ (0,1)^T,\ (-1,1)^T,\ (-1,0)^T,\ (0,-1)^T,\ (1,-1)^T.$$

They form exactly one positively oriented cycle. Hence the full sequence of $N_* = 6m$ directions winds around the node m times. Since the triangles around an interior node form one complete fan, the winding number is one. Therefore $m = 1$ and $N_* = 6$, contradicting $N_* \neq 6$.

The recurrence also gives

$$r_{i+3} = -r_i + e_i + e_{i+1}, \qquad r_{i+4} = -r_{i+1} + e_{i+1} + e_{i+2},$$

and consequently

$$|r_{i+3} + r_i| + |r_{i+4} + r_{i+1}| \leq Ch^{1+\alpha}.$$

For $K_i = \triangle(z, x_i, x_{i+1})$, set

$$R_i = \begin{pmatrix} | & | \\ r_i & r_{i+1} \\ | & | \end{pmatrix}, \qquad g_i = \nabla I_{K_i} q_{z,H}.$$

The interpolation conditions yield

$$g_i = \frac{1}{2} R_i^{-T} \begin{pmatrix} r_i^T H r_i \\ r_{i+1}^T H r_{i+1} \end{pmatrix}. \tag{3.7}$$

Define

$$\mathcal{G}_H(r,s) = \frac{1}{2} \begin{pmatrix} r & s \end{pmatrix}^{-T} \begin{pmatrix} r^T H r \\ s^T H s \end{pmatrix}.$$

Then $g_i = \mathcal{G}_H(r_i, r_{i+1})$ and $\mathcal{G}_H(-r,-s) = -\mathcal{G}_H(r,s)$.

By shape regularity and quasi-uniformity, the normalized pairs $(r/h, s/h)$ lie in a uniformly bounded shape-regular set. Since $\mathcal{G}_H(r,s) = h\mathcal{G}_H(r/h, s/h)$, its derivatives with respect to r, s are bounded by $C\|H\|_F$, uniformly in h. Thus $\mathcal{G}_H$ is uniformly Lipschitz with respect to its vertex vectors on the relevant set.

Set

$$\delta_1 = r_{i+3} + r_i, \qquad \delta_2 = r_{i+4} + r_{i+1},$$

so that $|\delta_1| + |\delta_2| \leq Ch^{1+\alpha}$. Along the segment

$$p_i(t) = -r_i + t\delta_1, \qquad q_i(t) = -r_{i+1} + t\delta_2, \qquad 0 \leq t \leq 1,$$

the vector lengths remain of order h, and

$$\det(p_i(t), q_i(t)) = \det(r_i, r_{i+1}) + O(h^{2+\alpha}) \geq \frac{c}{2} h^2$$

for sufficiently small h. The segment therefore remains in a uniformly shape-regular set. The Lipschitz estimate and the oddness of $\mathcal{G}_H$ then imply

$$|g_{i+3} + g_i| = |\mathcal{G}_H(r_{i+3}, r_{i+4}) - \mathcal{G}_H(-r_i, -r_{i+1})| \leq Ch^{1+\alpha}\|H\|_F.$$

Pairing g_1 with g_4, g_2 with g_5, and g_3 with g_6, we obtain

$$\left|\sum_{i=1}^{6} g_i\right| \le \sum_{i=1}^{3} |g_{i+3} + g_i| \le Ch^{1+\alpha}\|H\|_F.$$

Since $N_z = 6$, this proves (3.5). □

3.3 Superconvergence of the Recovered Gradient

Under the mesh conditions in Section 3.2, we establish the global superconvergence of the simple equal-weight averaging recovered gradient. The proof combines an interpolation superapproximation estimate, a supercloseness estimate for the discrete eigenfunction, and a local recovery estimate at regular interior nodes.

Lemma 3.2 (Superapproximation estimate for the finite element interpolation error)**.** *Let $u \in W^{3,\infty}(\Omega)$, and let $\mathcal{T}_h$ satisfy the mesh conditions in Section 3.2. Then, for any $v_h \in V_h$,*

$$|(\nabla(u - I_h u), \nabla v_h)_\Omega| \le Ch^{1+\alpha_0}\|u\|_{W^{3,\infty}(\Omega)}\|v_h\|_{1,\Omega}, \tag{3.8}$$

where α_0 is defined in (3.3)*, and C is independent of h.*

Proof. The mesh partition in Section 3.2 satisfies the $(\alpha, \widetilde{\sigma}_0)$ condition, with $\widetilde{\sigma}_0 = \min\{\sigma_0, 1\}$. Applying Lemma 2.1 of Xu and Zhang (2004) with the piecewise constant matrix $D_K = I$, whose boundedness and adjacent-element variation conditions are immediate, and using $u \in W^{3,\infty}(\Omega) \subset H^3(\Omega) \cap W^{2,\infty}(\Omega)$, gives

$$|(\nabla(u - I_h u), \nabla v_h)_\Omega| \le Ch^{1+\rho}\left(\|u\|_{H^3(\Omega)} + |u|_{W^{2,\infty}(\Omega)}\right)|v_h|_{1,\Omega},$$

where

$$\rho = \min\left\{\alpha, \frac{\widetilde{\sigma}_0}{2}, \frac{1}{2}\right\} = \alpha_0.$$

Since $|v_h|_{1,\Omega} \le \|v_h\|_{1,\Omega}$ and $\|u\|_{H^3(\Omega)} + |u|_{W^{2,\infty}(\Omega)} \le C\|u\|_{W^{3,\infty}(\Omega)}$, estimate (3.8) follows. □

Lemma 3.3 (Supercloseness estimate for the Steklov finite element eigenfunction)**.** *Let (λ, u) be an isolated simple Steklov eigenpair and (λ_h, u_h) the corresponding discrete eigenpair, with $\kappa > 0$ and $u \in W^{3,\infty}(\Omega)$. Assume*

$$b(u, u) = b(u_h, u_h) = 1,$$

and that the signs of u and u_h are chosen consistently. If the a priori estimates in (2.16) *and* (2.12) *hold, namely,*

$$|\lambda_h - \lambda| \le Ch^2, \qquad \|u - u_h\|_{0,\partial\Omega} \le Ch^{3/2},$$

then

$$\|u_h - I_h u\|_{1,\Omega} \le Ch^{1+\alpha_0}. \tag{3.9}$$

The constant is independent of h and may depend on u and its regularity norm.

Proof. Set $e_h = u_h - I_h u$. Subtracting the continuous and discrete variational equations gives, for any $v_h \in V_h$,

$$\begin{aligned} a(e_h, v_h) = a(u - I_h u, v_h) + (\lambda_h - \lambda)b(u_h, v_h) \\ + \lambda b(u_h - u, v_h). \end{aligned} \tag{3.10}$$

Taking $v_h = e_h$, Lemma 3.2 and the standard interpolation estimate yield

$$|a(u - I_h u, e_h)| \le Ch^{1+\alpha_0}\|e_h\|_{1,\Omega}. \tag{3.11}$$

Here we used $h^2 \le h^{1+\alpha_0}$, since $0 < \alpha_0 \le 1/2$ and $h \le 1$.

For the remaining terms, the boundary normalization, the trace inequality, and the two assumed a priori estimates give

$$|(\lambda_h - \lambda) b(u_h, e_h)| \le C h^2 \|e_h\|_{1,\Omega}, \tag{3.12}$$

and

$$|\lambda b(u_h - u, e_h)| \le C h^{3/2} \|e_h\|_{1,\Omega}. \tag{3.13}$$

Combining (3.10)–(3.13) and using the coercivity of $a(\cdot,\cdot)$, we obtain

$$c\|e_h\|_{1,\Omega}^2 \le C h^{1+\alpha_0} \|e_h\|_{1,\Omega},$$

because $h^2 + h^{3/2} \le C h^{1+\alpha_0}$. Dividing by $\|e_h\|_{1,\Omega}$ when $e_h \neq 0$ proves (3.9); the result is immediate if $e_h = 0$. □

Lemma 3.4 (Local recovery error estimate at regular interior nodes)**.** *Let $u \in W^{3,\infty}(\Omega)$, and let $\mathcal{T}_h$ satisfy the shape-regularity, quasi-uniformity, and local mesh conditions in Section 3.2. For the nodal interpolant $I_h u$ and the simple equal-weight averaging recovery operator G_h, there is a constant C independent of h such that, for every regular interior node $z \in \mathcal{Z}_{1,h}$ and sufficiently small h,*

$$|G_h I_h u(z) - \nabla u(z)| \le C h^{1+\alpha} \|u\|_{W^{3,\infty}(\Omega)}. \tag{3.14}$$

Proof. For a regular interior node z, let $H_z = D^2 u(z)$ and $q_{z,H_z}(x) = \frac{1}{2}(x-z)^T H_z (x-z)$. Taylor's formula gives, on the patch ω_z,

$$u(x) = u(z) + \nabla u(z)\cdot(x-z) + q_{z,H_z}(x) + R_z(x), \qquad |R_z(x)| \le C h^3 \|u\|_{W^{3,\infty}(\Omega)}.$$

Since linear interpolation reproduces affine functions,

$$\nabla I_K u = \nabla u(z) + \nabla I_K q_{z,H_z} + \nabla I_K R_z, \qquad K \in \omega_z.$$

The local cancellation estimate in Lemma 3.1 and the bound $\|H_z\|_F \le C\|u\|_{W^{2,\infty}(\Omega)}$ imply

$$\left| \frac{1}{N_z} \sum_{K \in \omega_z} \nabla I_K q_{z,H_z} \right| \le C h^{1+\alpha} \|u\|_{W^{3,\infty}(\Omega)}.$$

Moreover, the local interpolation gradient estimate gives

$$|\nabla I_K R_z| \le C h^{-1} \max_{a \in \mathcal{N}(K)} |R_z(a)| \le C h^2 \|u\|_{W^{3,\infty}(\Omega)}.$$

Averaging these estimates over ω_z and using $h^2 \le h^{1+\alpha}$ for $0 < \alpha \le 1$ proves (3.14). □

For the vector field ∇u, let $I_h(\nabla u)$ denote the vector-valued function obtained by applying nodal linear interpolation to each component.

Theorem 3.1 (Superconvergence of the simple equal-weight averaging recovered gradient)**.** *Let (λ, u) be an isolated simple Steklov eigenpair and (λ_h, u_h) the corresponding discrete eigenpair, with $\kappa > 0$ and $u \in W^{3,\infty}(\Omega)$. Assume that the mesh satisfies the conditions in Section 3.2, that both eigenfunctions are normalized in the boundary L^2 norm, and that their signs are chosen consistently. If the a priori estimates in* (2.16) *and* (2.12) *hold, then*

$$\|G_h u_h - \nabla u\|_{0,\Omega} \le C h^{1+\alpha_0}. \tag{3.15}$$

The constant is independent of h and may depend on u and its regularity norm.

Proof. By linearity of G_h, insert $G_h I_h u$ and $I_h(\nabla u)$ and apply the triangle inequality:

$$\begin{aligned}\|G_h u_h - \nabla u\|_{0,\Omega} \le \|G_h(u_h - I_h u)\|_{0,\Omega} \\ + \|G_h I_h u - I_h(\nabla u)\|_{0,\Omega} \\ + \|I_h(\nabla u) - \nabla u\|_{0,\Omega}.\end{aligned} \tag{3.16}$$

The stability of G_h and Lemma 3.3 give

$$\|G_h(u_h - I_h u)\|_{0,\Omega} \le C\|u_h - I_h u\|_{1,\Omega} \le Ch^{1+\alpha_0}.$$

For the second term, set $w_h = G_h I_h u - I_h(\nabla u)$ and use the partition $\Omega = \Omega_{1,h} \cup \widetilde{\Omega}_{2,h}$ from Section 3.2, up to element boundaries, where

$$\Omega_{1,h} = \bigcup_{K \in \mathcal{T}_{1,h}} \overline{K}.$$

If $K \in \mathcal{T}_{1,h}$, all its vertices z_i are regular interior nodes. Since the nodal basis functions are nonnegative and sum to one, Lemma 3.4 yields

$$|w_h(x)| = \left| \sum_{i=1}^{3} \big(G_h I_h u(z_i) - \nabla u(z_i)\big)\phi_i(x) \right| \le Ch^{1+\alpha_0}\|u\|_{W^{3,\infty}(\Omega)}, \quad x \in K.$$

Consequently,

$$\|w_h\|_{0,\Omega_{1,h}} \le Ch^{1+\alpha_0}\|u\|_{W^{3,\infty}(\Omega)}.$$

Now let $K \in \mathcal{T}_{2,h}$. At each vertex z_i of K, if z_i is regular, Lemma 3.4 gives

$$|G_h I_h u(z_i) - \nabla u(z_i)| \le Ch^{1+\alpha}\|u\|_{W^{3,\infty}(\Omega)} \le Ch\|u\|_{W^{3,\infty}(\Omega)}.$$

If z_i is exceptional or lies on the boundary, the standard linear interpolation gradient estimate on each $K' \in \omega_{z_i}$ gives

$$|\nabla I_{K'} u - \nabla u(z_i)| \le Ch\|u\|_{W^{2,\infty}(\Omega)}.$$

Taking the equal-weight average over $K' \in \omega_{z_i}$ gives the same $O(h)$ bound for $|G_h I_h u(z_i) - \nabla u(z_i)|$. Thus, by the partition-of-unity property,

$$|w_h(x)| \le Ch\|u\|_{W^{3,\infty}(\Omega)}, \qquad x \in K,$$

and hence

$$\|w_h\|_{0,\widetilde{\Omega}_{2,h}} \le Ch|\widetilde{\Omega}_{2,h}|^{1/2}\|u\|_{W^{3,\infty}(\Omega)}.$$

The area estimate in Section 3.2 gives $|\widetilde{\Omega}_{2,h}| \le Ch^{\widetilde{\sigma}_0}$, where $\widetilde{\sigma}_0 = \min\{\sigma_0, 1\}$. Since $\alpha_0 \le \widetilde{\sigma}_0/2$, it follows that

$$\|w_h\|_{0,\widetilde{\Omega}_{2,h}} \le Ch^{1+\widetilde{\sigma}_0/2}\|u\|_{W^{3,\infty}(\Omega)} \le Ch^{1+\alpha_0}\|u\|_{W^{3,\infty}(\Omega)}.$$

Combining the two regions, we obtain

$$\|G_h I_h u - I_h(\nabla u)\|_{0,\Omega} \le Ch^{1+\alpha_0}\|u\|_{W^{3,\infty}(\Omega)}. \tag{3.17}$$

Finally, componentwise interpolation of ∇u gives

$$\|I_h(\nabla u) - \nabla u\|_{0,\Omega} \le Ch^2\|u\|_{H^3(\Omega)} \le Ch^{1+\alpha_0}\|u\|_{W^{3,\infty}(\Omega)},$$

because $\alpha_0 \le 1/2$. Substitution of the three bounds into (3.16) proves (3.15). □

3.4 Recovery-Based Error Estimator

Based on the superconvergence result in Theorem 3.1, let

$$e = u - u_h, \qquad \delta_h = G_h u_h - \nabla u_h, \qquad \rho_h = G_h u_h - \nabla u.$$

Then

$$\nabla e = \delta_h - \rho_h.$$

By Theorem 3.1, we have

$$\|\rho_h\|_{0,\Omega} = \|G_h u_h - \nabla u\|_{0,\Omega} \le C h^{1+\alpha_0}.$$

We define the recovery-based error estimator by

$$\eta_h = \|G_h u_h - \nabla u_h\|_{0,\Omega} = \|\delta_h\|_{0,\Omega}.$$

Since it depends only on the finite element solution and its recovered gradient, η_h is computable without knowing the exact eigenfunction.

Furthermore, using $\nabla e = \delta_h - \rho_h$ and the reverse triangle inequality, we obtain

$$\begin{aligned} |\eta_h - \|\nabla e\|_{0,\Omega}| &= |\|\delta_h\|_{0,\Omega} - \|\delta_h - \rho_h\|_{0,\Omega}| \\ &\le \|\rho_h\|_{0,\Omega} \\ &\le C h^{1+\alpha_0}. \end{aligned}$$

Thus, the recovery-based estimator approximates the true finite element gradient error with a higher-order accuracy.

3.5 Recovery Estimate for the Eigenvalue Error

Under the boundary normalization $b(u,u) = b(u_h,u_h) = 1$, the eigenvalue error identity established in Section 2, together with $\nabla e = \delta_h - \rho_h$ and $\eta_h = \|\delta_h\|_{0,\Omega}$, gives

$$\begin{aligned} \lambda_h - \lambda &= \|\delta_h - \rho_h\|_{0,\Omega}^2 + \kappa^2 \|e\|_{0,\Omega}^2 - \lambda \|e\|_{0,\partial\Omega}^2 \\ &= \eta_h^2 - 2(\delta_h, \rho_h) + \|\rho_h\|_{0,\Omega}^2 \\ &\quad + \kappa^2 \|e\|_{0,\Omega}^2 - \lambda \|e\|_{0,\partial\Omega}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} &|(\lambda_h - \lambda) - \eta_h^2| \\ &\le 2|(\delta_h, \rho_h)| + \|\rho_h\|_{0,\Omega}^2 + \kappa^2 \|e\|_{0,\Omega}^2 + |\lambda| \|e\|_{0,\partial\Omega}^2. \end{aligned}$$

By the a priori estimates in Section 2, (2.9), (2.10), and (2.12), together with Theorem 3.1, we have

$$\begin{aligned} \|\nabla e\|_{0,\Omega} &\le C h, \\ \|e\|_{0,\Omega} &\le C h^2, \\ \|e\|_{0,\partial\Omega} &\le C h^{3/2}, \\ \|\rho_h\|_{0,\Omega} &\le C h^{1+\alpha_0}. \end{aligned}$$

Moreover, since $\delta_h = \nabla e + \rho_h$,

$$\|\delta_h\|_{0,\Omega} \le \|\nabla e\|_{0,\Omega} + \|\rho_h\|_{0,\Omega} \le C h.$$

Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}&|(\lambda_h - \lambda) - \eta_h^2| \\ &\le C\left(h^{2+\alpha_0} + h^{2+2\alpha_0} + h^4 + h^3\right) \\ &\le Ch^{2+\alpha_0},\end{aligned}$$

where $0 < \alpha_0 \le 1/2$ and $h \le 1$.

Thus, we obtain

$$\lambda_h - \lambda = \eta_h^2 + O(h^{2+\alpha_0}). \tag{3.18}$$

This establishes that the recovery-based estimator approximates the eigenvalue error up to a higher-order term, providing the basis for the asymptotic exactness result in the next section.

3.6 Asymptotic Exactness of the Error Estimator

Theorem 3.2 (Asymptotic exactness of the recovery-based error estimator)**.** *Let (λ, u) be the first isolated simple Steklov eigenpair, and let (λ_h, u_h) be the corresponding first finite element eigenpair. Assume that the hypotheses leading to* (3.18) *hold and that the eigenvalue error satisfies*

$$\lambda_h - \lambda \ge c_0 h^2,$$

where $c_0 > 0$ is independent of h. Then

$$\frac{\eta_h^2}{\lambda_h - \lambda} = 1 + O(h^{\alpha_0}). \tag{3.19}$$

In particular,

$$\lim_{h\to 0} \frac{\eta_h^2}{\lambda_h - \lambda} = 1. \tag{3.20}$$

Proof. By (3.18),

$$\left|(\lambda_h - \lambda) - \eta_h^2\right| \le Ch^{2+\alpha_0}.$$

Together with $\lambda_h - \lambda \ge c_0 h^2$, this gives

$$\begin{aligned}\left|\frac{\eta_h^2}{\lambda_h - \lambda} - 1\right| &= \frac{|\eta_h^2 - (\lambda_h - \lambda)|}{\lambda_h - \lambda} \\ &\le \frac{Ch^{2+\alpha_0}}{c_0 h^2} = Ch^{\alpha_0}.\end{aligned}$$

Since $\alpha_0 > 0$, letting $h \to 0$ proves the result. □

For the first Steklov eigenvalue, the conforming finite element approximation satisfies $\lambda_h \ge \lambda$ by the min–max principle. Define the effectivity index by

$$R_h = \frac{\eta_h^2}{|\lambda_h - \lambda|}. \tag{3.21}$$

Theorem 3.2 therefore implies

$$R_h \longrightarrow 1, \qquad h \to 0.$$

When the exact eigenvalue is unavailable, a sufficiently accurate fine-mesh approximation λ_{ref} can be used to define the numerical effectivity index

$$R_h^{\mathrm{ref}} = \frac{\eta_h^2}{|\lambda_h - \lambda_{\mathrm{ref}}|}. \tag{3.22}$$

This quantity approximates the exact effectivity index only when the reference eigenvalue error is negligible relative to $|\lambda_h - \lambda|$.

To improve the accuracy of the finite element eigenvalue, we define the postprocessed eigenvalue by

$$\widehat{\lambda}_h = \lambda_h - \eta_h^2. \tag{3.23}$$

Theorem 3.3 (Error estimate for the postprocessed eigenvalue)**.** *Assume that the preceding superconvergence conditions and the eigenvalue error estimate* (3.18) *hold. Then*

$$|\widehat{\lambda}_h - \lambda| \le Ch^{2+\alpha_0}, \tag{3.24}$$

where C is independent of h.

Proof. By (3.23),

$$\widehat{\lambda}_h - \lambda = (\lambda_h - \lambda) - \eta_h^2.$$

The estimate follows directly from (3.18). □

Thus, subtracting the recovery-based estimator cancels the leading term in the original eigenvalue error and improves the convergence order. In particular, when $\alpha_0 = 1/2$, the postprocessed eigenvalue error is $O(h^{5/2})$. Its numerical performance is examined in Section 4.

3.7 Section Summary

This section developed a simple equal-weight averaging gradient recovery operator and established its linearity, stability, and superconvergence properties. Under the prescribed mesh conditions, a local cancellation estimate was combined with the interpolation superapproximation and eigenfunction supercloseness estimates to obtain

$$\|G_h u_h - \nabla u\|_{0,\Omega} \le Ch^{1+\alpha_0}.$$

Based on this result, a recovery-based error estimator was constructed. By combining the superconvergence estimate with the eigenvalue error identity, we established

$$\lambda_h - \lambda = \eta_h^2 + O(h^{2+\alpha_0}),$$

which further yields its asymptotic exactness under the nondegeneracy condition. Finally, the postprocessed eigenvalue was shown to satisfy

$$|\widehat{\lambda}_h - \lambda| \le Ch^{2+\alpha_0}.$$

These results provide the theoretical foundation for the numerical experiments in Section 4.

4 Numerical Experiments

4.1 Numerical Model and Experimental Setup

We investigate the numerical performance of the proposed simple-averaging gradient recovery method on the unit square $\Omega = (0,1)^2$, with $\kappa = 1$ in the Steklov problem (1.1).

For this problem, the first eigenvalue and its corresponding eigenfunction are available analytically. Let $\mu = 1/\sqrt{2}$. Then

$$\lambda = \mu \tanh\left(\frac{\mu}{2}\right) \approx 0.240079085427227,$$

and the corresponding eigenfunction is

$$u(x,y) = C\cosh\left(\mu\left(x - \frac{1}{2}\right)\right)\cosh\left(\mu\left(y - \frac{1}{2}\right)\right),$$

where the boundary normalization $\|u\|_{L^2(\partial\Omega)} = 1$ gives

$$C = \left[4\cosh^2\left(\frac{\mu}{2}\right)\left(\frac{1}{2} + \frac{\sinh\mu}{2\mu}\right)\right]^{-1/2}.$$

The problem is discretized using the continuous piecewise linear finite element method described in Section 2. The interior degrees of freedom are eliminated when solving the generalized eigenvalue problem. The discrete eigenfunction is normalized in the boundary L^2 norm, with its sign chosen so that the sum of its boundary nodal values is positive.

The computational mesh is constructed by dividing the unit square into $n \times n$ congruent squares and splitting each square along its lower-left to upper-right diagonal. With $\ell = 1/n$, the maximum element diameter is

$$h = \sqrt{2}\ell = \frac{\sqrt{2}}{n}.$$

Every interior node has six neighboring vertices, and their position vectors satisfy the exact relation

$$r_i + r_{i+2} - r_{i+1} = 0, \qquad i = 1, \ldots, 6,$$

with cyclic indices. Thus, all interior nodes are regular, and

$$\mathcal{Z}_{2,h} = \varnothing, \qquad \Omega_{2,h} = \varnothing.$$

Consequently, we may take $\alpha = 1$ and $\sigma_0 = 1$. Together with the boundary-layer estimate $|\Omega_{\partial,h}| \leq Ch$, this gives

$$\widetilde{\sigma}_0 = 1, \qquad \alpha_0 = \frac{1}{2}.$$

Therefore, Theorem 3.1 yields

$$\|G_h u_h - \nabla u\|_{0,\Omega} \leq Ch^{3/2}.$$

The numerical experiments are performed on six successively refined meshes:

$$n = 4,\ 8,\ 16,\ 32,\ 64,\ 128, \qquad h = \frac{\sqrt{2}}{n}.$$

For each mesh, we compute the first discrete eigenvalue λ_h and define the eigenvalue error and recovery-based estimator by

$$E_h = |\lambda_h - \lambda|, \qquad \eta_h^2 = \|G_h u_h - \nabla u_h\|_{0,\Omega}^2.$$

The numerical results are used to examine eigenvalue convergence, gradient superconvergence, estimator exactness, and the accuracy of the postprocessed eigenvalue.

4.2 Convergence of the Steklov Eigenvalues

Table .1 reports the absolute errors of the first finite element eigenvalues and the corresponding convergence orders for different mesh sizes.

Table .1: Steklov eigenvalue errors and convergence orders

n	Number of nodes	h	$\lvert\lambda_h - \lambda\rvert$	Convergence order
4	25	0.353553	5.6155×10^{-4}	–
8	81	0.176777	1.4720×10^{-4}	1.9317
16	289	0.088388	3.7393×10^{-5}	1.9769
32	1089	0.044194	9.3955×10^{-6}	1.9927
64	4225	0.022097	2.3525×10^{-6}	1.9978
128	16641	0.011049	5.8839×10^{-7}	1.9994

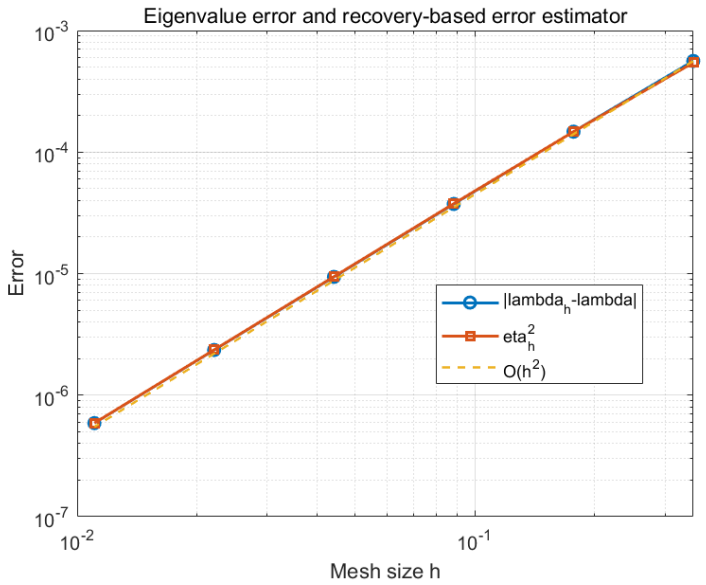


Figure 1: Convergence curves of the Steklov eigenvalue error and the recovery-based error estimator

As shown in Table .1 and Figure 1, the eigenvalue error decreases from 5.6155×10^{-4} to 5.8839×10^{-7} as the mesh is refined, with its convergence order approaching 2. The recovery-based estimator η_h^2 exhibits a similar $O(h^2)$ convergence trend. These results are consistent with the theoretical eigenvalue error estimate in Section 2 and provide a numerical basis for examining the asymptotic exactness of the estimator.

4.3 Convergence of the Recovered Gradient and the Error Estimator

We first examine the convergence behavior of the recovery-based error estimator. Table 2 reports the values of η_h^2 and the corresponding convergence orders for different mesh sizes.

Table .2: Convergence of the recovery-based error estimator

n	Number of nodes	h	η_h^2	Convergence order
4	25	0.353553	5.4414×10^{-4}	–
8	81	0.176777	1.4713×10^{-4}	1.8869
16	289	0.088388	3.7604×10^{-5}	1.9681
32	1089	0.044194	9.4405×10^{-6}	1.9940
64	4225	0.022097	2.3595×10^{-6}	2.0004
128	16641	0.011049	5.8937×10^{-7}	2.0013

Together with the second-order eigenvalue error estimate in Section 2, the expansion (3.18) is consistent with the approximately second-order behavior of η_h^2 observed in Table .2.

To compare the approximation accuracy of the original finite element gradient and the recovered gradient, we define

$$E_{\mathrm{FEM},h} = \|\nabla u_h - \nabla u\|_{0,\Omega}, \qquad E_{\mathrm{rec},h} = \|G_h u_h - \nabla u\|_{0,\Omega}.$$

Here, u is the analytical eigenfunction given in Section 4.1, and u_h is the corresponding finite element eigenfunction.

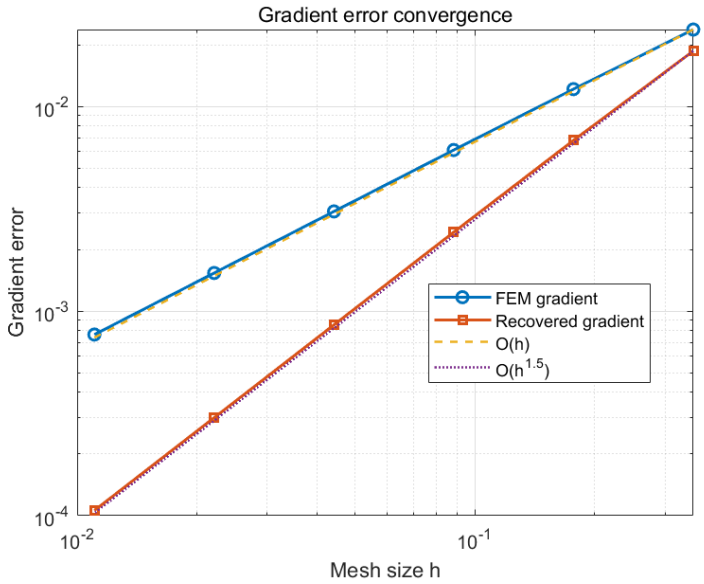


Figure .2: Convergence curves of the finite element gradient error and the simple-averaging recovered gradient error

Table .3: Errors and convergence orders of the finite element gradient and the simple-averaging recovered gradient

n	Finite element gradient error	Convergence order	Recovered gradient error	Convergence order
4	0.02372875	–	0.01863997	–
8	0.01213839	0.9671	0.006820877	1.4504
16	0.006115842	0.9890	0.002427173	1.4907
32	0.003065338	0.9965	0.0008558509	1.5038
64	0.001533798	0.9989	0.0003011983	1.5066
128	0.0007670653	0.9997	0.0001060611	1.5058

As shown in Table .3 and Figure 2, the convergence order of the original finite element gradient approaches 1, whereas that of the simple-averaging recovered gradient approaches 1.5. In particular, for $n = 128$, the corresponding errors are 0.0007670653 and 0.0001060611, respectively. The recovered gradient therefore provides a more accurate approximation to the exact gradient.

Moreover, its observed convergence rate is consistent with the theoretical estimate in Theorem 3.1, which gives $O(h^{1+\alpha_0})$. Since $\alpha_0 = 1/2$ in the present example, the predicted order is 1.5. These numerical results demonstrate the improved gradient approximation achieved by the simple-averaging recovery method on the uniform triangular meshes considered here.

4.4 Asymptotic Exactness of the Recovery-Based Error Estimator

To evaluate the accuracy of the recovery-based error estimator, we define the effectivity index by

$$R_h = \frac{\eta_h^2}{|\lambda_h - \lambda|}.$$

Since the exact eigenvalue is available for the present model, the denominator can be evaluated directly. The estimator is asymptotically exact if R_h approaches 1 as $h \to 0$.

Table .4 reports the effectivity indices on the six successively refined meshes.

Table 4: Asymptotic exactness of the recovery-based error estimator

n	Number of nodes	h	η_h^2	R_h
4	25	0.353553	5.4414×10^{-4}	0.96901
8	81	0.176777	1.4713×10^{-4}	0.99954
16	289	0.088388	3.7604×10^{-5}	1.0057
32	1089	0.044194	9.4405×10^{-6}	1.0048
64	4225	0.022097	2.3595×10^{-6}	1.0030
128	16641	0.011049	5.8937×10^{-7}	1.0017

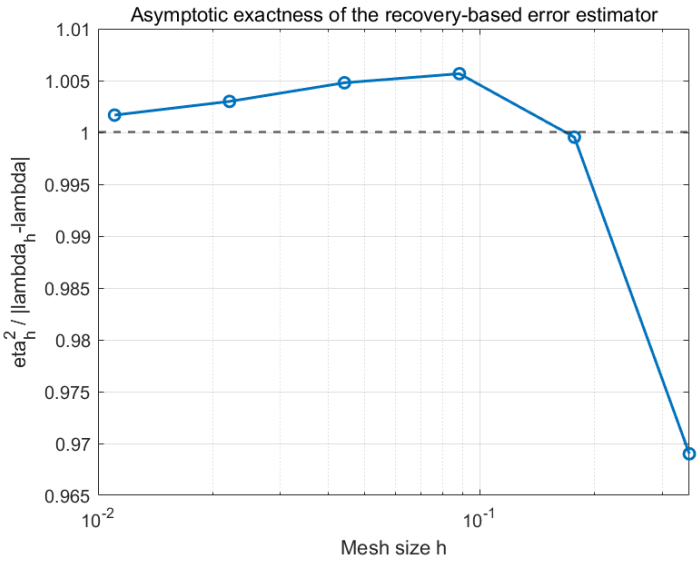


Figure 3: Effectivity curve of the recovery-based error estimator

As shown in Table .4 and Figure 3, the effectivity index remains close to 1 on the refined meshes. Although it is not monotone over all mesh levels, its deviation from 1 decreases steadily from $n = 16$ to $n = 128$. In particular, on the finest mesh, $R_h = 1.0017$, corresponding to a relative deviation of approximately 0.17%. These results provide numerical evidence that the proposed simple-averaging recovery-based error estimator is asymptotically exact for the uniform triangular mesh family considered here, in agreement with the theoretical result in Section 3.

4.5 Postprocessed Eigenvalue Accuracy

To further evaluate the improvement in eigenvalue accuracy achieved by the simple-averaging gradient recovery method, we consider the postprocessed eigenvalue defined by

$$\widehat{\lambda}_h = \lambda_h - \eta_h^2. \tag{4.1}$$

Here, η_h^2 is the recovery-based error estimator constructed in Section 3.

Table .5 compares the errors and convergence orders of the original and postprocessed eigenvalues on different meshes.

Table .5: Errors and convergence orders of the original and postprocessed eigenvalues

h	$\lvert\lambda_h - \lambda\rvert$	Convergence order	$\lvert\widehat{\lambda}_h - \lambda\rvert$	Convergence order
0.353553	5.6155×10^{-4}	–	1.7404×10^{-5}	–
0.176777	1.4720×10^{-4}	1.9317	6.7189×10^{-8}	8.0169
0.088388	3.7393×10^{-5}	1.9769	2.1179×10^{-7}	-1.6563
0.044194	9.3955×10^{-6}	1.9927	4.4995×10^{-8}	2.2348
0.022097	2.3525×10^{-6}	1.9978	7.0415×10^{-9}	2.6758
0.011049	5.8839×10^{-7}	1.9994	9.8348×10^{-10}	2.8399

The postprocessed eigenvalue error satisfies

$$\lvert\widehat{\lambda}_h - \lambda\rvert = \lvert(\lambda_h - \lambda) - \eta_h^2\rvert.$$

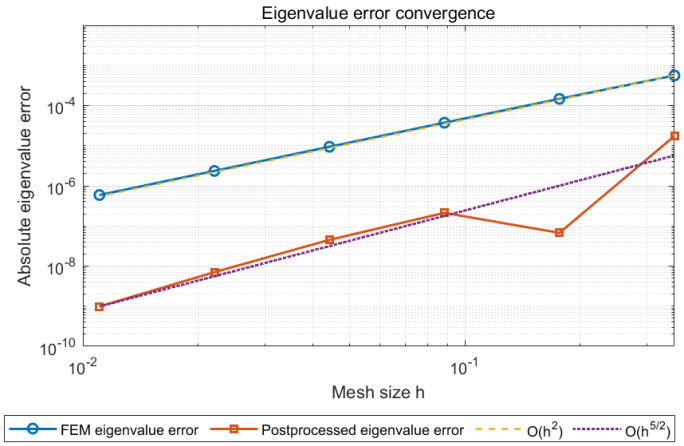


Figure .4: Error convergence curves of the original finite element and postprocessed eigenvalues

As shown in Table .5 and Figure .4, the original finite element eigenvalue error decreases at an approximately second-order rate. The postprocessed eigenvalue error is smaller than the original error at all six mesh levels, demonstrating an improvement in numerical accuracy. However, the postprocessed error is not monotone: it decreases substantially from the first to the second mesh level, increases at the third level, and then decreases on the subsequent meshes. This behavior results from the cancellation between the finite element eigenvalue error and the recovery-based estimator.

Over the successive refinements from $n = 16$ to 32, from $n = 32$ to 64, and from $n = 64$ to 128, the observed convergence orders are 2.2348, 2.6758, and 2.8399, respectively. Nevertheless, these values do not yet establish a stable asymptotic convergence rate. According to Theorem 3.3, when $\alpha_0 = 1/2$, the theoretical error bound is $O(h^{5/2})$. The numerical results demonstrate the improvement achieved by the postprocessing procedure, although further mesh refinement and numerical investigation are needed to assess its asymptotic convergence behavior.

4.6 Section Summary

Numerical experiments on uniform triangular meshes demonstrate that the finite element eigenvalue error converges at approximately second order, while the original and recovered gradient errors exhibit convergence orders close to 1 and 1.5, respectively. The recovery-based error estimator η_h^2 also shows approximately second-order convergence, and its effectivity index approaches 1 as the mesh is refined. Furthermore, the postprocessed eigenvalue is more accurate than the original finite element eigenvalue on all tested meshes, although its error exhibits nonmonotone behavior. These results support the theoretical analysis of gradient superconvergence and estimator asymptotic exactness under the prescribed mesh and regularity assumptions, while further investigation is needed to establish the asymptotic convergence behavior of the postprocessed eigenvalue.

5 Conclusions and Future Work

5.1 Conclusions

This paper investigates a simple-averaging gradient recovery method and its application to error estimation for the Steklov eigenvalue problem using continuous piecewise linear finite elements. A recovered gradient is constructed by assigning equal weights to the elementwise gradients in each nodal patch. Based on this recovery operator, a computable error estimator is developed, and its superconvergence properties and relationship with the eigenvalue error are analyzed. The main conclusions are summarized as follows.

(1) The variational formulation and finite element approximation of the Steklov eigenvalue problem are established. A priori error estimates for the discrete eigenfunction and eigenvalue are derived, and an eigenvalue error identity is obtained under boundary L^2 normalization. This identity shows that the squared gradient error provides the leading contribution to the eigenvalue error, thereby motivating the recovery-based estimation developed in this paper.

(2) A simple equal-weight nodal averaging recovery operator is constructed. At each mesh node, the recovered gradient is defined as the arithmetic average of the elementwise gradients over its nodal patch, and a continuous piecewise linear recovered gradient field is obtained through nodal interpolation. Under the prescribed mesh-structure and regularity assumptions, the recovered gradient satisfies the superconvergence estimate

$$\|G_h u_h - \nabla u\|_{0,\Omega} \leq C h^{1+\alpha_0}.$$

Based on this result, the recovery-based error estimator is defined by

$$\eta_h = \|G_h u_h - \nabla u_h\|_{0,\Omega}.$$

Combining the superconvergence estimate with the eigenvalue error identity yields

$$\lambda_h - \lambda = \eta_h^2 + O(h^{2+\alpha_0}).$$

Furthermore, for the first isolated simple Steklov eigenvalue, under the additional nondegeneracy condition $\lambda_h - \lambda \geq c_0 h^2$, the estimator is asymptotically exact:

$$\frac{\eta_h^2}{\lambda_h - \lambda} = 1 + O(h^{\alpha_0}).$$

The corresponding postprocessed eigenvalue, defined by

$$\widehat{\lambda}_h = \lambda_h - \eta_h^2,$$

satisfies

$$|\widehat{\lambda}_h - \lambda| \leq C h^{2+\alpha_0}.$$

(3) Numerical experiments are conducted on the unit square using uniform triangular meshes. The results show that the finite element eigenvalue error converges at approximately second order, while the original finite element gradient error and the recovered gradient error exhibit convergence orders close to 1 and 1.5, respectively. The squared recovery-based estimator η_h^2 also exhibits approximately second-order convergence, and its effectivity index approaches 1 as the mesh is refined. These observations provide numerical evidence supporting the theoretical superconvergence and asymptotic exactness results for the mesh family considered.

Moreover, the postprocessed eigenvalue error is smaller than the original finite element eigenvalue error at all six tested mesh levels. However, its convergence behavior is nonmonotone, and the observed convergence orders do not yet establish a stable asymptotic rate. Thus, the numerical results demonstrate an improvement in eigenvalue accuracy while leaving its detailed asymptotic behavior for further investigation.

Declaration of AI Use

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5.2 Limitations and Future Work

The present analysis relies on sufficient solution regularity and structured-mesh assumptions, including shape regularity, quasi-uniformity, and the approximate parallelogram condition at regular interior nodes. The numerical experiments are limited to the unit square with $\kappa = 1$ on uniform triangular meshes. Future work will investigate nonuniform and unstructured meshes, domains with reduced regularity, adaptive refinement based on recovery indicators, higher-order finite elements, and extensions to other elliptic eigenvalue problems.

Competing Interests

Authors have declared that no competing interests exist.

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