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ON SCHOUTEN-VAN-KAMPEN CONNECTION IN AN INDEFINITE SASAKIAN MANIFOLDS

Abstract The object of present paper is to study a Schouten-Van-Kampen connection in an-Indefinite Sasakian manifolds. In this paper, we study A Schouten-Van-Kampen connection in an Indefinite Sasakian manifolds whose projective curvature tensor satisfies certain curvature conditions. Additionally, we have analyzed certain geometric properties of conformal curvature tensor admitting Schouten-Van-Kampen connection.

Keywords: Indefinite Sasakian manifolds, Schouten-Van-Kampen connection, quasi-Projectively flat-Indefinite Sasakian manifolds,Projectively flat-Indefinite Sasakian manifolds, space like,time like,conformal curvature tensor.

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1. Introduction

The geometry of manifolds equipped with indefinite metrics has received significant attention because of its fundamental role in mathematical physics and the theory of relativity. Over the years, several researchers have developed different structures in this setting. In 1976, Sato [14] introduced the notion of an almost paracontact structure on a differentiable manifold as a counterpart of the classical almost contact structure[3],[15] with close connections to almost product geometry. Unlike almost contact manifolds, which are necessarily odd-dimensional, almost paracontact manifolds may occur in both even and odd dimensions.

Earlier, Takahashi[20] investigated almost contact manifolds endowed with an associated pseudo-Riemannian metric. Later, Matsumoto[10] modified the characteristic vector field ξ and 1-form η in the almost paracontact framework and equipped the resulting manifold with a Lorentzian metric, thereby defining a Lorentzian almost paracontact manifold.

Subsequently, Bejancu and Duggal[1] introduced the real hypersurfaces of indefinite Kahler manifolds , while Xufeng and Xiaoli[21] proved the class of ϵ-Sasakian manifolds.recently, De and Sarkar[5] proposed the concept of ϵ-Kenmotsu manifolds and examined several important curvature properties, including conformal flatness, Weyl semi-symmetry, and η-recurrence. Also, On quarter-Symmetric Metric connection in an Indefinite Sasakian manifolds studied by Singh et al.[17]. In addition, several other researchers have made noteworthy contributions to this field [18][19],[4].

The Schouten–van Kampen connection has emerged as a significant geometric tool in the study of non-holomorphic manifolds and related differential structures. Due to its intrinsic compatibility with parallel transport, it provides a natural and effective framework for investigating differentiable manifolds endowed with an affine connection together with complementary distributions[2],[7],[16]. Over the past decades, this connection has attracted considerable attention owing to its rich geometric characteristics and wide applicability. Notably, Olszak [12] established several fundamental results concerning the adaptation of the Schouten–van Kampen connection to almost para-contact metric manifolds, thereby extending its geometric relevance. Motivated by these developments, numerous researchers have subsequently explored its curvature properties, structural behavior, and applications across various classes of manifolds. These investigations have significantly contributed to the advancement of modern differential geometry and have revealed new directions for further research[6],[9],[11] and [23]. Kumhar B.[8] et al. studied ϵ-Kenmotsu manifolds admitting Schouten-Van-Kampen connection and G.Pandey et al.[13]studied Impact of Schouten-Van-Kampen connection on LP-Sasakian manifolds.

In present work, we study On Schouten-Van-Kampen connection on an Indefinite-sasakian manifolds.We obtain several identities relating the curvature tensor corresponding to these two connections.In addition we analyze

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certain geometric properties of the projective curvature tensor and conformal curvature tensor associated with the Schouten-Van-kampen connection on an Indefinite-Sasakian manifolds.

2. **Indefinite Sasakian Manifolds**

An odd-dimensional semi-Riemannian manifold M^n is called an indefinite almost contact manifold if it admits an indefinite almost contact structure (ϕ, ξ, η) consisting of a $(1,1)$-tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi \tag{2.1}$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \tag{2.2}$$

If g is a semi-Riemannian metric with (ϕ, ξ, η), that is,

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \epsilon\eta(X_1)\eta(X_2) \tag{2.3}$$

$$g(X_1, \xi) = \epsilon\eta(X_1), \quad g(\xi, \xi) = \epsilon = \pm 1 \tag{2.4}$$

$$g(X_1, \phi X_2) = -g(\phi X_1, X_2) \tag{2.5}$$

for all vector fields X_1 and X_2 on M^n, then (M^n, g) becomes an indefinite almost contact metric manifold equipped with an indefinite almost contact structure (ϕ, ξ, η).

An indefinite almost contact metric manifold M^n is called normal if

$$N_\phi + d\eta \otimes \xi = 0, \tag{2.6}$$

where N_ϕ is the Nijenhuis tensor field.

An indefinite normal contact metric manifold M^n is called an indefinite Sasakian manifold if

$$(\nabla_{X_1}\eta)(X_2) = \epsilon g(X_1, \phi X_2), \tag{2.7}$$

$$\nabla_{X_1}\xi = \epsilon\phi X_1, \tag{2.8}$$

$$(\nabla_{X_1}\phi)(X_2) = -g(X_1, X_2)\xi + \epsilon\eta(X_2)X_1. \tag{2.9}$$

In view of equation (2.4), ξ is never a light like vector field on M^n.

Also, in an indefinite Sasakian manifold M^n the following relations hold:

$$\eta(R(X_1, X_2)X_3) = \epsilon\left[g(X_1, X_3)\eta(X_2) - g(X_2, X_3)\eta(X_1)\right], \tag{2.10}$$

$$R(X_1, X_2)\xi = \epsilon\left[\eta(X_2)X_1 - \eta(X_1)X_2\right], \tag{2.11}$$

$$R(\xi, X_1)X_2 = \epsilon\left[g(X_1, X_2)\xi - \eta(X_2)X_1\right], \tag{2.12}$$

$$S(X_2, \xi) = -(n - \epsilon)\eta(X_2), \tag{2.13}$$

$$Q\xi = -(n - \epsilon)\xi. \tag{2.14}$$

$$S(\phi X_2, \phi X_3) = S(X_2, X_3) + \epsilon(n - \epsilon)\eta(X_2)\eta(X_3), \tag{2.15}$$

for any vector fields X_1, X_2 and X_3, where R and S are the semi-Riemannian curvature tensor and Ricci tensor respectively.

3. Some curvature properties of an Indefinite Sasakian manifolds admitting Schouten-Van-Kampen connection

Throughout this section we associate $\star$ with the quantities with respect to the Schouten-Van- Kampen connection.The Schouten-van-Kampen $\nabla^\star$ associated to the Levi-Civita connection ∇ is given by[12]

$$\nabla^\star_{X_1} X_2 = \nabla_{X_1} X_2 - \eta(X_2)\nabla_{X_1}\xi + (\nabla_{X_1}\eta)(X_2)\xi, \tag{3.1}$$

for any vector fields X_1, X_2 on M^n.

Using equations (2.4) and (2.5) the above equation yields,

$$\nabla^\star_{X_1} X_2 = \nabla_{X_1} X_2 - \epsilon\eta(X_2)\phi(X_1) + \epsilon g(X_1, \phi X_2)\xi, \tag{3.2}$$

Taking $X_2 = \xi$ in equations (3.2)and (2.4), we obtain

$$\nabla^\star_{X_1}\xi = 0. \tag{3.3}$$

We now calculate the Riemann curvature tensor $R^\star$ using equation (3.2), as follows:

$$R^\star(X_1, X_2)X_3 = \nabla^\star_{X_1}\nabla^\star_{X_2} X_3 - \nabla^\star_{X_2}\nabla^\star_{X_1} X_3 - \nabla^\star_{[X_1, X_2]} X_3. \tag{3.4}$$

$$\begin{aligned}\nabla^{\star}_{X_1}(\nabla^{\star}_{X_2}X_3) &= \nabla_{X_1}(\nabla_{X_2}X_3) + \epsilon g(\nabla_{X_1}X_2,\phi X_3)\xi - g(X_1,X_3)\eta(X_2)\xi \\ &+ \eta(X_3)g(X_2,X_1)\xi + \epsilon g(X_2,\phi\nabla_{X_1}X_3)\xi - g(X_1,\phi X_3)\phi X_2 \\ &- \epsilon\eta(\nabla_{X_1}X_3)\phi X_2 + \epsilon\eta(X_3)g(X_1,X_2)\xi - \eta(X_2)\eta(X_3)X_1 \\ &- \epsilon\eta(X_3)\phi\nabla_{X_1}X_2 - \epsilon\eta(\nabla_{X_2}X_3)\phi X_1 + \eta(X_3)\eta(\phi X_2)\phi X_1 \\ &+ \epsilon g(X_1,\phi(\nabla_{X_2}X_3))\xi + \eta(X_3)g(X_1,X_2)\xi - \epsilon(\eta(X_3)\eta(X_2)\eta(X_1)\xi. \end{aligned} \tag{3.5}$$

$$\begin{aligned}\nabla^{\star}_{X_2}(\nabla^{\star}_{X_1}X_3) &= \nabla_{X_2}(\nabla_{X_1}X_3) - g(X_2,\phi X_3)\phi X_1 - \epsilon\eta(\nabla_{X_2}X_3)\phi X_1 \\ &+ \epsilon\eta(X_3)g(X_2,X_1)\xi - \eta(X_3)\eta(X_1)X_2 - \epsilon\eta(X_3)\phi(\nabla_{X_2}X_1) \\ &+ \epsilon g(\nabla_{X_2}X_1,\phi X_3)\xi - g(X_2,X_3)\eta(X_1)\xi + \eta(X_3)g(X_1,X_2)\xi \\ &+ \epsilon g(X_1,\phi(\nabla_{X_2}X_3))\xi + \eta(X_3)\eta(\phi X_1)\phi(X_2) \\ &+ \epsilon g(X_2,\phi(\nabla_{X_1}X_3))\xi + \eta(X_3)g(X_2,X_1)\xi - \epsilon\eta(X_3)\eta(X_1)\eta(X_2)\xi. \end{aligned} \tag{3.6}$$

$$\begin{aligned}\nabla^{\star}_{[X_1,X_2]}X_3 &= \nabla_{[X_1,X_2]}X_3 - \epsilon\eta(X_3)\phi(\nabla_{X_1}X_2) + \epsilon\eta(X_3)\phi(\nabla_{X_2}X_1) \\ &+ \epsilon g(\nabla_{X_1}X_2,\phi X_3)\xi - \epsilon g(\nabla_{X_2}X_1,\phi X_3)\xi. \end{aligned} \tag{3.7}$$

$$\begin{aligned}R^{\star}(X_1,X_2)X_3 &= R(X_1,X_2)X_3 + g(X_2,\phi X_3)\phi X_1 - g(X_1,\phi X_3)\phi X_2 \\ &+ (g(X_2,X_3)\eta(X_1) - g(X_1,X_3)\eta(X_2))\xi \\ &+ (\eta(X_1)X_2) - \eta(X_2)X_1)\eta(X_3). \end{aligned} \tag{3.8}$$

Taking $Z{=}\xi$ in (3.8),we get

$$R^{\star}(X_1,X_2)\xi = (\epsilon-1)(\eta(X_2)X_1 - \eta(X_1)X_2) \tag{3.9}$$

$$S^{\star}(X_2,X_3) = S(X_2,X_3) + (1-n)\eta(X_2)\eta(X_3) \tag{3.10}$$

contracting the equation (3.10)

$$Q^{\star}X_2 = QX_2 + \epsilon(1-n)\eta(X_2)\xi. \tag{3.11}$$

$$\begin{aligned}R^{\star}(X_2,X_3)X_1 &= R(X_2,X_3)X_1 + g(X_3,\phi X_1)\phi X_2 - g(X_2,\phi X_1)\phi X_3 \\ &+ (g(X_3,X_1)\eta(X_2) - g(X_2,X_1)\eta(X_3))\xi \\ &+ (\eta(X_2)X_3 - \eta(X_3)X_2)\eta(X_1). \end{aligned} \tag{3.12}$$

Next,

$$\begin{aligned}R^{\star}(X_3,X_1)X_2 =& R(X_3,X_1)X_2 + g(X_1,\phi X_2)\phi X_3 - g(X_3,\phi X_2)\phi X_1 \\ &+ (g(X_1,X_2)\eta(X_3) - g(X_3,X_2)\eta(X_1))\xi + (\eta(X_3)X_1 - \eta(X_1)X_3)\eta(X_2). \end{aligned} \tag{3.13}$$

Adding the equations(3.8), (3.12) and (3.13)

$$\begin{aligned} &R^\star(X_1,X_2)X_3+R^\star(X_2,X_3)X_1+R^\star(X_3,X_1)X_2\\ &=2(g(X_1,\phi X_2)\phi X_3+g(X_2,\phi X_3)\phi X_1+g(X_3,\phi X_1)). \end{aligned} \tag{3.14}$$

Theorem 3.1. *The first Bianchi Identity on an indefinite Sasakian manifolds admitting Schouten-Van-Kampen connection is the equation (3.14).*

Now, taking the inner product with U in equation (3.8)

$$\begin{aligned} R^\star(X_1,X_2,X_3,U) =&R(X_1,X_2,X_3,U)+g(X_2,\phi X_3)g(\phi X_1,U)\\ &-g(X_1,\phi X_3)g(\phi X_2,U)+(g(X_2,X_3)\eta(X_1)-g(X_1,X_3)\eta(X_2))\epsilon\eta(U)\\ &+(\eta(X_1)g(X_2,U)-\eta(X_2)g(X_1,U))\eta(X_3). \end{aligned} \tag{3.15}$$

Interchanging X_1 and X_2 in the equation (3.15),we obtain

$$\begin{aligned} R^\star(X_2,X_1,X_3,U) =& R(X_2,X_1,X_3,U)+g(X_1,\phi X_3)g(\phi X_2,U)\\ &-g(X_2,\phi X_3)g(\phi X_1,U)+(g(X_1,X_3)\eta(X_2)-g(X_2,X_3)\eta(X_1))\epsilon\eta(U)\\ &+(\eta(X_2)g(X_1,U)-\eta(X_1)g(X_2,U))\eta(X_3). \end{aligned} \tag{3.16}$$

Adding Equations (3.15) and (3.16) with the fact that $R(X_1,X_2,X_3,U)+R(X_2,X_1,X_3,U)=0$, we obtain

$$R^\star(X_1,X_2,X_3,U)+R^\star(X_2,X_1,X_3,U)=0. \tag{3.17}$$

Again interchanging X_3 and U in the equation 3.15

$$\begin{aligned} R^\star(X_1,X_2,U,X_3) =&R(X_1,X_2,U,X_3)+g(X_2,\phi U)g(\phi X_1,X_3)\\ &-g(X_1,\phi U)g(\phi X_2,X_3)+(g(X_2,U)\eta(X_1)-g(X_1,U)\eta(X_2))\epsilon\eta(X_3)\\ &+(\eta(X_1)g(X_2,X_3)-\eta(X_2)g(X_1,X_3))\eta(U). \end{aligned} \tag{3.18}$$

Now,adding equations 3.15 and 3.18 with the fact $R(X_1,X_2,X_3,U)+R(X_1,X_2,U,X_3)=0$, we get

$$\begin{aligned} R^\star(X_1,X_2,X_3,U)+R^\star(X_1,X_2,U,X_3) =&(\epsilon+1)[(g(X_2,X_3)\eta(X_1)-g(X_1,X_3)\eta(X_2))\eta(U)\\ &+(g(X_2,U)\eta(X_1)-g(X_1,U)\eta(X_2))\eta(X_3)]. \end{aligned} \tag{3.19}$$

Again interchanging pair of slots in the equation (3.15),we have

$$R^\star(X_3,U,X_1,X_2) = R(X_3,U,X_1,X_2) + \epsilon(g(U,X_1)\eta(X_3) - g(X_3,X_1)\eta(U))\eta(X_2) + (\eta(X_3)g(U,X_2) - \eta(U)g(X_3,X_2))\eta(X_1). \tag{3.20}$$

Now, subtracting the equation (3.20) from the equation (3.15) with fact $R(X_1,X_2,X_3,U)$-$R(X_3,U,X_1,X_2)$=0,we obtain the following:

$$R^\star(X_1,X_2,X_3,U) - R^\star(X_3,U,X_1,X_2) = (\epsilon+1)(g(X_2,X_3)\eta(X_1)\eta(U) - g(X_1,U)\eta(X_2)\eta(X_3)). \tag{3.21}$$

Thus in view of equations (3.17),(3.19) and (3.21), we can state that:

Theorem 3.2. *The Riemann curvature tensor of type (0,4) of a SVK connection in an Indefinite Sasakian manifolds is*

1. *Skew symmetric in first two slots.*
2. *Last two slots is not skew symmetric.*
3. *pair of slots is not symmetric.*

Next, suppose $R^\star(X_1,X_2,X_3) = 0$, which by the virtue of equation (3.8), we get

$$R(X_1,X_2)X_3 = g(X_1,\phi X_3)\phi X_2 - g(X_2,\phi X_3)\phi X_1 + (g(X_1,X_3)\eta(X_2) - g(X_2,X_3)\eta(X_1))\xi + (\eta(X_2)X_1 - \eta(X_1)X_2)\eta(X_3). \tag{3.22}$$

Taking the inner product of the above equation with ξ, we get

$$\epsilon\eta(R(X_1,X_2)X_3 = \epsilon(g(X_1,X_3)\eta(X_2) - g(X_2,X_3)\eta(X_1)) \tag{3.23}$$

which, by virtue (2.4)

$$R(X_1,X_2)X_3 = g(X_1,X_3)X_2 - g(X_2,X_3)X_1. \tag{3.24}$$

The above equation can be written as:

$$R(X_1,X_2,X_3) = g(X_1,X_3)g(X_2,U) - g(X_2,X_3)g(X_1,U). \tag{3.25}$$

Thus, we can state the following:

Theorem 3.3. *The curvature tensor on an Indefinite Sasakian manifolds equipped with Schouten-Van-Kampen connection vanishesh then the manifolds is of constant curvature.*

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Now, Suppose Ricci tensor of Schouten-Van-Kampen connection in M^n i.e.

$$S^\star(X_2, X_3) = 0. \tag{3.26}$$

Using equation (3.10) in equation (3.25)

$$S(X_2, X_3) = -(1-n)\eta(X_2)\eta(X_3). \tag{3.27}$$

Theorem 3.4. *If the Ricci tensor on an Indefinite sasakian manifolds equipped with Schouten-Van-Kampen connection vanishes then the manifolds M^n is an η-Einstein manifolds.*

4. Quasi-Projectively flat-Indefinite Sasakian manifolds with respect to Schouten-van-Kampen connection

The projective curvature tensor [22]$P^\star$ with respect to Schouten-Van-Kampen connection is defined by

$$P^\star(X_1, X_2)X_3 = R^\star(X_1, X_2)X_3 - \frac{1}{n-1}(S^\star(X_2, X_3)X_1 - S^\star(X_1, X_3)X_2). \tag{4.1}$$

Definition 4.1. *An Indefinite Sasakian manifolds M^n is said to be quasi-Projectively flat with respect to Schouten-Van-Kampen connection if*

$$g(P^\star(\phi X_1, X_2)X_3, \phi W) = 0. \tag{4.2}$$

where $P^\star$ is the projective curvature tensor with respect to Schouten-Van-Kampen connection In view of the equation (4.1),we have

$$g(P^\star(X_1, X_2)X_3, W) = g(R^\star(X_1, X_2)X_3, W) - \frac{1}{n-1}(S^\star(X_2, X_3)g(X_1, W) - S^\star(X_1, X_3)g(X_2, W)). \tag{4.3}$$

Putting $X_1=\phi X_1$ and $W=\phi W$ in the above equation we get,

$$g(P^\star(\phi X_1, X_2)X_3, \phi W) = g(R^\star(\phi X_1, X_2)X_3, \phi W) - \frac{1}{n-1}(S^\star(X_2, X_3)g(\phi X_1, \phi W) - S^\star(\phi X_1, X_3)g(X_2, \phi W)). \tag{4.4}$$

Now, suppose that M^n is quasi-projectively-flat with respect to Schouten-Van-Kampen connection

Then by the virtue of equation (4.2) and (4.4), we obtain

$$g(R^{\star}(\phi X_1, X_2)X_3, \phi W) = \frac{1}{n-1}(S^{\star}(X_2, X_3)g(\phi X_1, \phi W) - S^{\star}(\phi X_1, X_3)g(X_2, \phi W)). \tag{4.5}$$

Using equations (3.8) and (3.10), we get

$$\begin{aligned} g(R(\phi X_1, X_2)X_3, \phi W) =& \frac{1}{n-1}(S(X_2, X_3)g(\phi X_1, \phi W) - S(\phi X_1, X_3)g(X_2, \phi W)) \\ & g(X_2, \phi X_3)g(X_1, \phi W) + g(\phi X_1, \phi X_3)g(\phi X_2, \phi W). \end{aligned} \tag{4.6}$$

Let $(e_1, e_2, ...e_{n-1}, \xi)$ be a local orthonormal basis of vector fields in M^n, then $(\phi e_1, \phi e_2, ...\phi e_{n-1}, \xi)$ is also a local orthonormal basis of M^n. Putting $X_1 = W = e_i$ in the equation (4.6) and taking summation over $(i, 1 \leq i \leq n-1)$, we get

$$\begin{aligned} \sum_{i=n-1}^{n} g(R(\phi e_i, X_2)X_3, \phi e_i) =& \frac{1}{n-1}\sum_{i=n-1}^{n}(S(X_2, X_3)g(\phi e_i, \phi e_i) - S(\phi e_i, X_3)g(X_2, \phi e_i)) \\ & \sum_{i=n-1}^{n}(g(X_2, \phi X_3)g(e_i, \phi e_i) + g(\phi e_i, \phi X_3)g(\phi X_2, \phi e_i)). \end{aligned} \tag{4.7}$$

Also,

$$\sum_{i=n-1}^{n} g(\phi e_i, \phi e_i) = n-1. \tag{4.8}$$

$$\sum_{i=n-1}^{n}(S(\phi e_i, X_3)g(X_2, \phi e_i)) = S(X_2, X_3). \tag{4.9}$$

$$\sum_{i=n-1}^{n}(g(\phi e_i, \phi X_3)g(\phi X_2, \phi e_i)) = g(\phi X_3, \phi X_2) \tag{4.10}$$

$$\sum_{i=n-1}^{n} g(R(\phi e_i, X_2)X_3, \phi e_i) = \frac{n-2}{n-1}(S(X_2, X_3) + g(\phi X_2, \phi X_3)). \tag{4.11}$$

$$S(X_2, X_3) = (n-1)(g(X_2, X_3) + \epsilon\eta(X_2)\eta(X_3)). \tag{4.12}$$

Theorem 4.2. *A quasi-Projectively flat Indefinite Sasakian manifolds admitting Schouten-Van-Kampen connection is an η-Einstein manifolds.*

5. ϕ-projectively flat-Indefinite Sasakian manifolds admitting Schouten-Van-Kampen connection

Definition 5.1. *An Indefinite Sasakian manifolds equipped with Schouten-Van-Kampen connection is said to be ϕ-projectively flat, if*

$$\phi^2(P^\star(\phi X_1, \phi X_2))\phi X_3 = 0. \tag{5.1}$$

where,$P^\star$ is the projective curvature tensor of the manifolds M^n with respect to Schouten-van-Kampen connection assume that M^n be a ϕ- projectively flat an Indefinite Sasakian manifolds with respect to Schouten-Van-Kampen connection.

It is easy to show that $\phi^2(P^\star(\phi X_1, \phi X_2)\phi X_3 = 0$, holds if and only if

$$g(P^\star(\phi X_1, \phi X_2)\phi X_3, \phi W) = 0. \tag{5.2}$$

for any $X_1, X_2, X_3, W \in TM^n$. Putting $X_2 = \phi X_2$ and $X_3 = \phi X_3$ in the equation (4.4), we obtain

$$\begin{aligned} g(p^\star(\phi X_1, \phi X_2)\phi X_3, \phi W) = g(R^\star(\phi X_1, \phi X_2)\phi X_3, \phi W) - \frac{1}{n-1}(S^\star(\phi X_2, \phi X_3)g(\phi X_1, \phi W) \\ - (S^\star(\phi X_1, \phi X_3)g(\phi X_2, \phi W)). \end{aligned} \tag{5.3}$$

From equations (5.2), (5.3),(3.8) and (3.10), we get

$$g(R^\star(\phi X_1, \phi X_2)\phi X_3, \phi W) = \frac{1}{n-1}(S^\star(\phi X_2, \phi X_3)g(\phi X_1, \phi W) - S^\star(\phi X_1, \phi X_3)g(\phi X_2, \phi W)). \tag{5.4}$$

Again, using the equation (2.15), we have

$$\begin{aligned} g(R(\phi X_1, \phi X_2)\phi X_3, \phi W) = \frac{1}{n-1}(S(\phi X_2, \phi X_3)g(\phi X_1, \phi W) - S(\phi X_1, \phi X_3)g(\phi X_2, \phi W)) \\ - g(\phi X_2, X_3)g(X_1, \phi W) + g(\phi X_1, X_3)g(X_2, \phi W). \end{aligned} \tag{5.5}$$

Let $(e_1, e_2, ...e_{n-1}, \xi)$ be a local orthonormal basis of vector fields in M^n, then $(\phi e_1, \phi e_2, ...\phi e_{n-1}, \xi)$ is also a local orthonormal basis of M^n. Putting $X_1 = W = e_i$ in the equation (5.5) and summuing over $(i, 1 \leq i \leq n-1)$,

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we get

$$\sum_{i=n-1}^{n} g(R(\phi e_i, \phi X_2)\phi X_3, \phi e_i) = \frac{1}{n-1}\sum_{i=n-1}^{n}(S(\phi X_2, \phi X_3)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi X_3)g(\phi X_2, \phi e_i)) \\ \sum_{i=n-1}^{n}(g(\phi e_i, X_3)g(X_2, \phi e_i)). \tag{5.6}$$

Using the equations (4.8),(4.9) and (4.10),

$$S(\phi X_2, \phi X_3) = (n-1)g(X_2, X_3). \tag{5.7}$$

From equation (2.15), we obtain

$$S(X_2, X_3) = (n-1)g(X_2, X_3) - \epsilon(n-\epsilon)\eta(X_2)\eta(X_3). \tag{5.8}$$

which is of the form

$$S(X_2, X_3) = ag(X_2, X_3) - b\eta(X_2)\eta(X_3). \tag{5.9}$$

Thus, we can state that:

Theorem 5.2. *ϕ-Projectively flat Indefinite Sasakian manifolds equipped with Schouten-van-Kampen connection is an η-Einstein manifolds.*

6. Conformal curvature tensor of Schouten-Van-Kampen connection

Conformal curvature tensor[17] $C^\star$ of Schouten-Van-Kampen connection $\nabla^\star$ in M^n is defined

$$C^\star(X_1, X_2)X_3 = R^\star(X_1, X_2)X_3 - \frac{1}{n-2}(S^\star(X_2, X_3)X_1 \\ - S^\star(X_1, X_3)X_2 + g(X_2, X_3)Q^\star X_1 - g(X_1, X_3)Q^\star X_2) \\ + \frac{r^\star}{(n-1)(n-2)}(g(X_2, X_3)X_1 - g(X_1, X_3)X_2). \tag{6.1}$$

Using Equations (3.8),(3.10),(3.11) and (3.12) in above equation, we obtain

$$
\begin{aligned}
C^{\star}(X_1,X_2)X_3 &= R(X_1,X_2)X_3 + g(X_2,\phi X_3)\phi X_1 - g(X_1,\phi X_3)\phi X_2 \\
&+ (g(X_2,X_3)\eta(X_1) - g(X_1,X_3)\eta(X_2))\xi + (\eta(X_1)X_2 - \eta(X_2)X_1)\eta(X_3) \\
&- \frac{1}{n-2}((S(X_2,X_3) + (1-n)\eta(X_2)\eta(X_3))X_1 - (S(X_1,X_3) \\
&+ (1-n)\eta(X_1)\eta(X_3))X_2 + g(X_2,X_3)QX_1 + (1-n)\epsilon\eta(X_1)\xi g(X_2,X_3) - g(X_1,X_3)QX_2 \\
&- (1-n)\epsilon\eta(X_2)\xi g(X_1,X_3)) + \frac{r^{\star}}{(n-1)(n-2)}(g(X_2,X_3)X_1 - g(X_1,X_3)X_2).
\end{aligned} \tag{6.2}
$$

Which yields

$$
\begin{aligned}
C^{\star}(X_1,X_2)X_3 &= C(X_1,X_2)X_3 + g(X_2,\phi X_3)\phi X_1 - g(X_1,\phi X_3)\phi X_2 \\
&+ (g(X_2,X_3)\eta(X_1) - g(X_1,X_3)\eta(X_2))\xi + (\eta(X_1)X_2 - \eta(X_2)X_1)\eta(X_3) \\
&- \frac{1}{n-2}((1-n)\eta(X_2)\eta(X_3)X_1 - (1-n)\eta(X_1)\eta(X_3)X_2 \\
&+ (1-n)\epsilon\eta(X_1)g(X_2,X_3)\xi - (1-n)\epsilon\eta(X_2)g(X_1,X_3)\xi),
\end{aligned} \tag{6.3}
$$

where $C(X_1,X_2)X_3$ is the Conformal curvature tensor of connection ∇ in m^n defined as

$C(X_1,X_2)X_3$

$$
\begin{aligned}
&= R(X_1,X_2)X_3 - \frac{1}{n-2}(S(X_2,X_3)X_1 - S(X_1,X_3)X_2 \\
&+ g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2) + (\frac{r+\epsilon(1-n)}{(n-1)(n-2)})(g(X_2,X_3)X_1 - g(X_1,X_3)X_2).
\end{aligned} \tag{6.4}
$$

Now, interchanging X_1 and X_2 in equation (6.2) with the fact that $R(X_1,X_2)X_3 + R(X_2,X_1)X_3 = 0$, we have

$$C^{\star}(X_1,X_2)X_3 + C^{\star}(X_2,X_1)X_3 = 0. \tag{6.5}$$

Hence, we can state as follows:

Theorem 6.1. *In an Indefinite Sasakian manifolds admitting Schouten-Van-Kampen connection, we have* $C^{\star}(X_1,X_2)X_3 + C^{\star}(X_2,X_1)X_3 = 0$.

Conclusion

The present work investigates curvature properties of indefinite Sasakian manifolds endowed with a Schouten-Van-Kampen connection. Relations between the Schouten-Van-Kampen connection and the Levi-Civita connection are used to obtain expressions involving the corresponding Riemann curvature tensor, Ricci tensor, and Ricci operator. The study also considers algebraic properties of the curvature tensor and examines the consequences of selected vanishing-curvature conditions. Within the framework developed in the manuscript, quasi-projectively flat and (\phi)-projectively flat conditions lead to Ricci-tensor expressions having the form associated with an (\eta)-Einstein manifold. In addition, the conformal curvature tensor of the Schouten-Van-Kampen connection is expressed in relation to the corresponding curvature quantities of the underlying indefinite Sasakian structure. Overall, the analysis illustrates the influence of the Schouten-Van-Kampen connection on projective, conformal, and Ricci curvature properties in indefinite Sasakian geometry and provides a basis for further theoretical investigation of related geometric structures and curvature conditions.

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