UNDER PEER REVIEW

Clarkson-Kruskal Similarity Reductions and Exact Solutions of the (2+1)-Dimensional Sawada-Kotera-Kadomtsev-Petviashvili Equation

Abstract: The (2+1)-dimensional Sawada-Kotera-Kadomtsev Petviashvili (SKKP) equation governs nonlinear wave propagation in fluid mechanics and plasma physics, especially low water wave dynamics. This study presents the first application of the Clarkson-Kruskal (CK) direct technique to the SKKP equation, which is difficult because of its fifth order dispersive term and higher-degree nonlinearities. We construct explicit equations for z and β by adding specific assumptions and using well known remarks of CK direct method.This allow us to systematically reduce the governing equation.we obtain our results in two cases.In the first case the problem reduces to family of ordinary differential equation when $z_x \neq 0$ and $z_y \neq 0$. This yields previously unreported soliton,cnoidal periodic wave and rational line-lump solutions, as well as new similarity reductions.The line-lump solutions exhibit movable, continuous blow-up along a straight line, this phenomenon that his not been reported anywhere.In second case we obtain ordinary differential equation and a Jacobi elliptic function solution with non-stationary amplitude growth along the y-direction when $z_x = 0$ and $z_y \neq 0$.The amplitude of this solution is spatially modified by arbitrary time dependent function.The graphical depiction highlights physical behavior not previously reported in the literature on this equation and displays the associated wave shapes. Every derived solution depends on arbitrary time and free parameter functions.

Keywords: The SKKP Equation, the Clarkson-Kruskal (CK) direct technique, new similarity reductions, and exact solutions.

1 Introduction

The (2+1)-dimensional Sawada-Kotera-Kadomtsev-Petviashvili (SKKP) equation can be expressed as

$$(u_t + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x + u_{xxxxx})_x + u_{yy} = 0 \tag{1}$$

where u is a real function of x, y, and t, three independent variables. Equation (1) was introduced by Wazwaz [1] by embedding the Sawada-Kotera (SK) hierarchy within the Kadomtsev-Petviashvili (KP) framework, and it is completely integrable,this equation plays a substantial role in fluid mechanics, where it is used to describe the evolution of weakly dispersive, weakly two-dimensional wave fields, with particular relevance to shallow-water wave phenomena.
Several methods have since been applied to Eq.(1) to construct exact solutions.To prove that the equation accepts an endless number of conservation laws,Wazwaz [1] first utilized the tanh-coth approach to get a solution with one soliton, and then they used the Hirota bilinear method to get solutions with N solitons. Xin et al [2] used the classical direct symmetry approach to develop the corresponding conservation laws, as well as symmetry reductions and group-invariant and exact solutions for various functions, such as Jacobi elliptic, hyperbolic, and trigonometric.Using modified Bell polynomials,Xu et al. [3] created the bilinear version of the problem, from which N-soliton solutions were obtained. They then used the Riemann theta function to produce quasi-periodic wave solutions.More recently, Wang [4] employed the Hirota bi-linear method with symbolic computation, applying multiple test functions to extract bright- and dark-type lump solutions as well as sin-cosh-type breather solutions; in the same study, bright, periodic, and singular travelling-wave solutions were obtained using the Kudryashov method, the Bernoulli sub-equation function method (BSEFM), and the extended Wang direct mapping method(EWDMM).
A wide range of techniques have been developed for the analysis of nonlinear partial differential equations (NLPDEs), such as Lie symmetry analysis [5], the Hirota bilinear method [6–8],the Adomian decomposition method (ADM) [9, 10], the Clarkson-Kruskal (CK) direct method [11], the auxiliary equation method [12–14],and hybrid approaches that combine the Backlund transformation and the Hirota bilinear method [15–19].

To achieve this purpose, we investigate the Clarkson-Kruskal (CK) direct technique. The Clarkson-Kruskal (CK) direct technique, proposed by Clarkson and Kruskal [11], is an effective method for obtaining similarity reductions and accurate solutions to NLPDEs. Many scientists use this method to examine similarity reductions and exact solutions of NLPDEs [11, 20–29]. Using the CK direct technique yields results for a certain type of partial differential equation, In general the solution can be expressed as follows, using the (2 + 1)-dimension:

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)\omega(z(x, y, t)) \tag{2}$$

where α,β and z are functions of x,y and t to compute and ω satisfies a reduced equation which is lower than the dimension of the original problem.

We are motivated to apply the CK direct method to Eq.(1) because, relative to the Lie symmetry method and the Hirota bilinear technique, it is capable of generating a large number of solution, types including lump-type, soliton, and elliptic function solutions, from a single, unified reduction procedure. This flexibility, however, comes at a computational cost: determining explicit expressions for β and z becomes considerably more difficult in the presence of high-order dispersive terms such as u_{xxxxx} and nonlinear terms such as uu_{xxx}, $u_x u_{xx}$ and $u^2 u_x$.The CK direct technique has not been used for the SKKP equation due to its computational complexity. In the present study, we address this gap by introducing suitable simplifying assumptions and invoking established remarks of the CK method to obtain closed-form expressions for β and z, which in turn allow the reduction procedure to be carried through to completion.

This study produces two separate cases.In the first, where $z_x \neq 0$ and $z_y \neq 0$ The SKKP equation simplifies to a collection of ordinary differential equations under the conditions $A = 0, B = 0, C = 0, D = 0, E = 0, F = 0, G = 0$ and we obtain novel similarity reductions given in Eqs.(30),(46).We present new exact solutions for lump-type, elliptic function, and solitary wave, defined in terms of arbitrary t-functions and free parameters.These are presented as Eq.(31), Eq.(34),Eq.(65), Eq.(68), and Eq.(71) to the best of our knowledge,these solutions are novel to the literature. In the second case, where $z_x = 0$ and $z_y \neq 0$ the equation reduces to an ordinary differential equation of Painleve I type, given as the new similarity reduction Eq.(87) from which the new Jacobi elliptic function solution Eq.(88) derived.This solution likewise depends on arbitrary functions of t and several parameters and, as far as we are aware, has not been reported elsewhere.

Beyond their formal construction, these solutions correspond to physically distinct wave behaviors. The line-lump solution describes a lump wave whose crest propagates along the x-direction; the presence of a rational function term in this solution produces a movable, continuous blow-up along a straight line, a phenomenon not previously documented for the SKKP equation (Fig.(1),Fig.(5) (a)-(c)).The soliton solution propagates stably in the positive x-direction, retaining its waveform, amplitude, and energy over long-distance propagation (Fig.(2),Fig.(3) (a)-(c)). The cnoidal periodic wave travels in the positive x-direction at constant speed and amplitude, governed by the parameter c_1, with energy fully conserved during propagation (Fig.(4),(a)-(c)).The Jacobi elliptic function solution (88)is distinctive in that its amplitude is linearly modulated by the spatial coordinates x and y through the arbitrary time-dependent functions η(t), θ(t), σ(t),giving rise to propagation along the positive y-axis with a non-stationary amplitude profile a feature not previously observed in Jacobi elliptic solutions of the SKKP equation (Fig.(6), (a)-(c)).None of these solutions has appeared in previously published work, and together they disclose novel physical phenomena in the SKKP equation-governed wave propagation.

The detailed outline of the paper is organized as follows. In Section 2, through the CK direct method,we discuss and obtain the similarity reductions, and exact solutions of the SKKP equation in the case of $z_x \neq 0$, $z_y \neq 0$ and $z_x = 0$,$z_y \neq 0$. Finally, the conclusion and discussion is given in Section 3.

2 Symmetry reductions and novel exact solutions

The SKKP equation's exact solutions and similarity reductions are presented in this section. putting Eq.(2) into Eq.(1) and collecting the coefficients of monomials of ω and its derivatives gives:

$$\begin{aligned}&\gamma_0\omega''''' + \gamma_1\omega + \gamma_2\omega' + \gamma_3\omega'' + \gamma_4\omega''' + \gamma_5\omega'''' + \gamma_6\omega''''' + \gamma_7\omega''''\omega + \gamma_8\omega''\omega + \gamma_9\omega'\omega \\ &+ \gamma_{10}(\omega')^2\omega + \gamma_{11}(\omega''')\omega' + \gamma_{12}\omega''(\omega^2) + \gamma_{13}\omega'\omega^2 + \gamma_{14}\omega'''\omega + \gamma_{15}\omega''\omega' + \gamma_{16}(\omega')^2 \\ &+ \gamma_{17}(\omega)^2 + \gamma_{18}\omega^3 + \gamma_{19}(\omega'')^2 + \gamma_{20} = 0\end{aligned} \tag{3}$$

where γ_i, $(i = 1, 2, 3, ...20)$ see A for details.

and ' := d/dz. To make Eq.(3) an ordinary differential equation for ω, the coefficient ratios of various derivatives and powers of ω must only be functions of z. if $z_x \neq 0$ and $z_y \neq 0$ these conditions read

$$\gamma_i = \gamma_0\Gamma_i(z) \;\; (i = 1, 2, 3, ...20) \tag{4}$$

Where $\Gamma_i(z)$ (i=1,2,3,...20) are the functions of z will be determined later.

In the calculation of $\alpha(x, y, t)$, $\beta(x, y, t)$, z(x, y, t) and $\Gamma_i(z)(i = 1, 2, 3, ..., 20)$, the following four remarks may be used:
Remark 1: if $\alpha(x, y, t)$ has the form $\alpha = \alpha_0(x, y, t) + \beta(x, y, t)\Omega(z)$, then we can take $\Omega = 0$ (by substituting $\omega(z) \longrightarrow \omega(z) - \Omega(z)$).
Remark 2: if $\beta(x, y, t)$ has the form $\beta = \beta_0(x, y, t)\Omega(z)$, then we can take $\Omega = \Omega_0 = constant$ (by substituting $\omega(z) \longrightarrow \omega(z)\Omega_0/\Omega(z)$).
Remark 3: if $z(x, y, t)$ is determined by an equation of the form $\Omega(z) = z_0(x, y, t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ (by substituting $z \longrightarrow \Omega^{-1}z$)).
Remark 4: We reserve English and Greek letters for indeterminate functions of z (or x, y, t) so that after performing operations (differentiation, integration, exponentiation, rescaling, etc.), the outcome will be represented by the same letter. For example, the derivative of $\Gamma(z)$ will be named $\Gamma(z)$.

2.1 First case(A) $z_x \neq 0$,$z_y \neq 0$

First case(A1):Here we will use $A = 0, B = 0, C = 0, , D = 0, E = 0, F = 0$

In this case we employ the coefficient of ω''''' (Specially $\beta(z_x)^6$) as the normalizing coefficient, the other coefficients must follow the form $\beta(z_x)^6)\tau_i(z)$ as defined in Eq.(4).
Compare the coefficients of ω''''' (i=0) with $\omega''''\omega$ (i=7) see (A) for details and using Remark(ii),we obtain:

$$\beta = (z_x)^2 \tag{5}$$

Compare the coefficients of ω'''''' (i=0) with ω'''' (i=6) see (A) for details and using two time integration we get:

$$z = x\theta(y,t) + \sigma(y,t) \tag{6}$$

where θ and σ both are the functions of y,t.
Compare the Coefficients of ω'''''' (i=0) and ω'''' (i=5) see (A) for details and using Remark(i), we derive:

$$\alpha = 0 \tag{7}$$

By putting Eq.(5), Eq.(6) and Eq.(7) in (A) we derive:

$$\begin{aligned}
&\Gamma_4(z) = \Gamma_5(z) = \Gamma_6(z) = \Gamma_8(z) = \Gamma_9(z) = \Gamma_{13}(z) = \Gamma_{14}(z) = \Gamma_{15}(z) = \Gamma_{16}(z) = \Gamma_{17}(z) \\
&= \Gamma_{17}(z) = \Gamma_{18}(z) = \Gamma_{20}(z) = 0 \\
&\Gamma_0(z) = 1,\ \Gamma_7(z) = \Gamma_{19}(z) = 15,\ \Gamma_{10}(z) = 90, \Gamma_{11}(z) = 30,\ \Gamma_{12}(z) = 45 \\
&\Gamma_1(z) = 2\theta\theta_{yy} + 2\theta_y^2,\ \Gamma_2(z) = 4x\theta\theta_y^2 + x\theta^2\theta_{yy} + 4\theta\theta_y\sigma_y + \theta^2\sigma_{yy} + 3\theta^2\theta_t \\
&\Gamma_3(z) = \theta^3(x\theta_t + \sigma_t) + \theta^2(x\theta_y + \sigma_y)^2
\end{aligned} \tag{8}$$

Considering Eq.(4) with i=1, i=2, i=3 under the results in (5), (6), (7),we derive:

$$\theta^8\Gamma_1(z) = 2\theta\theta_{yy} + 2\theta_y^2 \tag{9}$$

$$\theta^8\Gamma_2(z) = 4x\theta\theta_y^2 + x\theta^2\theta_{yy} + 4\theta\theta_y\sigma_y + \theta^2\sigma_{yy} + 3\theta^2\theta_t \tag{10}$$

$$\theta^8\Gamma_3(z) = \theta^3(x\theta_t + \sigma_t) + \theta^2(x\theta_y + \sigma_y)^2 \tag{11}$$

This implies that $\Gamma_1(z) = A,\ \Gamma_2(z) = Bz + C,\ \Gamma_3(z) = Dz^2 + Ez + F$
Here A ,B,C,D,E and F are all constants.
using Eq.(8) and the above results in Eq.(3) we will get ODE of ω(z)

$$\omega'''''' + A\omega + (Bz+C)\omega' + (Dz^2+Ez+F)\omega'' + 15\omega''''\omega + 90(\omega')^2\omega + 30\omega'''\omega' + 45\omega''\omega^2 + 15(\omega'')^2 = 0 \tag{12}$$

From Eq.(9) we get

$$\theta\theta_{yy} = \frac{\theta^8 A}{2} - \theta_y^2 \tag{13}$$

From Eq.(10) we get.

$$\theta^8[B(x\theta + \sigma) + C] = 4x\theta\theta_y^2 + x\theta^2\theta_{yy} + 4\theta\theta_y\sigma_y + \theta^2\sigma_{yy} + 3\theta^2\theta_t \tag{14}$$

$$z = x\theta(y,t) + \sigma(y,t) \tag{15}$$

we make a comparison of the coefficients of x and constant terms in Eq.(14) we get.

$$-B\theta^8 + \theta\theta_{yy} + 4\theta_y^2 = 0 \tag{16}$$

$$-B\theta^7\sigma - \theta^7 C + \theta\sigma_{yy} + 3\theta\theta_t + 4\theta_y\sigma_y = 0 \tag{17}$$

From Eq.(11) we get.

$$\theta^8[D(x\theta+\sigma)^2 + E(x\theta+\sigma) + F] = \theta^3(x\theta_t + \sigma_t) + \theta^2(x\theta_y + \sigma_y)^2 \tag{18}$$

Compare the coefficients of x^2 , x and constant terms in Eq.(18) we derive.

$$\theta^{10}D = \theta^2\theta_y^2 \tag{19}$$

$$\theta\theta_t + 2\theta_y\sigma_y = \theta^7(2D\sigma + E) \tag{20}$$

$$\theta^6(D\sigma^2 + E\sigma + F) = \theta\sigma_t + \sigma_y^2 \tag{21}$$

Substituting Eq.(13) into Eq.(16) we obtain:

$$\theta_y = \theta^4(\sqrt{\frac{B}{3} - \frac{A}{6}}) \tag{22}$$

Equation (19) and Equation (22) are same by setting $D = (\sqrt{\frac{B}{3} - \frac{A}{6}})$ in Equation (19)
Implies Eq.(22), Eq.(17) Eq.(20) and Eq.(21) we obtain:

$$\theta_y = \theta^4(\sqrt{\frac{B}{3} - \frac{A}{6}}) \tag{23}$$

$$-B\theta^7\sigma - C\theta^7 + \theta\sigma_{yy} + 3\theta\theta_t + 4\theta_y\sigma_y = 0 \tag{24}$$

$$-2D\theta^7\sigma - E\theta^7 + \theta\theta_t + 2\theta_y\sigma_y = 0 \tag{25}$$

$$\theta\sigma_t + \sigma_y^2 - D\theta^6\sigma^2 - E\theta^6\sigma - F\theta^6 = 0 \tag{26}$$

The solution of Eq.(23), Eq.(24), Eq.(25) and Eq.(26) is

$$\theta(y,t) = c_1, \sigma(y,t) = c_2 y - \frac{c_2^2 t}{c_1} + c_3 \tag{27}$$

Here θ and σ are the functions of y,t and c_1,c_2,c_3 are all constants
General similarity reduction of the SKKP equation (1) is given by

$$u(x,y,t) = c_1^2\omega(z) \tag{28}$$

$$z(x,y,t) = x\theta(y,t) + \sigma(y,t) \tag{29}$$

Where $\omega(z)$ satisfies a special ODE which is derived from Eq.(12) By using $A = B = C = D = E = F = 0$ in Eq.(12) we obtain:

$$\omega'''''' + 15\omega''''\omega + 90(\omega')^2\omega + 30\omega'''\omega' + 45\omega''\omega^2 + 15(\omega'')^2 = 0 \tag{30}$$

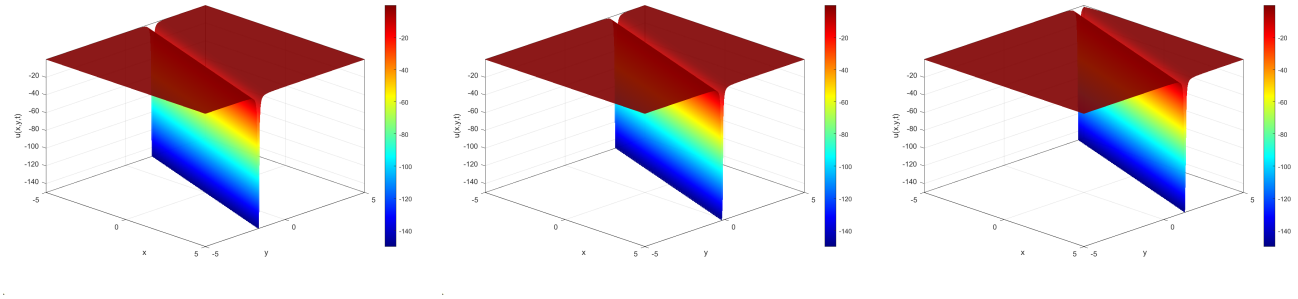
where $\omega(z) = \frac{-2}{(z-c)^2}$ and c is constant satisfies Eq.(30) The similarity reduction of the SKKP equation(1) is

$$u(x,y,t) = \frac{-2c_1^2}{(xc_1 + c_2y - \frac{c_2^2t}{c_1} + c_3 - c)^2} \tag{31}$$

$$z(x,y,t) = xc_1 + c_2y - \frac{c_2^2t}{c_1} + c_3 \tag{32}$$

Figure.(1), illustrates the solution (31) with the free parameters set as $c = c_1 = c_3 = 1$ and $c_2 = 3$, evaluated at $t = 0, t = 0.5$, and $t = 1$.

The figure illustrates the evolution of line-lump solution (31),showing the lump wave whose crest propagates in the x-direction.Further more due to rational term a movable and continuous blow-up phenomenon appears along a straight line.To our knowledge, such a solution with movable and continuous blow-up has not been reported for the SKKP equation before.This behavior is displayed in(Fig.(1) (a)-(c))



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$

Figure 1: Evolution of solution (31) with free parameters $c = c_1 = c_3 = 1$, $c_2 = 3$ and $t = 0$, $t = 0.5$, $t = 1$.

Also $\omega(z) = 2k^2sech^2(k(z-c)) + \frac{2(-5-\sqrt{5})k^2}{15}$ where c and k are constants satisfies

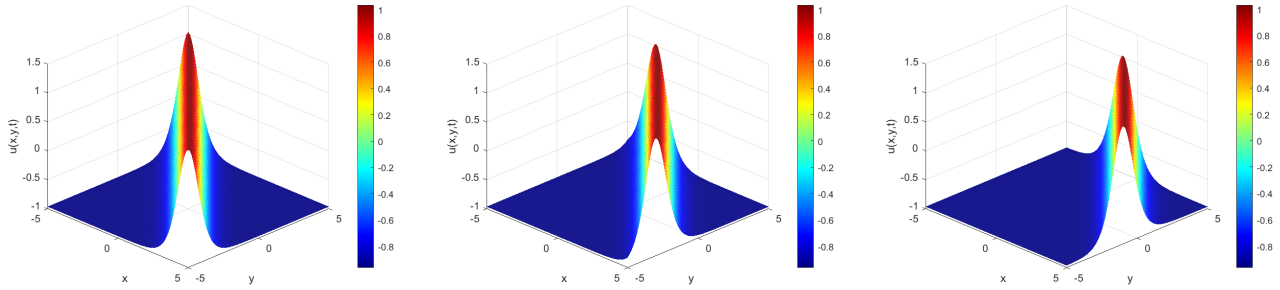
$$\omega'''''' + 15\omega''''\omega + 90(\omega')^2\omega + 30\omega'''\omega' + 45\omega''\omega^2 + 15(\omega'')^2 = 0 \tag{33}$$

The Similarity reduction of SKKP equation(1) is

$$u(x,y,t) = c_1^2(2k^2 sech^2(k(xc_1 + c_2 y - \frac{c_2^2 t}{c_1} + c_3 - c)) + \frac{2(-5-\sqrt{5})k^2}{15}) \tag{34}$$

$$z(x,y,t) = xc_1 + c_2 y - \frac{c_2^2 t}{c_1} + c_3 \tag{35}$$

Figure.(2),illustrates the solution (34) with free parameters set as $c = 0$ $c_1 = c_2 = 1$, $c_3 = 0$,$k = 1$ with $t = 0$, $t = 2$ and $t = 4$.
The solution describes a soliton, characterized physically by steady motion along positive x-direction. Even over a long propagation distances, it maintains its original waveform and amplitude, while its energy remains conserved. These features are shown in (Fig.(2) (a)-(c))



(a) $t = 0$ (b) $t = 2$ (c) $t = 4$

Figure 2: Evolution of solution (34) with free parameters $c = 0$ $c_1 = c_2 = 1$, $c_3 = 0$,$k = 1$ and $t = 0$, $t = 2$, $t = 4$.

First case(A2): $A = 0, B = 0, C = 0, , D = 0, E = 0, F = 0, G = 0$

Compare the coefficients of ω'''''' (i=0) with ω'''' (i=5 see (A) for details) here we will not use Remark(i) and setting $\Gamma_5(z) = 1$ we derive:

$$\alpha = \frac{z_x^2}{15} \tag{36}$$

Also we know that

$$\beta = z_x^2 \tag{37}$$

$$z = x\theta(y,t) + \sigma(y,t) \tag{38}$$

putting Eq.(36) , Eq.(37) , Eq.(38) in (A) we obtain: Eq.(39)-(44):

$$\Gamma_4(z) = \Gamma_6(z) = \Gamma_8(z) = \Gamma_{10}(z) = \Gamma_{14}(z) = \Gamma_{15}(z) = \Gamma_{16}(z) = \Gamma_{18}(z) = 0 \tag{39}$$

$$\Gamma_0(z) = 1, \Gamma_5(z) = 1, \Gamma_7(z) = 45, \Gamma_9(z) = 6, \Gamma_{11}(z) = 30, \Gamma_{12}(z) = 15, \Gamma_{13}(z) = 90, \Gamma_{17}(z) = 6, \Gamma_{19}(z) = 15 \tag{40}$$

$$\Gamma_1(z) = 2\theta\theta_{yy} + 2\theta_y^2 \tag{41}$$

$$\Gamma_2(z) = \theta(\theta\theta_{yy}x + 4\theta_y^2 x + 3\theta\theta_t + \theta\sigma_{yy} + 4\theta_y\sigma_y) \tag{42}$$

$$\Gamma_3(z) = \frac{\theta^8}{5} + \theta^3(x\theta_t + \sigma_t) + \theta^2(x\theta_y + \sigma_y)^2 \tag{43}$$

$$\Gamma_{20}(z) = \frac{2\theta_y^2}{15} + \frac{2\theta\theta_{yy}}{15} \tag{44}$$

From Eq.(41), (42), (43), (44) implies that $\Gamma_1(z) = A$, $\Gamma_2(z) = Bz + C$, $\Gamma_3(z) = Dz^2 + Ez + F$ and $\Gamma_{20}(z) = G$
Here A,B,C,D,E,F and G are all constants
using Eq.(40) and above results in Eq.(3) we will get ODE of $\omega(z)$.

$$\omega''''' + A\omega + (Bz + C)\omega' + (Dz^2 + Ez + F)\omega'' + \omega'''' + 15\omega''''\omega + 6\omega''\omega + 90\omega'^2\omega + 45\omega''\omega^2 + 30\omega'''\omega' + 6\omega'^2 + 15\omega''^2 + G = 0 \tag{45}$$

By putting $A = B = C = D = E = F = G = 0$ in Eq.(45) we obtain:

$$\omega''''' + \omega'''' + 15\omega''''\omega + 6\omega''\omega + 90\omega'^2\omega + 45\omega''\omega^2 + 30\omega'''\omega' + 6\omega'^2 + 15\omega''^2 = 0 \tag{46}$$

By comparing Eq.(41) with normalizing coefficient (βz_x^6) we get.

$$\frac{\theta^8 A}{2} - \theta_y^2 - \theta\theta_{yy} = 0 \tag{47}$$

By comparing Eq.(42) with normalizing coefficient (βz_x^6) we get.

$$\theta^8(B(x\theta + \sigma) + C) = \theta(\theta_{yy}\theta x + 4\theta_y^2 x + 4\theta_y\sigma_y + 3\theta\theta_t + \theta\sigma_{yy}) \tag{48}$$

Compare the coefficient of x and constant terms in Eq.(48) we get.

$$\theta^8 B - \theta\theta_{yy} - 4\theta_y^2 = 0 \tag{49}$$

$$\theta^7(B\sigma + C) - \theta\sigma_{yy} - 3\theta\theta_t - 4\theta_y\sigma_y = 0 \tag{50}$$

By comparing Eq.(43) with normalizing coefficient (βz_x^6) we get.

$$\theta^8(D(x\theta + \sigma)^2 + E(x\theta + \sigma) + F) = \frac{\theta^8}{5} + \theta^3(x\theta_t + \sigma_t) + \theta^2(x\theta_y + \sigma_y)^2 \tag{51}$$

Compare the coefficient of x^2, x and constant terms in Eq.(51) we get.

$$D\theta^8 - \theta_y^2 = 0 \tag{52}$$

$$2D\theta^7\sigma + E\theta^7 - \theta\theta_t - 2\theta_y\sigma_y = 0 \tag{53}$$

$$D\theta^6\sigma^2 + E\theta^6\sigma + F\theta^6 - \frac{\theta^6}{5} - \theta\sigma_t - \sigma_y^2 = 0 \tag{54}$$

By comparing Eq.(44) with normalizing coefficient (βz_x^6) we get.

$$\frac{15\theta^8 G}{2} - \theta_y^2 - \theta\theta_{yy} = 0 \tag{55}$$

From Eq.(55) $\theta\theta_{yy} = \frac{15\theta^8 G}{2} - \theta_y^2$ put in Eq.(47) we obtain:

$$\theta^8(\frac{A}{2} - \frac{15G}{2}) = 0 \tag{56}$$

From Eq.(52) $\theta_y^2 = D\theta^8$ put in Eq.(49) we obtain:

$$\theta^8 B - \theta\theta_{yy} - 4\theta^8 D = 0 \tag{57}$$

Now for finding θ and σ we will use the following equations

$$\theta^8(\frac{A}{2} - \frac{15G}{2}) = 0 \tag{58}$$

$$\theta^8 B - \theta\theta_{yy} - 4\theta^8 D = 0 \tag{59}$$

$$D\theta^6\sigma^2 + E\theta^6\sigma + F\theta^6 - \frac{\theta^6}{5} - \theta\sigma_t - \sigma_y^2 = 0 \tag{60}$$

$$2D\theta^7\sigma + E\theta^7 - \theta\theta_t - 2\theta_y\sigma_y = 0 \tag{61}$$

$$\theta^7(B\sigma + C) - \theta\sigma_{yy} - 3\theta\theta_t - 4\theta_y\sigma_y = 0 \tag{62}$$

From Eq.(58), Eq.(59), Eq.(60), Eq.(61), Eq.(62) by using maple software we obtain:

$$\theta = c_1, \sigma = \frac{-c_1^5 t}{5} + c_2 \tag{63}$$

here c_1, and c_2 are constant.
Where $\omega(z) = -\frac{1}{3} + \frac{sech^2}{2}(\frac{z+c}{2})$ where c is any real constant satisfies

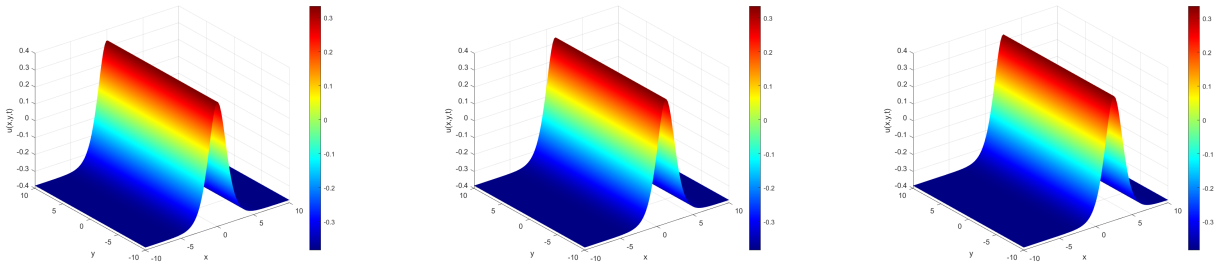
$$\omega'''''' + \omega'''' + 15\omega''''\omega + 6\omega''\omega + 90\omega'^2\omega + 45\omega''\omega^2 + 30\omega'''\omega' + 6\omega'^2 + 15\omega''^2 = 0 \tag{64}$$

The Similarity Reduction of SKKP Equation(1) is

$$u(x,y,t) = \frac{c_1^2}{15} + c_1^2(-\frac{1}{3} + \frac{sech^2}{2}(\frac{z+c}{2})) \tag{65}$$

$$z = xc_1 + \frac{-c_1^5 t}{5} + c_2 \tag{66}$$

Figure.(3),illustrates the solution (65) with free parameters set as $c = 0$, $c_1 = 1$, $c_2 = 0$,and $t = 0$, $t = 5$, $t = 10$.
The solution describes a soliton, characterized physically by steady motion along positive x-direction. Even over a long propagation distances, it maintains its original waveform and amplitude, while its energy remains conserved. These features are shown in (Fig.(3) (a)-(c))



(a) $t = 0$ (b) $t = 3$ (c) $t = 6$

Figure 3: Evolution of solution (65) with free parameters $c = 0$, $c_1 = 1.2$, $c_2 = 0$,and $t = 0$, $t = 3$, $t = 6$.

Also $\omega(z) = \frac{1}{5} + cn^2(z, \frac{1}{\sqrt{2}})$ where cn(z,k) is the jacobi elliptic cosine with modulus k=$\frac{1}{\sqrt{2}}$ satisfies

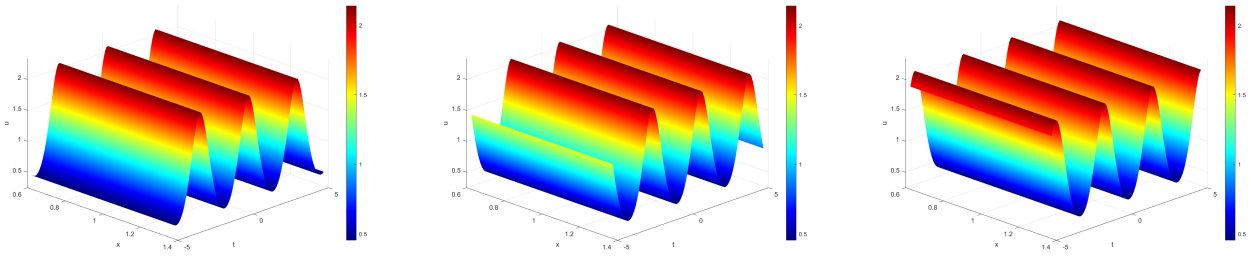
$$\omega'''''' + \omega'''' + 15\omega''''\omega + 6\omega''\omega + 90\omega'^2\omega + 45\omega''\omega^2 + 30\omega'''\omega' + 6\omega'^2 + 15\omega''^2 = 0 \tag{67}$$

The similarity reduction of SKKP Equation (1) is

$$u(x,y,t) = \frac{c_1^2}{15} + c_1^2(\frac{1}{5} + cn^2(xc_1 + \frac{-c_1^5 t}{5} + c_2, \frac{1}{\sqrt{2}})) \tag{68}$$

$$z = xc_1 + \frac{-c_1^5 t}{5} + c_2 \tag{69}$$

Figure.(4), illustrates the solution (68) with free parameters set as $c_1 = 1$, $c_2 = 0$, $k = \frac{1}{\sqrt{2}}$,and $t = 0$, $t = 1.5$, $t = 3$.
This wave is a cnoidal a periodic traveling wave which propagate in the positive x-direction with constant speed and amplitude which is control by parameter c_1. The Energy of the wave is remains conserved during propagation shown in (Fig.(4) (a)-(c))



(a) $t = 0$ (b) $t = 1.5$ (c) $t = 3$

Figure 4: Evolution of solution (68) with free parameters $c_1 = 1$, $c_2 = 0$, $k = \frac{1}{\sqrt{2}}$,and $t = 0$, $t = 1.5$, $t = 3$.

Also the function $\omega(z) = -4\wp(z; \frac{1}{60}, g) - \frac{1}{15}$ where $\wp$ satisfies the Weierstrass equation $(\wp')^2 = 4\wp^3 - \frac{1}{60}\wp - g$ satisfy the sixth-order nonlinear differential equation (67) for any real parameter g.
here we will use the degenerate case of the Weierstrass elliptic function and we will obtain the folluwing soliton form :
$\omega(z) = -\frac{1+\sqrt{5}}{15} + \frac{1}{\sqrt{5}} sech^2(\frac{z}{2\sqrt[4]{5}})$
where $\omega(z)$ satisfies (67) this soliton solution is similar to old soliton solution which is given in Eq.(65)

Also $\omega(z) = -\frac{2}{(z-c)^2}$ Here c is arbitrary constant and $\omega(z)$ satisfies

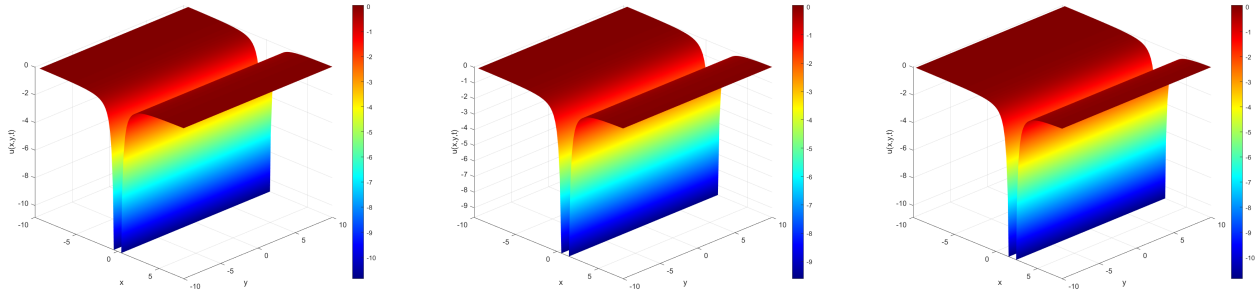
$$\omega'''''' + \omega'''' + 15\omega''''\omega + 6\omega''\omega + 90\omega'^2\omega + 45\omega''\omega^2 + 30\omega'''\omega' + 6\omega'^2 + 15\omega''^2 = 0 \tag{70}$$

The similarity reduction of SKKP Equation(1) is

$$u(x, y, t) = \frac{c_1^2}{15} + c_1^2\left(\frac{-2}{(xc_1 + \frac{-c_1^5 t}{5} + c_2 - c)^2}\right) \tag{71}$$

$$z = xc_1 + \frac{-c_1^5 t}{5} + c_2 \tag{72}$$

Figure.(5),illustrates the solution (71) with free parameters at $c = 0$, $c_1 = 1$, $c_2 = 0$,and $t = 0$, $t = 5$, $t = 10$.
It exhibits the evolution of the line-lump solution(71),showing the propagation of the lump wave with wave crest along the x direction .In addition,due to the presence of the rational function term,a movable and continuous blow-up phenomenon occurs on a straight line.To our knowledge, such a solution with movable and continuous blow-up has not been appears for the SKKP equation, shown in (Fig.(5) (a)-(c))



(a) $t = 0$ (b) $t = 5$ (c) $t = 10$

Figure 5: Evolution of solution (71) with free Parameters $c = 0$, $c_1 = 1$, $c_2 = 0$,and $t = 0$, $t = 5$, $t = 10$.

2.2 Second case $z_x = 0$,$z_y \neq 0$

In this case,Eq.(3) can be written as

$$\gamma_0\omega'' + \gamma_1\omega + \gamma_2\omega' + \gamma_3\omega^3 + \gamma_4\omega^2 + \gamma_5 = 0 \tag{73}$$

where γ_i, $(i = 1, ..., 5)$ see B for details.
Similarly, we will use the coefficient of ω'' (βz_y^2) a normalizing coefficient so the other coefficients must take the form $\Gamma_i(z)\beta z_y^2$ i.e.

$$\gamma_i = \gamma_0\Gamma_i(z), \quad (i = 1, ..., 5). \tag{74}$$

To reduce the calculation difficulty and get reduction of the equation, we can assume $\beta(x, y, t) = xa(y, t) + b(y, t)$ Here a and b is arbitrary function of y,t. for convenience to make our calculation easy we let $a = z_y$
Compare the Coefficients of ω'' (i=0) with ω' (i=2 see (B) for details) and put $\beta(x, y, t) =$

$xz_y + b(y,t)$ we obtain:

$$z_y z_t + 2(xz_{yy} + b_y)z_y + xz_y z_{yy} + bz_{yy} = \Gamma_2(z)(xz_y + b)z_y^2 \tag{75}$$

Compare the coefficients of x and constant terms in Eq.(75)
we obtain:

$$3z_y z_{yy} = z_y^3 \Gamma_2(z) \tag{76}$$

$$z_y z_t + 2z_y b_y + bz_{yy} = \Gamma_2(z) z_y^2 b \tag{77}$$

From Eq.(76) we obtain:

$$z(y,t) = \eta(t)y + \sigma(t) \tag{78}$$

Where $\eta(t)$, $\sigma(t)$ are orbitrary function of t.
From Eq.(77) by setting $\Gamma_2(z) = 0$ we obtain:

$$b(y,t) = \theta(t) - \frac{\eta'(t)y^2}{4} - \frac{\sigma'(t)y}{2} \tag{79}$$

So,

$$\beta(x,y,t) = ax + \theta(t) - \frac{\eta'(t)y^2}{4} - \frac{\sigma'(t)y}{2} \tag{80}$$

By comparing coefficient of ω^2 (i=0) with ω''(i=4) by using Remark(i) we derive:

$$\alpha = 0 \tag{81}$$

From the solutions in Eq.(78) ,Eq.(80) and Eq.(81) we obtain:

$$\begin{aligned} &\Gamma_1(z) = 0,\ \Gamma_2(z) = 0,\ \Gamma_4(z) = 0,\ \Gamma_5(z) = 0 \\ &\Gamma_0(z) = (ax + \theta(t) - \frac{\sigma'(t)y}{2}) \\ &\Gamma_3(z) = 90(ax + \theta(t) - \frac{\sigma'(t)y}{2})a^2 \end{aligned} \tag{82}$$

From Eq.(82) compare $\Gamma_0(z)$, $\Gamma_3(z)$ with normalizing coefficient (βz_y^2)we obtain:

$$\Gamma_0(z) = 1,\ \ \Gamma_3(z) = 90 \tag{83}$$

Compare the coefficients of ω^3(i=3) and ω''(i=0) we obtain.

$$45\beta_{xx}^2 + 90\beta\beta_x^2 = \Gamma_3(z)\beta z_y^2 \tag{84}$$

By putting $\beta(x,y,t) = xa + b(y,t)$ and $\Gamma_3(z) = 90$ in Eq.(84) we obtain:

$$a = z_y = \eta(t) \tag{85}$$

Using Eq.(85) we can write Eq.(78) and Eq.(80) as

$$\beta(x, y, t) = \eta(t)x + \theta(t) - \frac{\sigma'(t)y}{2}, \;\; z = \eta(t)y + \sigma(t), \tag{86}$$

From Eq.(73) we obtain ODE of ω(z) as

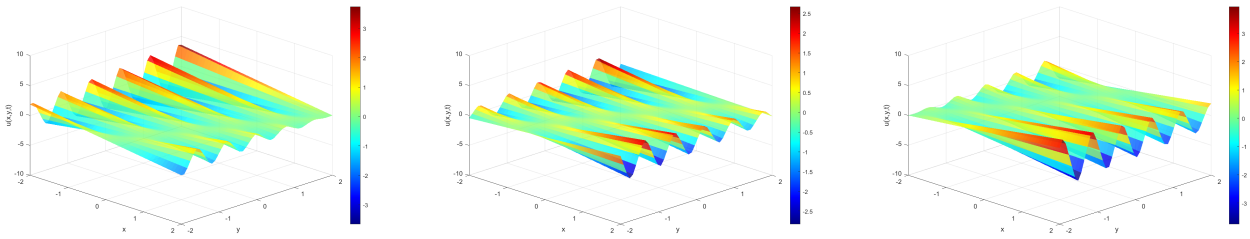
$$\omega'' + 90\omega^3 = 0 \tag{87}$$

Where $\omega(z) = (Acn(\sqrt{90}(z-c), \frac{1}{\sqrt{2}}))$ where $A = 1$ or -1 and ω(z) satisfies Eq.(87)
The Similarity reduction of SKKP Equation(1) will takes the form:

$$u(x, y, t) = A(\eta(t)x + \theta(t) - \frac{\sigma'(t)y}{2})(cn(\sqrt{90}(z-c), \frac{1}{\sqrt{2}})) \tag{88}$$

$$z = \eta(t)y + \sigma(t) \tag{89}$$

Figure.(6),illustrates the solution (88) with free parameters at $\eta(t) = 1$, $\theta(t) = t$,$\sigma(t) = -t$, $\sigma'(t) = -1$, $A = 1$, $c = 1$ with $t = -1$, $t = 0$ and $t = 1$.
A Jacobi elliptic function solution of the form (88) is remarkably unique because its amplitude is linearly modulated by the spatial coordinates x and y through arbitrary time-dependent functions η(t), θ(t), σ(t),causing the wave to propagate along the positive y-axis with a non-stationary amplitude profile a feature never before observed in any Jacobi elliptic solution for the SKKP equation. Energy in a wave is related to how tall the wave is (its amplitude). Taller the wave more is energy shown in (Fig.(6) (a)-(c))



(a) $t = -1$ (b) $t = 0$ (c) $t = 1$

Figure 6: Evolution of solution (88) with $\eta(t) = 1$, $\theta(t) = t$,$\sigma(t) = -t$, $\sigma'(t) = -1$, $A = 1$, $c = 1$ and $t = -1$, $t = 0$, $t = 1$.

Physically through the analysis of the above solutions, The propagation dynamics of these solutions offers valuable insights into the evolution of fluids waves governed by the SKKP equation.

3 Conclusion And Discussion

In this manuscript, we have successfully implemented the Clarkson-Kruskal (CK) direct method for the first time to investigate the (2+1)-dimensional SKKP equation,effectively circumventing the computational obstacles posed by its fifth-order dispersion and high-order nonlinear terms through special assumptions and the CK remarks.
Our results are divided into two main cases.In the first case ($z_x \neq 0, z_y \neq 0$),the equation was reduced to two distinct subcase ODEs Eq.(30),Eq.(46).From these reductions, we derived a series of novel exact solutions: rational line-lump solutions exhibiting a movable and continuous blow-up along a straight line Eq.(31) and Eq.(71),new stable soliton solutions Eq.(34), Eq.(65),and a cnoidal periodic wave Eq.(68).In second case ($z_x = 0.z_y \neq 0$),we obtained the reduction (87) and the remarkably unique Jacobi elliptic function solution(88),characterized by an amplitude linearly modulated by the spatial coordinates x and y through arbitrary time-dependent functions,leading to non-stationary amplitude propagation along the y-axis a feature never before observed in any Jacobi elliptic solution for the SKKP equation. All these solutions involve arbitrary functions of time and several parameters, and none have been previously reported.

Declarations

Conflict of interest/Competing interests

The authors declare that they have no conflicts of interest.

Data availability

No datasets were generated or analyzed during the current study.

Appendix A The form of $\boldsymbol{\gamma}_i$

$$\gamma_0 = \beta z_x^6 \tag{A}$$

$\gamma_1 = 180\alpha\alpha_x\beta_x + 90\alpha\beta\alpha_{x,x} + \beta_{t,x} + \beta_{x,x,x,x,x,x} + \beta_{y,y} + 90\beta\alpha_x^2 + 30\alpha_x\beta_{x,x,x}$
$+ 30\beta_x\alpha_{x,x,x} + 45\alpha^2\beta_{x,x} + 15\alpha\beta_{x,x,x,x} + 15\beta\alpha_{x,x,x,x} + 30\alpha_{x,x}\beta_{x,x}$

$\gamma_2 = 15\alpha(x,y,t)\,\beta(x,y,t)\,z_{x,x,x,x} + 60\alpha_{x,x}\beta_x z_x + 30\alpha_{x,x}\beta(x,y,t)\,z_{x,x} + 90\alpha_x\beta_{x,x}z_x + 90\alpha_x\beta_x z_{x,x} +$
$30\alpha_x\beta(x,y,t)\,z_{x,x,x} + 30\beta(x,y,t)\,z_x\alpha_{x,x,x} + 90\alpha(x,y,t)^2\,\beta_x z_x + 45\alpha(x,y,t)^2\,\beta(x,y,t)\,z_{x,x} +$
$60\alpha(x,y,t)\,\beta_{x,x,x}z_x + 90\alpha(x,y,t)\,\beta_{x,x}z_{x,x} + 60\alpha(x,y,t)\,\beta_x z_{x,x,x} + 180\alpha(x,y,t)\,\alpha_x\beta(x,y,t)\,z_x + \beta_t z_x +$
$6\beta_{x,x,x,x,x}z_x + 6\beta_x z_{x,x,x,x,x} + \beta(x,y,t)\,z_{x,x,x,x,x,x} + 15\beta_{x,x,x,x}z_{x,x} + 20\beta_{x,x,x}z_{x,x,x} + 15\beta_{x,x}z_{x,x,x,x} + \beta_x z_t +$
$\beta(x,y,t)\,z_{t,x} + 2\beta_y z_y + \beta(x,y,t)\,z_{y,y}$

$\gamma_3 = 60\beta_{x,x,x}z_x z_{x,x} + 45\alpha(x,y,t)\,\beta(x,y,t)\,z_{x,x}^2 + 30\alpha_{x,x}\beta(x,y,t)\,z_x^2 + 90\alpha_x\beta_x z_x^2 +$
$45\alpha(x,y,t)^2\,\beta(x,y,t)\,z_x^2 + 90\alpha(x,y,t)\,\beta_{x,x}z_x^2 + 60\beta_x z_{x,x}z_{x,x,x} + 30\beta_x z_x z_{x,x,x,x} + 15\beta(x,y,t)\,z_{x,x}z_{x,x,x,x} +$
$6\beta(x,y,t)\,z_x z_{x,x,x,x,x} + \beta(x,y,t)\,z_x z_t + 60\beta_{x,x}z_x z_{x,x,x} + 90\alpha_x\beta(x,y,t)\,z_x z_{x,x} + 180\alpha(x,y,t)\,\beta_x z_x z_{x,x} +$
$60\alpha(x,y,t)\,\beta(x,y,t)\,z_x z_{x,x,x} + 15\beta_{x,x,x,x}z_x^2 + \beta(x,y,t)\,z_y^2 + 45\beta_{x,x}z_{x,x}^2 + 10\beta(x,y,t)\,z_{x,x,x}^2$

$\gamma_4 = 30\alpha_x\beta(x,y,t)\,z_x^3 + 60\alpha(x,y,t)\,\beta_x z_x^3 + 60\beta_x z_x^2 z_{x,x,x} + 15\beta(x,y,t)\,z_x^2 z_{x,x,x,x} + 90\beta_{x,x}z_x^2 z_{x,x} +$
$90\beta_x z_x z_{x,x}^2 + 60\beta(x,y,t)\,z_x z_{x,x,x}z_{x,x} + 90\alpha(x,y,t)\,\beta(x,y,t)\,z_x^2 z_{x,x} + 15\beta(x,y,t)\,z_{x,x}^3 + 20\beta_{x,x,x}z_x^3$

$\gamma_5 = 15\alpha(x,y,t)\,\beta(x,y,t)\,z_x^4 + 20\beta(x,y,t)\,z_x^3 z_{x,x,x} + 60\beta_x z_x^3 z_{x,x} + 45\beta(x,y,t)\,z_x^2 z_{x,x}^2 + 15\beta_{x,x}z_x^4$

$\gamma_6 = 15\beta z_x^4 z_{x,x} + 6\beta_x z_x^5$

$\gamma_7 = 15\beta^2 z_x^4$

$\gamma_8 = 45\beta^2 z_{x,x}^2 + 90\beta_x^2 z_x^2 + 90\alpha\,\beta^2 z_x^2 + 120\beta z_x^2\beta_{x,x} + 60\beta^2 z_x z_{x,x,x} + 270\beta z_x\beta_x z_{x,x}$

$\gamma_9 = 15\beta^2 z_{x,x,x,x} + 90\beta_x^2 z_{x,x} + 180\beta^2\alpha_x z_x + 150\beta_x\beta_{x,x}z_x + 90\beta_x\beta z_{x,x,x} + 90\beta z_x\beta_{x,x,x} + 90\alpha\,\beta^2 z_{x,x} + 120\beta\beta_{x,x}z_{x,x} + 360\alpha\beta_x\beta z_x$

$\gamma_{10} = 90\beta^3 z_x^2$

$\gamma_{11} = 30\beta^2 z_x^4$

$\gamma_{12} = 45\beta^3 z_x^2$

$\gamma_{13} = 45\beta^3 z_{x,x} + 270\beta^2\beta_x z_x$

$\gamma_{14} = 90\beta_x\beta z_x^3 + 90\beta^2 z_x^2 z_{x,x}$

$\gamma_{15} = 150\beta_x\beta z_x^3 + 120\beta^2 z_x^2 z_{x,x}$

$\gamma_{16} = 90\alpha\,\beta^2 z_x^2 + 90\beta z_x^2\beta_{x,x} + 30\beta^2 z_x z_{x,x,x} + 150\beta z_x\beta_x z_{x,x} + 60\beta_x^2 z_x^2 + 15\beta^2 z_{x,x}^2$

$\gamma_{17} = 180\beta\alpha_x\beta_x + 90\alpha\beta\beta_{x,x} + 15\beta_{x,x}^2 + 90\alpha\beta_x^2 + 30\beta_x\beta_{x,x,x} + 45\beta^2\alpha_{x,x} + 15\beta\beta_{x,x,x,x}$

$\gamma_{18} = 90\beta\beta_x^2 + 45\beta^2\beta_{x,x}$

$\gamma_{19} = 15\beta^2 z_x^4$

$\gamma_{20} = 90\alpha\alpha_x^2 + 30\alpha_x\alpha_{x,x,x} + 45\alpha^2\alpha_{x,x} + 15\alpha\alpha_{x,x,x,x} + 15\alpha_{x,x}^2 + \alpha_{x,x,x,x,x,x} + \alpha_{t,x} + \alpha_{y,y}$

Appendix B The form of $\boldsymbol{\gamma}_i$

$$\gamma_0 = \beta z_y^2 \tag{B}$$

$\gamma_1 = 180\alpha\alpha_x\beta_x + 90\alpha\beta\alpha_{x,x} + \beta_{t,x} + \beta_{x,x,x,x,x,x} + \beta_{y,y} + 90\beta\alpha_x^2 + 30\alpha_x\beta_{x,x,x} + 30\beta_x\alpha_{x,x,x} + 45\alpha^2\beta_{x,x} + 15\alpha\beta_{x,x,x,x} + 15\beta\alpha_{x,x,x,x} + 30\alpha_{x,x}\beta_{x,x}$

$\gamma_2 = \beta_x z_t + 2\beta_y z_y + \beta z_{y,y}$

$\gamma_3 = 90\beta\beta_x^2 + 45\beta^2\beta_{x,x}$

$\gamma_4 = 180\beta\alpha_x\beta_x + 90\alpha\beta\beta_{x,x} + 15\beta_{x,x}^2 + 90\alpha\beta_x^2 + 30\beta_x\beta_{x,x,x} + 45\beta^2\alpha_{x,x} + 15\beta\beta_{x,x,x,x}$

$$\gamma_5 = 90\alpha\alpha_x^2 + 30\alpha_x\alpha_{x,x,x} + 45\alpha^2\alpha_{x,x} + 15\alpha\alpha_{x,x,x,x} + 15\alpha_{x,x}^2 + \alpha_{x,x,x,x,x,x} + \alpha_{t,x} + \alpha_{y,y}$$

Declaration of AI Use

This manuscript was prepared through the intellectual contributions of the author(s) to the study design, data analysis, interpretation of results, and scholarly content. ChatGPT, Claude and Perplexity were used solely to assist with grammatical refinement and reference formatting during manuscript revision. The author(s) accept full responsibility for the accuracy, integrity, and final content of the manuscript.

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