

Five-Point Implicit Linear Multistep Method For Numerical Computation of Higher Order Differential Equation

Abstract

This paper develops a five-point Implicit Linear Multistep Method (ILMM) for the numerical solution of fourth-order ordinary differential equations. The method is formulated through an interpolation and collocation procedure, employing the basic power series and its derivatives, respectively. The resulting continuous formulation is evaluated at appropriately selected points to derive the corresponding discrete implicit linear multistep scheme. The performance of the proposed scheme is further investigated through two numerical test problems with known exact solutions. The numerical solutions are compared with the exact solutions both numerically and graphically.

Keywords: higher order ODE, interpolation and collocation, implicit linear multistep method
1.Introduction

Higher-order ordinary differential equations (ODEs) arise naturally in numerous areas of science and engineering, including control theory, mechanics, fluid dynamics, and mathematical modelling. Consequently, the development of accurate and computationally efficient numerical techniques for their solution continues to attract considerable attention from researchers. In particular, the numerical treatment of higher-order initial value problems (IVPs) is important because their direct solution can provide computational advantages over approaches based on their reformulation as systems of first-order differential equations.

Consider the general fourth-order initial value problem

$$y^{(4)} = f(x, y, y', y'', y'''), \quad a \leq x \leq b,$$

subject to the initial conditions

$$y(a) = \gamma_0, \quad y'(a) = \gamma_1, \quad y''(a) = \gamma_2, \quad y'''(a) = \gamma_3. \quad (1)$$

A conventional approach to solving equation (1) is to transform the fourth-order equation into an equivalent system of four first-order ordinary differential equations and subsequently apply an appropriate numerical integration scheme. Although this approach is widely used, the transformation increases the number of dependent variables and may result in additional computational operations, thereby increasing the computational effort and time required to obtain the numerical solution [1–5].

An alternative approach is to solve the higher-order differential equation directly without first reducing it to a system of first-order equations. Previous studies have demonstrated that direct methods based on implicit LMMs can provide accurate and efficient numerical solutions for higher-order ordinary differential equations [6,7]. Furthermore, continuous collocation-based multistep methods have been extensively investigated for the direct numerical solution of higher-order ODEs [8–9].

Several continuous linear multistep methods have subsequently been proposed for the direct solution of fourth-order ordinary differential equations [10–14,16]. Numerical quadrature [19-36] is applicable to ILMM because the model involves integrals of continuous functions that often cannot be evaluated analytically. It provides an efficient way to compute these integrals accurately, enabling the practical implementation of the ILMM. The B-spline

collocation techniques [37-57] are essential for ILMM because they provide a flexible and accurate way to approximate unknown continuous functions using piecewise polynomial basis functions. Spline collocations convert the ILMM's continuous equations into a finite system of algebraic equations, making them computationally tractable. Fractional calculus and integral transforms [58-68] extend classical ODEs to capture memory and non-local effects in complex dynamical systems whereas integral transforms simplify ODEs by converting differential equations into algebraic or integral forms that are easier to solve. Block methods [69-74] are essential in ILMM because they compute multiple solution points simultaneously when solving ODEs. Recent works on numerical computations like [75-78] are based on fractional and boundary value problems. They improve the efficiency and accuracy of the numerical solution process. They also provide a systematic framework for handling complex ODE systems within ILMM

Motivated by these considerations, this paper proposes a five-point fully implicit linear multistep method for the direct numerical solution of fourth-order ordinary differential equations. The method is constructed using interpolation and collocation techniques based on the basic power series and its derivatives, respectively. The proposed formulation is designed to eliminate the need for reducing the original fourth-order equation to a system of first-order equations and to avoid the additional predictor–corrector procedure. Its numerical performance is demonstrated through selected test problem, and the computed solutions are compared with the corresponding exact solutions. The numerical results indicate that the proposed method provides accurate and efficient approximations for the considered fourth-order initial value problems.

The manuscript is potentially significant to the scientific community because it focuses on developing or applying a numerical method for solving higher-order differential equations, which frequently arise in mathematical modelling, engineering, physics, and other scientific disciplines.

The remainder of this paper is organized as follows. Section 2 presents the derivation of the proposed five-point implicit linear multistep method. The analysis of the resulting scheme is discussed in Section 3. The order and error constant are derived in section 4. The numerical experiments and comparisons with exact solutions is presented in Section 5. Finally, the conclusions of the study are given in Section 6.

2. Derivation of the method

According to [15], a k -step linear multistep method (LMM) for an n -th-order ordinary differential equation can be expressed in the form

$$\sum_{j=0}^k \alpha_j y_{n+j} = h^n \sum_{j=0}^k \beta_j f_{n+j}, \quad (2)$$

where α_j and β_j are uniquely determined constants satisfying

$$\alpha_k = 1, \alpha_0 + \beta_0 \neq 0,$$

and n denotes the order of the differential equation.

By employing interpolation and collocation techniques, equation (2) can be extended to a continuous linear multistep method of the form

$$Y(x) = \sum_{j=0}^k \alpha_j(x) y_{n+j} + h^n \sum_{j=0}^k \beta_j(x) f_{n+j},$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuously differentiable functions of x . The continuous formulation provides a framework for obtaining numerical approximations to higher-order ordinary differential equations directly, without first reducing them to an equivalent system of first-order equations. Moreover, continuous collocation methods facilitate the construction

of numerical approximations over an interval and provide a convenient means of assessing the approximation error.

To obtain an approximate solution of the fourth-order initial value problem (1), consider a polynomial approximation of degree m in the form

$$Y(x) = \sum_{j=0}^m a_j (x - x_n)^j, \quad x_n \leq x \leq x_{n+k}, \quad (3)$$

Differentiating the polynomial approximation in equation (3) four times with respect to x yields

$$Y^{(4)}(x) = \sum_{j=4}^m j(j-1)(j-2)(j-3)a_j(x-x_n)^{j-4}. \quad (4)$$

The coefficients a_j , $j = 0, 1, \dots, m$, are determined by imposing suitable interpolation and collocation conditions. For the interpolation conditions, we consider the mesh points

$$x = x_n, x_{n+1}, \dots, x_{n+k-1},$$

where $x_{n+j} = x_n + jh$ and h denotes the step size. Substituting these points into equation (3) gives

$$\begin{aligned} Y(x_n) &= y_n = a_0, \\ Y(x_{n+1}) &= y_{n+1} = a_0 + a_1h + a_2h^2 + a_3h^3 + \dots + a_mh^m, \end{aligned}$$

and, in general,

$$Y(x_{n+i}) = y_{n+i} = a_0 + ia_1h + i^2a_2h^2 + i^3a_3h^3 + \dots + i^ma_mh^m, \quad i = 0, 1, \dots, k-1.$$

Consequently, at the last interpolation point, we obtain

$$Y(x_{n+k-1}) = y_{n+k-1} = a_0 + (k-1)a_1h + (k-1)^2a_2h^2 + \dots + (k-1)^ma_mh^m. \quad (5)$$

The collocation conditions are then imposed on the fourth derivative given in equation (4). Specifically, collocating at the points

$$x = x_n, x_{n+1}, \dots, x_{n+k},$$

and using

$$Y^{(4)}(x_{n+i}) = f_{n+i},$$

we obtain

$$f_{n+i} = \sum_{j=4}^m j(j-1)(j-2)(j-3)a_j(x_{n+i} - x_n)^{j-4}, \quad i = 0, 1, \dots, k. \quad (6)$$

In particular, at the final collocation point x_{n+k} ,

$$f_{n+k} = Y^{(4)}(x_{n+k}) = \sum_{j=4}^m j(j-1)(j-2)(j-3)a_j(x_{n+k} - x_n)^{j-4}. \quad (7)$$

continuous formulation and, subsequently, the discrete five-point fully implicit linear multistep method. For the construction of the proposed method, we consider equations (5) and (7) with $k = 5$ and $m = 10$. The resulting interpolation and collocation equations are assembled into the matrix system for determining the unknown polynomial coefficient and further employing the matrix inversion technique, the unknown coefficients are expressed in terms of the nodal solution values y_{n+j} and the corresponding function values f_{n+j} . Substituting these expressions into equation (3), we obtain the following continuous formulation of the proposed method:

$$\begin{aligned} Y(x) = & \left(1 + \frac{47(x-x_n)}{60h} - \frac{401(x-x_n)^2}{60h^2} + \frac{275(x-x_n)^3}{48h^3} + \frac{21(x-x_n)^5}{5h^5} + \frac{959(x-x_n)^6}{300h^6} - \frac{9(x-x_n)^7}{8h^7} + \right. \\ & \left. \frac{17(x-x_n)^8}{80h^8} - \frac{(x-x_n)^9}{48h^9} + \frac{(x-x_n)^{10}}{1200h^{10}} \right) y_n + \left(\frac{6(x-x_n)}{5h} - \frac{6(x-x_n)^2}{5h^2} - \frac{277(x-x_n)^3}{12h^3} + \frac{84(x-x_n)^5}{5h^5} - \right. \\ & \left. \frac{959(x-x_n)^6}{75h^6} + \frac{9(x-x_n)^7}{2h^7} - \frac{17(x-x_n)^8}{20h^8} + \frac{(x-x_n)^9}{12h^9} - \frac{(x-x_n)^{10}}{300h^{10}} \right) y_{n+1} + \left(\frac{24(x-x_n)}{5h} - \frac{1403(x-x_n)^2}{50h^2} + \right. \end{aligned}$$

$$\begin{aligned}
& \left(\frac{279(x-x_n)^3}{8h^3} - \frac{126(x-x_n)^5}{5h^5} + \frac{959(x-x_n)^6}{50h^6} - \frac{9(x-x_n)^7}{2h^7} + \frac{51(x-x_n)^8}{40h^8} - \frac{(x-x_n)^9}{8h^9} + \frac{(x-x_n)^{10}}{200h^{10}} \right) y_{n+2} + \\
& \left(-\frac{58(x-x_n)}{5h} + \frac{977(x-x_n)^2}{50h^2} - \frac{281(x-x_n)^3}{12h^3} + \frac{84(x-x_n)^5}{5h^5} - \frac{959(x-x_n)^6}{75h^6} + \frac{9(x-x_n)^7}{2h^7} - \frac{17(x-x_n)^8}{20h^8} + \right. \\
& \left. \frac{(x-x_n)^9}{12h^9} - \frac{(x-x_n)^{10}}{300h^{10}} \right) y_{n+3} + \left(\frac{21(x-x_n)}{20h} - \frac{501(x-x_n)^2}{100h^2} + \frac{283(x-x_n)^3}{48h^3} - \frac{21(x-x_n)^5}{5h^5} + \frac{959(x-x_n)^6}{300h^6} - \right. \\
& \left. \frac{9(x-x_n)^7}{2h^7} + \frac{17(x-x_n)^8}{80h^8} - \frac{(x-x_n)^9}{48h^9} + \frac{(x-x_n)^{10}}{1200h^{10}} \right) y_{n+4} + \left(-\frac{79(x-x_n)}{10080h} + \frac{28261(x-x_n)^2}{100800h^2} - \right. \\
& \left. \frac{32359(x-x_n)^3}{725760h^3} + \frac{(x-x_n)^4}{24h^4} - \frac{179(x-x_n)^5}{7200h^5} + \frac{521(x-x_n)^6}{54000h^6} - \frac{97(x-x_n)^7}{40320h^7} + \frac{149(x-x_n)^8}{403200h^8} - \frac{23(x-x_n)^9}{725760h^9} + \right. \\
& \left. \frac{(x-x_n)^{10}}{864000h^{10}} \right) f_n \\
& + \left(-\frac{1013(x-x_n)}{2800h} + \frac{263647(x-x_n)^2}{216000h^2} - \frac{442433(x-x_n)^3}{362880h^3} + \frac{153(x-x_n)^5}{200h^5} - \frac{20461(x-x_n)^6}{36000h^6} - \frac{3977(x-x_n)^7}{20160h^7} - \right. \\
& \left. \frac{133(x-x_n)^8}{3600h^8} + \frac{1307(x-x_n)^9}{362880h^9} - \frac{31(x-x_n)^{10}}{216000h^{10}} \right) f_{n+1} \\
& + \left(-\frac{18917(x-x_n)}{25200h} + \frac{5057729(x-x_n)^2}{1512000h^2} - \frac{1395881(x-x_n)^3}{362880h^3} + \frac{817(x-x_n)^5}{300h^5} - \frac{7019(x-x_n)^6}{3375h^6} - \right. \\
& \left. \frac{14813(x-x_n)^7}{20160h^7} - \frac{28073(x-x_n)^8}{201600h^8} + \frac{4967(x-x_n)^9}{362880h^9} - \frac{79(x-x_n)^{10}}{144000h^{10}} \right) f_{n+2} \\
& + \left(-\frac{41(x-x_n)}{225h} + \frac{28073(x-x_n)^2}{31500h^2} - \frac{23959(x-x_n)^3}{22680h^3} + \frac{169(x-x_n)^5}{225h^5} - \frac{1919(x-x_n)^6}{3375h^6} + \frac{143(x-x_n)^7}{720h^7} - \right. \\
& \left. \frac{3749(x-x_n)^8}{100800h^8} + \frac{41(x-x_n)^9}{11340h^9} - \frac{31(x-x_n)^{10}}{216000h^{10}} \right) f_{n+3} \\
& + \left(-\frac{47(x-x_n)}{16800h} - \frac{55817(x-x_n)^2}{302400h^2} + \frac{17041(x-x_n)^3}{725760h^3} - \frac{13(x-x_n)^5}{800h^5} + \frac{23(x-x_n)^6}{2000h^6} - \frac{29(x-x_n)^7}{8064h^7} + \right. \\
& \left. \frac{229(x-x_n)^8}{403200h^8} - \frac{31(x-x_n)^9}{725760h^9} + \frac{(x-x_n)^{10}}{864000h^{10}} \right) f_{n+4} \\
& + \left(-\frac{3(x-x_n)}{3600h} + \frac{23(x-x_n)^2}{12096h^2} - \frac{883(x-x_n)^3}{362880h^3} + \frac{(x-x_n)^5}{600h^5} - \frac{(x-x_n)^6}{864h^6} + \frac{(x-x_n)^7}{2880h^7} - \frac{(x-x_n)^8}{20160h^8} + \right. \\
& \left. \frac{(x-x_n)^9}{362880h^9} \right) f_{n+5}
\end{aligned}$$

Substitution at $x = x_{n+5}$ gives the discrete solution as

$$y_{n+5} - y_n + 5(y_{n+1} - y_{n+4}) + 10(y_{n+3} - y_{n+2}) = h^4 \left[\frac{1}{720} f_n - \frac{125}{720} f_{n+1} - \left(\frac{350}{720} f_{n+2} - \frac{350}{720} f_{n+3} \right) + \frac{125}{720} f_{n+4} - \frac{1}{720} f_{n+5} \right] \quad (8)$$

3. Analysis of the Method

The discrete scheme obtained in equation (8) can be written in the standard form of a linear multistep method as

$$\sum_{j=0}^5 \alpha_j y_{n+j} = h^4 \sum_{j=0}^5 \beta_j f_{n+j}. \quad (9)$$

To determine the order of accuracy and the associated error constant of the proposed method, equation (9) is expressed in terms of its linear difference operator. Define the difference operator $L[y, (x)h]$ by

$$L[y, (x)h] = \sum_{j=0}^5 [\alpha_j y(x_n + jh) - h^4 \beta_j f(x_n + jh)]. \quad (10)$$

Here, $y(x)$ is assumed to possess sufficiently many continuous derivatives in the interval under consideration. Since the differential equation is of fourth order,

$$f(x) = y^{(4)}(x).$$

Expanding $y(x_n + jh)$ and $f(x_n + jh)$ about x_n using Taylor's series and collecting the corresponding powers of h , the difference operator can be expressed as:

$$L[y, (x)h] = \gamma_0 y(x) + \gamma_1 h y'(x) + \gamma_2 h^2 y''(x) + \gamma_3 h^3 y'''(x) + \gamma_4 h^4 y^{(4)}(x) + \dots + \gamma_{p+2} h^{p+2} y^{(p+2)}(x) + \dots \quad (11)$$

The coefficients γ_i , $i = 0, 1, 2, \dots$, are determined from the coefficients α_j and β_j of the proposed linear multistep formula.

4. Order and error constant

Consider the proposed implicit linear multistep method for the fourth-order differential equation $y^{(4)} = f(x, y)$, given by in equation (8)

In the standard form

$$\sum_{j=0}^5 \alpha_j y_{n+j} = h^4 \sum_{j=0}^5 \beta_j f_{n+j},$$

the coefficients are from equation (8)

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-1, 1, 5, -10, 10, -51)$$

$$(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = \frac{1}{720} (1, -125, -350, 350, 125, -1).$$

To determine the order of the method, expand y_{n+j} and $f_{n+j} = y^{(4)}(x_{n+j})$ in Taylor series about x_n . The resulting local truncation error can be expressed as

$$L[y(x_n)] = \sum_{q=0}^{\infty} C_q h^q y^{(q)}(x_n)$$

where, for $q \geq 4$,

$$C_q = \frac{1}{q!} \sum_{j=0}^5 j^q \alpha_j - \frac{1}{(q-4)!} \sum_{j=0}^5 j^{q-4} \beta_j.$$

The corresponding characteristic polynomials are

$$\rho(\xi) = \sum_{j=0}^5 \alpha_j \xi^j = (\xi - 1)^5$$

and

$$\sigma(\xi) = \sum_{j=0}^5 \beta_j \xi^j = \frac{1}{720} (1 - 125\xi - 350\xi^2 + 350\xi^3 + 125\xi^4 - \xi^5).$$

The order conditions obtained from the above Taylor expansion are

$$C_0 = C_1 = \dots = C_{10} = 0,$$

whereas the first non-vanishing coefficient is

$$C_{11} = \frac{1}{11!} \sum_{j=0}^5 j^{11} \alpha_j - \frac{1}{7!} \sum_{j=0}^5 j^7 \beta_j = \frac{1}{3024} \neq 0.$$

For completeness, the relevant moment sums are

$$\sum_{j=0}^5 j^{11} \alpha_j = 29607600,$$

and

$$\sum_{j=0}^5 j^7 \beta_j = \frac{11210}{3}.$$

Consequently, the local truncation error begins with

$$L[y(x_n)] = \frac{1}{3024} h^{11} y^{(11)}(x_n) + O(h^{12}).$$

Since the underlying differential equation is of fourth order, the first non-zero term h^{11} corresponds to a method of order

$$p + 3 = 10$$

and hence $p = 7$

Therefore, the proposed implicit linear multistep method is a seventh-order method, and its corresponding error constant is

$$C_{11} = 0.00033069$$

5. Numerical Example.

The numerical solution obtained using the proposed method is compared with the corresponding exact solution for given test problem. The numerical results show that the proposed method produces small absolute errors for the test problems considered. Detailed numerical results, including the computed solutions and corresponding errors, are reported in the Table-1.

Example-1

$$y^{iv} + 5y'' + 4y = 90 \sin 4x$$

With $y(0) = 1, y'(0) = 2, y''(0) = -1, y'''(0) = -32, h = 0.1, 0 \leq x \leq 1$.

Exact solution is $y = \cos(x) + \frac{1}{2} \sin 4x$ [17,18].

Table-1 (Comparison between exact and approximate solution of Example-1)

x	Exact Solution	Approx. Solution	Absolute Error
0.1	1.189713336432350	1.189713336432351	1.1101e-15
0.2	1.338744623291002	1.338744623291016	1.3988e-14
0.3	1.421356032109218	1.421356032115081	5.8610e-12
0.4	1.420847795523637	1.420847795519521	4.1181e-12
0.5	1.332231275303213	1.332231275121098	1.8211e-10
0.6	1.163067205185253	1.163067205253167	6.7913e-11
0.7	0.932336262362440	0.932336263626655	1.2641e-09
0.8	0.667519637633374	0.667519653526932	1.5893e-08
0.9	0.400349746623237	0.400349894653455	1.4802e-07
1.0	0.161901058214175	0.161902146316597	1.0880e-06

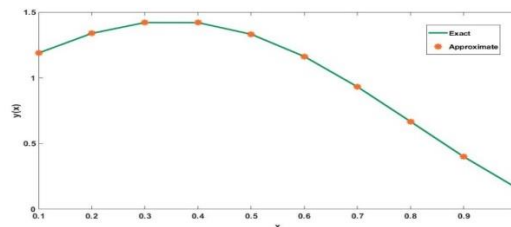


Figure 1

Figure 1: Comparison of Exact and Approximate Value of Example -1

Example -2 (Comparison between exact and approximate solution of Example-2)

$$y^{(iv)} = 16y + 64(e^{2x} + e^{-2x})$$

$$y_0''' = -16, y_0'' = 20, y_0' = 24, y_0 = 1, x \in [0,1], h = 0.1$$

The analytical solution of the IVP by [17,18] is given by

$$y = 16 \sin 2x - \frac{1}{2}(3e^{2x} + 5e^{-2x}) + 2x(e^{2x} - e^{-2x})$$

Table-2 (Comparison between exact and approximate solution of Example-2)

x	Exact Solution (E)	Approximate Solution (A)	Absolute Error ($ E - A $)
0.1	3.473966439192	3.473966439192	0
0.2	5.997358406207	5.997358406207	0
0.3	8.437114762547	8.437114762547	0
0.4	10.683678043446	10.683678043447	1.0001e-12
0.5	12.656214004453	12.656214004452	1.0001e-12
0.6	14.308142774140	14.308142774139	1.0001e-12
0.7	15.632932343466	15.632932343464	2.0002e-12
0.8	16.670187757298	16.670187757299	1.0019e-12
0.9	17.512165555130	17.512165555138	8.0007e-12
1.0	18.310954520294	18.310954520353	5.9000e-11

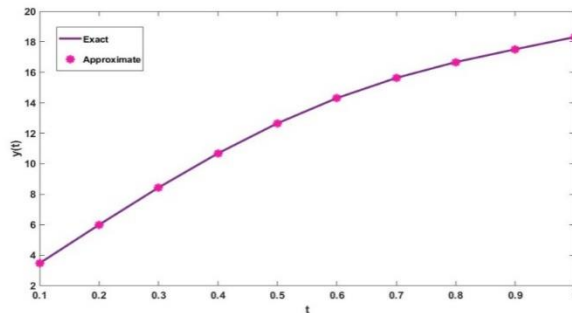


Figure 2

Figure 2: Comparison of Exact and Approximate solutions of Example -2

6.Conclusions

The ILMM technique is employed for the numerical solution of fourth-order ordinary differential equations. The absolute errors obtained using the proposed scheme are evaluated through both numerical computations and graphical representations. The corresponding results are presented in Tables 1 and 2, while Figures 1 and 2 illustrate the error distributions for different values of x . The numerical results and graphical observations demonstrate that the proposed method yields improved approximations, highlighting its effectiveness and potential for solving the considered problem. The consistency between the tabulated data and graphical representations further supports the reliability of the proposed approach. Future research may focus on extending the method to more challenging problems involving larger computational domains, nonlinear differential equations, and systems of ordinary differential equations. In addition, its applicability and computational performance could be explored in the context of partial differential equations and problems defined over variable, nonuniform, or irregular intervals. Such extensions would provide valuable insights into the versatility, robustness, and broader applicability of the proposed scheme across diverse mathematical and computational problems, thereby establishing a foundation for further theoretical developments and practical applications.

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