

Asymptotic Symbols and Fredholm Theory for Variable-Coefficient Finite-Band Nörlund-Type Operators on c_0 and c

Abstract

This paper develops an operator-theoretic framework for a class of variable-coefficient finite-band Nörlund-type averaging operators acting on the classical Banach sequence spaces c_0 and c . Instead of assuming that the matrix becomes exactly translation invariant after finitely many rows, the coefficients are allowed to vary indefinitely, subject only to convergence toward a limiting finite band. For a fixed integer $m \geq 1$, we consider

$$(\mathcal{N}x)_n = \sum_{j=0}^{\min\{m,n\}} a_{j,n} x_{n-j}, \quad a_{j,n} \longrightarrow a_j,$$

and associate with $\mathcal{N}$ the asymptotic polynomial symbol

$$a(z) = a_0 + a_1 z + \cdots + a_m z^m.$$

The first main result is a compactness **characterisation**: the convergence of the band coefficients is equivalent to compact equivalence between $\mathcal{N}$ and the shift polynomial $a(S)$. The resulting perturbation is generally compact of infinite rank, so the framework strictly contains the eventually constant finite-band case. Using Atkinson's theorem, the Calkin algebra, polynomial spectral mapping, and a direct Fredholm analysis of the unilateral right shift, it is **proved** that

$$\sigma_{\text{ess}}(\mathcal{N}) = a(\mathbb{T})$$

on both c_0 and c . If $\lambda \notin a(\mathbb{T})$ and $N_a(\lambda)$ denotes the number, counted with multiplicity, of zeros of $a(z) - \lambda$ in the open unit disk, then

$$\text{ind}(\mathcal{N} - \lambda I) = -N_a(\lambda) = -\text{wind}(a(\mathbb{T}), \lambda).$$

We further obtain a **localisation** theorem for the full spectrum,

$$a(\overline{\mathbb{D}}) \subseteq \sigma(\mathcal{N}) \subseteq a(\overline{\mathbb{D}}) \cup \{a_{0,n} : n \geq 0\},$$

and consequently an exact formula $\sigma(\mathcal{N}) = a(\overline{\mathbb{D}})$ whenever every diagonal coefficient $a_{0,n}$ lies in $a(\overline{\mathbb{D}})$. For **normalised** nonnegative averaging coefficients, the limiting symbol satisfies $a(1) = 1$, the essential spectral radius is one, and the constant sequence is an embedded eigenvector at $\lambda = 1$ on c . Two variable-coefficient examples are developed in detail, including a three-term family whose difference from its limiting Toeplitz operator is compact but of infinite rank.

Keywords: Nörlund-type operator, Fredholm index, compact perturbation, asymptotic symbol, essential spectrum, unilateral shift.

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1 Introduction

Infinite matrix transformations occupy a natural interface between summability theory and operator theory. Classical summability methods convert a scalar sequence into a new sequence through a lower-triangular matrix, while functional analysis allows the same transformation to be studied as a bounded operator on Banach sequence spaces. This viewpoint brings questions of boundedness, invertibility, point spectrum, essential spectrum, Fredholmness and index into the theory of summability matrices. Standard foundations may be found in (Hardy, 1949) and (Wilansky, 2000). Nörlund and weighted-mean matrices are particularly important examples. Recent work has continued to investigate their spectral behaviour on classical sequence spaces. Ochieng et al. (2025) studied the spectrum of a Nörlund matrix on c_0 , whereas Yeboah and Iddrisu (2026) considered a particular three-term Nörlund averaging matrix on c and c_0 . Yadav and Mahto (2026) further examined the spectra of lower-triangular double-band infinite matrices on ℓ^1 , reinforcing the relevance of structured infinite matrices in spectral studies on sequence spaces. These studies provide a useful starting point for a broader question: what operator-theoretic structure persists when the finite-band weights are not constant, but only approach constant limiting values? This question is also connected with Toeplitz and band-dominated operator theory. Toeplitz methods encode translation-invariant tails through a scalar or matrix-valued symbol; see (Böttcher & Silberman, 2006). More generally, limit-operator and band-dominated frameworks identify behaviour at infinity as a decisive ingredient in Fredholm theory (Rabinovich et al., 2004; Seidel, 2014; Hagger et al., 2016). Related recent operator-theoretic studies continue to examine structural and spectral properties of bounded operator classes and essential properties of Toeplitz-type operators in Hilbert- and Banach-space settings (Nyongesa & Wanjala, 2025; Fulsche, 2024; Karlovych & Shargorodsky, 2025). This paper explores an explicit version of this concept for a concrete class of one-sided, finite-memory averaging matrices on the spaces c_0 and c . The essential step is to replace exact eventual stationarity by asymptotic stationarity. For a fixed bandwidth m , let

$$(\mathcal{N}x)_n = \sum_{j=0}^{\min\{m,n\}} a_{j,n}x_{n-j}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

where, for each fixed j ,

$$a_{j,n} \longrightarrow a_j. \quad (1.2)$$

The associated limiting polynomial

$$a(z) = \sum_{j=0}^m a_j z^j \quad (1.3)$$

will be called the asymptotic symbol. If S denotes the unilateral right shift, then $a(S)$ is the translation-invariant operator determined by the limiting band. **The central approach is based on the exact compact decomposition**

$$\mathcal{N} = a(S) + K, \quad K \in \mathcal{K}(X), \quad X \in \{c_0, c\}. \quad (1.4)$$

Unlike the eventually constant case, K need not have finite rank. Indeed, one of the main examples has coefficients that differ from their limits in every row, so K is compact but infinite rank. This distinction is the principal structural extension studied in this paper. The contribution can be summarised as follows.

- (i) A variable-coefficient finite-band Nörlund-type class is introduced on c_0 and c , together with boundedness and limit-preservation results.
- (ii) A sharp compactness criterion is proved: for finite-band operators, convergence of every band coefficient to its limiting value is equivalent to compactness of the difference from the limiting shift polynomial.
- (iii) The finite-rank and infinite-rank cases are separated. The perturbation is finite rank exactly when the coefficient bands are eventually constant.
- (iv) The asymptotic symbol is shown to determine the Calkin-algebra class of the operator, and hence its Fredholm essential spectrum.

- (v) A self-contained Fredholm analysis of $S - \alpha I$ on c_0 and c is given and used to obtain the exact index formula

$$\text{ind}(\mathcal{N} - \lambda I) = -\#\{z \in \mathbb{D} : a(z) = \lambda\}, \quad \lambda \notin a(\mathbb{T}). \quad (1.5)$$

- (vi) The ordinary spectrum is **localised** by combining the index calculation with the lower-triangular structure. In particular,

$$a(\overline{\mathbb{D}}) \subseteq \sigma(\mathcal{N}) \subseteq a(\overline{\mathbb{D}}) \cup \{a_{0,n} : n \geq 0\}. \quad (1.6)$$

- (vii) For **normalised** nonnegative averaging matrices, limit preservation, the essential spectral radius, and the embedded eigenvalue at 1 on c are obtained immediately from the symbol.
- (viii) Detailed three-term and four-term variable-coefficient examples illustrate the theory and show how exact eventual stationarity can be replaced by genuinely variable tails.

A useful conceptual summary is displayed in [figure 1](#).

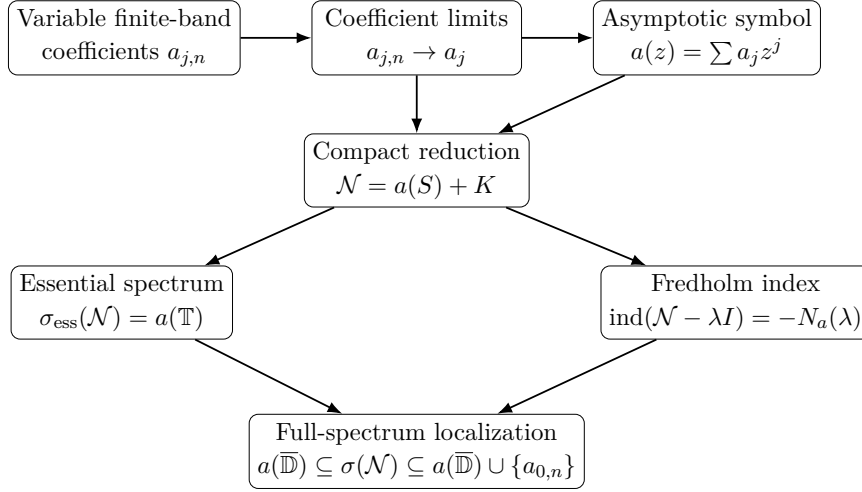


Figure 1: Logical structure of the asymptotic-symbol approach.

Throughout this paper, essential spectrum means the Fredholm essential spectrum

$$\sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}. \quad (1.7)$$

This convention is especially convenient because Atkinson's theorem identifies Fredholmness with invertibility in the Calkin algebra ([Atkinson, 1951](#); [Conway, 1990](#); [Kato, 1995](#)).

Research gap. Within the literature reviewed in this manuscript, the specific setting of finite-band Nörlund-type operators on c_0 and c with row-dependent coefficients that vary indefinitely while converging bandwise to fixed limits is not developed. In particular, the distinction between eventual stationarity and asymptotic stationarity motivates a direct compact-perturbation and Fredholm analysis for this class.

Objective. The objective of this paper is to establish an operator-theoretic framework for variable-coefficient finite-band Nörlund-type operators on c_0 and c by relating bandwise coefficient convergence to compact perturbations of a limiting shift polynomial and then using this structure to determine the Fredholm essential spectrum, the Fredholm index outside it, and a **localisation** of the ordinary spectrum.

2 Sequence spaces, the unilateral shift, and the operator class

We index sequences by

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}. \quad (2.1)$$

Let

$$\ell^\infty = \left\{ x = (x_n)_{n \geq 0} : \sup_{n \geq 0} |x_n| < \infty \right\}, \quad (2.2)$$

and define

$$c_0 = \{x \in \ell^\infty : x_n \rightarrow 0\}, \quad c = \{x \in \ell^\infty : x_n \text{ converges}\}. \quad (2.3)$$

Both are complex Banach spaces under the supremum norm

$$\|x\|_\infty = \sup_{n \geq 0} |x_n|. \quad (2.4)$$

The n th canonical unit vector is denoted by e_n and

$$\mathbf{1} = (1, 1, 1, \dots) \in c. \quad (2.5)$$

Definition 2.1 (Unilateral right shift). *For $X = c_0$ or $X = c$, define*

$$S(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots). \quad (2.6)$$

Then $S \in \mathcal{B}(X)$ and $\|S\| = 1$.

Definition 2.2 (Variable-coefficient finite-band operator). *Fix $m \in \mathbb{N}$. Let*

$$\{a_{j,n} : n \geq j\}, \quad j = 0, 1, \dots, m, \quad (2.7)$$

be complex coefficient sequences. The associated lower-triangular finite-band operator is

$$(\mathcal{N}x)_n = \sum_{j=0}^{\min\{m,n\}} a_{j,n} x_{n-j}, \quad n \geq 0. \quad (2.8)$$

Equation (2.8) is called a finite-band Nörlund-type operator. If in addition

$$a_{j,n} \geq 0 \quad \text{and} \quad \sum_{j=0}^{\min\{m,n\}} a_{j,n} = 1 \quad (n \geq 0), \quad (2.9)$$

*then $\mathcal{N}$ is called a **normalised** finite-band averaging operator.*

The matrix of [equation \(2.8\)](#) has the form

$$\mathcal{N} = \begin{pmatrix} a_{0,0} & 0 & 0 & 0 & 0 & \cdots \\ a_{1,1} & a_{0,1} & 0 & 0 & 0 & \cdots \\ a_{2,2} & a_{1,2} & a_{0,2} & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & & \\ a_{m,m} & \cdots & a_{2,m} & a_{1,m} & a_{0,m} & \cdots \\ 0 & a_{m,m+1} & \cdots & a_{2,m+1} & a_{1,m+1} & a_{0,m+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}, \quad (2.10)$$

where the nonzero entries remain confined to the main diagonal and the first m subdiagonals.

Assumption 2.3 (Asymptotic band convergence). *For every $j = 0, 1, \dots, m$, assume that there exists $a_j \in \mathbb{C}$ such that*

$$a_{j,n} \longrightarrow a_j \quad (n \rightarrow \infty). \quad (2.11)$$

Definition 2.4 (Asymptotic symbol). **Under equation (2.11), the polynomial**

$$a(z) = a_0 + a_1 z + \dots + a_m z^m \quad (2.12)$$

is called the asymptotic symbol of $\mathcal{N}$.

Remark 2.5. *The terminology “Nörlund-type” is deliberate. Classical Nörlund means are generated by a fixed weight sequence and its partial sums (Hardy, 1949; Wilansky, 2000). Equation (2.8) retains the finite-memory lower-triangular averaging structure but allows row-dependent weights. The general complex-coefficient formulation is convenient for spectral analysis; the normalised nonnegative subclass recovers the averaging interpretation.*

3 Boundedness, invariance, and preservation of limits

This section begins with elementary mapping properties.

Proposition 3.1 (Boundedness). *Suppose*

$$M_* = \sup_{n \geq 0} \sum_{j=0}^{\min\{m,n\}} |a_{j,n}| < \infty. \quad (3.1)$$

Then equation (2.8) defines a bounded operator on ℓ^∞ , c_0 , and c , and

$$\|\mathcal{N}\| \leq M_*. \quad (3.2)$$

In particular, the convergence hypothesis of equation (2.11) automatically implies $M_ < \infty$.*

Proof. For $x \in \ell^\infty$,

$$|(\mathcal{N}x)_n| \leq \sum_{j=0}^{\min\{m,n\}} |a_{j,n}| |x_{n-j}| \leq M_* \|x\|_\infty. \quad (3.3)$$

Taking the supremum over n proves boundedness on ℓ^∞ . If $x \in c_0$, then for each $j \leq m$, $x_{n-j} \rightarrow 0$. Since only finitely many coefficients occur in each row and they are uniformly bounded, $(\mathcal{N}x)_n \rightarrow 0$. Hence $\mathcal{N}(c_0) \subseteq c_0$. If $x \in c$ and $x_n \rightarrow L$, then

$$(\mathcal{N}x)_n - L \sum_{j=0}^m a_j = \sum_{j=0}^m a_{j,n} (x_{n-j} - L) + L \sum_{j=0}^m (a_{j,n} - a_j) \quad (3.4)$$

for all sufficiently large n . Both terms tend to zero, so

$$(\mathcal{N}x)_n \longrightarrow L \sum_{j=0}^m a_j. \quad (3.5)$$

Thus $\mathcal{N}(c) \subseteq c$. Finally, because m is finite and each sequence $a_{j,n}$ converges, every coefficient sequence is bounded, which gives $M_* < \infty$. $\square$

Corollary 3.2 (Normalised averaging case). *If $\mathcal{N}$ is normalised and nonnegative, then*

$$\|\mathcal{N}\|_{\mathcal{B}(c_0)} = \|\mathcal{N}\|_{\mathcal{B}(c)} = 1, \quad (3.6)$$

and $\mathcal{N}$ preserves limits:

$$x_n \rightarrow L \implies (\mathcal{N}x)_n \rightarrow L. \quad (3.7)$$

Moreover,

$$\mathcal{N}\mathbf{1} = \mathbf{1} \quad \text{on } c. \quad (3.8)$$

Proof. Every row has absolute row sum one, so $\|\mathcal{N}\| \leq 1$. For c , equality follows directly from $\mathcal{N}\mathbf{1} = \mathbf{1}$. For c_0 , choose a sufficiently long finitely supported sequence whose entries are all 1 across one complete row-window; the corresponding output coordinate is 1, hence $\|\mathcal{N}\| \geq 1$. Limit preservation follows from [equation \(3.5\)](#) because the limiting coefficients satisfy $\sum_{j=0}^m a_j = 1$. $\square$

Remark 3.3. *The preceding corollary gives the summability interpretation of the operator class: normalised finite-band matrices preserve ordinary convergence. The spectral results below, however, do not require positivity.*

4 Compactness of asymptotic band perturbations

The main structural question is whether variable coefficients that converge along each band can be separated from their limiting operator by a compact term.

Lemma 4.1 (Vanishing diagonal multipliers). *Let $d = (d_n)_{n \geq 0}$ satisfy $d_n \rightarrow 0$, and define*

$$(M_d x)_n = d_n x_n. \quad (4.1)$$

Then M_d is compact on both c_0 and c .

Proof. For $N \geq 0$, define the finite-rank truncation

$$(M_d^{(N)} x)_n = \begin{cases} d_n x_n, & 0 \leq n \leq N, \\ 0, & n > N. \end{cases} \quad (4.2)$$

Then

$$\|M_d - M_d^{(N)}\| = \sup_{n > N} |d_n| \longrightarrow 0. \quad (4.3)$$

Hence M_d is the operator-norm limit of finite-rank operators and is therefore compact. $\square$

For each $j = 0, \dots, m$, define

$$d_j(n) = \begin{cases} a_{j,n} - a_j, & n \geq j, \\ 0, & n < j. \end{cases} \quad (4.4)$$

Then $d_j(n) \rightarrow 0$.

Theorem 4.2 (Compact asymptotic reduction). *Let $X \in \{c_0, c\}$ and assume [equation \(2.11\)](#). Then*

$$\mathcal{N} = a(S) + K, \quad K = \sum_{j=0}^m M_{d_j} S^j \in \mathcal{K}(X). \quad (4.5)$$

Moreover, if P_N denotes truncation to the first $N + 1$ coordinates, then

$$\|K - P_N K\| \leq \sum_{j=0}^m \sup_{n>N} |d_j(n)|. \quad (4.6)$$

Proof. For $n \geq 0$,

$$(a(S)x)_n = \sum_{j=0}^{\min\{m,n\}} a_j x_{n-j}. \quad (4.7)$$

Consequently,

$$((\mathcal{N} - a(S))x)_n = \sum_{j=0}^{\min\{m,n\}} (a_{j,n} - a_j) x_{n-j} = \sum_{j=0}^m (M_{d_j} S^j x)_n. \quad (4.8)$$

Thus the algebraic identity in [equation \(4.5\)](#) holds. Each M_{d_j} is compact by the previous lemma. Since S^j is bounded, $M_{d_j} S^j$ is compact. A finite sum of compact operators is compact, proving $K \in \mathcal{K}(X)$. Finally,

$$\|(I - P_N)Kx\|_\infty \leq \sum_{j=0}^m \sup_{n>N} |d_j(n)| \|x\|_\infty, \quad (4.9)$$

which yields [equation \(4.6\)](#). $\square$

The next theorem shows that coefficient convergence is not merely sufficient; within a fixed finite bandwidth it is exactly the condition needed for compact equivalence with a constant band.

Theorem 4.3 (Compactness characterisation). *Fix constants $a_0, \dots, a_m$ and let $\mathcal{N}$ be a bounded finite-band operator of the form [equation \(2.8\)](#). Then, on $X = c_0$ or $X = c$,*

$$\mathcal{N} - a(S) \in \mathcal{K}(X) \quad (4.10)$$

if and only if

$$a_{j,n} \longrightarrow a_j \quad \text{for every } j = 0, \dots, m. \quad (4.11)$$

Proof. The “if” direction is [equation \(4.5\)](#). For the converse, put $K = \mathcal{N} - a(S)$ and suppose K is compact. The canonical vectors e_n converge weakly to zero in c_0 . They also converge weakly to zero in c , because the restriction of every functional in c^* to c_0 belongs to $(c_0)^* = \ell^1$. Compactness therefore implies

$$\|K e_n\|_\infty \longrightarrow 0. \quad (4.12)$$

For a fixed $j \in \{0, \dots, m\}$, the $(n + j)$ th coordinate of $K e_n$ is exactly

$$(K e_n)_{n+j} = a_{j,n+j} - a_j. \quad (4.13)$$

Hence

$$|a_{j,n+j} - a_j| \leq \|K e_n\|_\infty \longrightarrow 0. \quad (4.14)$$

Replacing $n + j$ by k proves $a_{j,k} \rightarrow a_j$. $\square$

Corollary 4.4 (Compact equivalence of two variable-band operators). *Let $\mathcal{N}_A$ and $\mathcal{N}_B$ have*

the same bandwidth m , with coefficients $a_{j,n}$ and $b_{j,n}$. Then

$$\mathcal{N}_A - \mathcal{N}_B \in \mathcal{K}(X) \quad (4.15)$$

if and only if

$$a_{j,n} - b_{j,n} \longrightarrow 0 \quad (j = 0, \dots, m). \quad (4.16)$$

In particular, two asymptotically finite-band operators with the same limiting coefficients have the same Calkin-algebra class.

Proposition 4.5 (Finite-rank versus infinite-rank perturbation). *Under [equation \(2.11\)](#), the compact operator $K = \mathcal{N} - a(S)$ is finite rank if the coefficient bands are eventually constant, that is, if there exists N such that*

$$a_{j,n} = a_j \quad (n \geq N, j = 0, \dots, m). \quad (4.17)$$

Conversely, for a finite-band matrix, if K has finite rank then only finitely many rows can contain nonzero deviations from the limiting band. Thus K is finite rank exactly when the tail is eventually constant.

Proof. If all deviations vanish beyond row N , then the range of K is contained in the finite-dimensional span of $e_0, \dots, e_{N+m}$. Conversely, suppose infinitely many rows contain a nonzero deviation. From these rows choose recursively a sequence of nonzero columns whose supports are separated by more than m . Because a column of a bandwidth- m lower-triangular matrix is supported in at most $m + 1$ consecutive coordinates, the selected nonzero column vectors have pairwise disjoint supports and are therefore linearly independent. The range of K is then infinite dimensional, contradicting finite rank. $\square$

Remark 4.6. *The previous proposition cleanly separates two domains. Exact stationarity after finitely many rows gives a finite-rank perturbation, whereas asymptotic stationarity without eventual equality typically gives a compact perturbation of infinite rank. The latter is the genuinely variable-coefficient setting emphasised in this paper.*

5 The asymptotic symbol in the Calkin algebra

Let $\mathcal{K}(X)$ denote the compact operators on X and

$$\mathcal{C}(X) = \mathcal{B}(X)/\mathcal{K}(X) \quad (5.1)$$

the Calkin algebra, with quotient map $\pi_X : \mathcal{B}(X) \rightarrow \mathcal{C}(X)$.

Definition 5.1 (Asymptotic symbol map). *For an asymptotically finite-band operator $\mathcal{N}$, define*

$$\Sigma(\mathcal{N}) = a, \quad (5.2)$$

where a is the polynomial in [equation \(2.12\)](#).

Proposition 5.2 (Uniqueness of the asymptotic symbol). *If $a(S) - b(S)$ is compact on c_0 or c , where a and b are polynomials, then $a = b$. Consequently, the asymptotic symbol is uniquely determined by the Calkin-algebra class within the polynomial shift algebra.*

Proof. Let

$$r(z) = a(z) - b(z) = \sum_{j=0}^d r_j z^j. \quad (5.3)$$

For each n ,

$$r(S)e_n = \sum_{j=0}^d r_j e_{n+j}. \quad (5.4)$$

Hence

$$\|r(S)e_n\|_\infty = \max_{0 \leq j \leq d} |r_j|. \quad (5.5)$$

If $r(S)$ were compact, the weak convergence $e_n \rightarrow 0$ would force $\|r(S)e_n\|_\infty \rightarrow 0$. Therefore every $r_j = 0$, so $a = b$. $\square$

Proposition 5.3 (Algebraic **behaviour** of the symbol). *Let $\mathcal{N}_1, \mathcal{N}_2$ be asymptotically finite-band operators with symbols a and b . Then, modulo compact operators,*

$$\mathcal{N}_1 + \mathcal{N}_2 \equiv (a + b)(S), \quad \mathcal{N}_1 \mathcal{N}_2 \equiv (ab)(S). \quad (5.6)$$

Equivalently,

$$\pi_X(\mathcal{N}_1 + \mathcal{N}_2) = (a + b)(\pi_X(S)), \quad \pi_X(\mathcal{N}_1 \mathcal{N}_2) = (ab)(\pi_X(S)). \quad (5.7)$$

Proof. By [equation \(4.5\)](#),

$$\mathcal{N}_1 = a(S) + K_1, \quad \mathcal{N}_2 = b(S) + K_2, \quad (5.8)$$

with K_1, K_2 compact. Addition is immediate. For multiplication,

$$\mathcal{N}_1 \mathcal{N}_2 = a(S)b(S) + a(S)K_2 + K_1b(S) + K_1K_2. \quad (5.9)$$

The last three terms are compact, while $a(S)b(S) = (ab)(S)$. $\square$

Remark 5.4. *This proposition formalises the role of the asymptotic symbol. Addition and multiplication of operators become ordinary polynomial addition and multiplication after passing to the Calkin algebra. This is the one-sided finite-band analogue of the symbol philosophy that underlies Toeplitz and band-dominated operator theory (Böttcher & Silbermann, 2006; Rabinovich et al., 2004; Seidel, 2014).*

6 Fredholm preliminaries

Definition 6.1 (Fredholm operator). *A bounded operator $T \in \mathcal{B}(X)$ is Fredholm if*

$$\dim \operatorname{Ker} T < \infty, \quad \operatorname{Ran} T \text{ is closed}, \quad \operatorname{codim} \operatorname{Ran} T < \infty. \quad (6.1)$$

Its Fredholm index is

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Ran} T. \quad (6.2)$$

Definition 6.2 (Fredholm essential spectrum). *For $T \in \mathcal{B}(X)$,*

$$\sigma_{\operatorname{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}. \quad (6.3)$$

Theorem 6.3 (Atkinson's theorem; Atkinson, 1951). *For a complex Banach space X and $T \in \mathcal{B}(X)$,*

$$T \text{ is Fredholm} \iff \pi_X(T) \text{ is invertible in } \mathcal{C}(X). \quad (6.4)$$

Consequently,

$$\sigma_{\text{ess}}(T) = \sigma_{\mathcal{C}(X)}(\pi_X(T)). \quad (6.5)$$

We also use the standard compact-stability and index-additivity properties (Conway, 1990; Kato, 1995; Gohberg & Kreĭn, 1969):

$$\begin{aligned} T \text{ Fredholm, } K \in \mathcal{K}(X) &\implies T + K \text{ Fredholm and } \text{ind}(T + K) = \text{ind } T, \\ A, B \text{ Fredholm} &\implies AB \text{ Fredholm and } \text{ind}(AB) = \text{ind } A + \text{ind } B. \end{aligned}$$

7 Fredholm theory of the unilateral shift on c_0 and c

Because every asymptotic symbol is a polynomial in S , the elementary factors $S - \alpha I$ are the building blocks of the index calculation.

7.1 The shift on c_0

Let S_0 denote the unilateral right shift on c_0 .

Theorem 7.1 (Shift Fredholm theorem on c_0). *For $\alpha \in \mathbb{C}$,*

$$S_0 - \alpha I \quad (7.1)$$

is Fredholm precisely when $|\alpha| \neq 1$, and

$$\text{ind}(S_0 - \alpha I) = \begin{cases} -1, & |\alpha| < 1, \\ 0, & |\alpha| > 1. \end{cases} \quad (7.2)$$

Therefore

$$\sigma_{\text{ess}}(S_0) = \mathbb{T}. \quad (7.3)$$

Proof. If $|\alpha| > 1$, then $\|\alpha^{-1}S_0\| < 1$, so

$$(S_0 - \alpha I)^{-1} = -\frac{1}{\alpha} \sum_{k=0}^{\infty} \alpha^{-k} S_0^k. \quad (7.4)$$

Thus the operator is invertible and has index zero. Now assume $|\alpha| < 1$. Define

$$\phi_\alpha(y) = \sum_{n=0}^{\infty} \alpha^n y_n, \quad y \in c_0. \quad (7.5)$$

This is bounded because

$$|\phi_\alpha(y)| \leq \frac{\|y\|_\infty}{1 - |\alpha|}. \quad (7.6)$$

If $y = (S_0 - \alpha I)x$, then

$$y_0 = -\alpha x_0, \quad y_n = x_{n-1} - \alpha x_n \quad (n \geq 1), \quad (7.7)$$

and by successive cancellation, $\phi_\alpha(y) = 0$. Conversely, if $\phi_\alpha(y) = 0$, define

$$x_n = \sum_{j=1}^{\infty} \alpha^{j-1} y_{n+j}. \quad (7.8)$$

Then $x \in c_0$ and $(S_0 - \alpha I)x = y$. Hence

$$\text{Ran}(S_0 - \alpha I) = \text{Ker } \phi_\alpha, \quad (7.9)$$

which is closed of codimension one. The kernel is trivial, so the index is -1 . Finally suppose $|\alpha| = 1$. For $N \geq 1$, define

$$b_n^{(N)} = \begin{cases} n/N, & 0 \leq n \leq N, \\ (2N - n)/N, & N \leq n \leq 2N, \\ 0, & n \geq 2N, \end{cases} \quad x_n^{(N)} = \alpha^{-n} b_n^{(N)}. \quad (7.10)$$

Then $\|x^{(N)}\|_\infty = 1$, while

$$\|(S_0 - \alpha I)x^{(N)}\|_\infty \leq \frac{1}{N} \rightarrow 0. \quad (7.11)$$

The operator is injective, so if its range were closed it would be bounded below, a contradiction. Hence it is not Fredholm on the unit circle. $\square$

7.2 The shift on c

Let S_c denote the shift on c . Every $x \in c$ decomposes uniquely as

$$x = y + L\mathbf{1}, \quad y \in c_0, \quad L = \lim_{n \rightarrow \infty} x_n. \quad (7.12)$$

Thus

$$c \cong c_0 \oplus \mathbb{C}. \quad (7.13)$$

Theorem 7.2 (Shift Fredholm theorem on c). *For $\alpha \in \mathbb{C}$,*

$$\sigma_{\text{ess}}(S_c) = \mathbb{T}, \quad (7.14)$$

and, whenever $|\alpha| \neq 1$,

$$\text{ind}(S_c - \alpha I) = \begin{cases} -1, & |\alpha| < 1, \\ 0, & |\alpha| > 1. \end{cases} \quad (7.15)$$

Proof. Define

$$J : c_0 \oplus \mathbb{C} \rightarrow c, \quad J(y, L) = y + L\mathbf{1}. \quad (7.16)$$

Since

$$S_c \mathbf{1} = \mathbf{1} - e_0, \quad (7.17)$$

one obtains

$$J^{-1} S_c J = \begin{pmatrix} S_0 & -e_0 \\ 0 & 1 \end{pmatrix}. \quad (7.18)$$

The off-diagonal term is rank one. Therefore $S_c - \alpha I$ is a finite-rank perturbation of

$$(S_0 - \alpha I) \oplus (1 - \alpha). \quad (7.19)$$

Every operator on the one-dimensional space $\mathbb{C}$ is Fredholm with index zero, including the zero operator. Compact stability and direct-sum additivity therefore reduce both Fredholmness and index to the c_0 component. $\square$

8 Essential spectrum from the asymptotic symbol

The compact reduction is now combined with the Calkin-algebra spectral mapping theorem.

Theorem 8.1 (Essential spectrum theorem). *Let $\mathcal{N}$ act on $X = c_0$ or $X = c$, and suppose its band coefficients satisfy [equation \(2.11\)](#). If*

$$a(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad (8.1)$$

then

$$\sigma_{\text{ess}}(\mathcal{N}) = a(\mathbb{T}) = \{a(e^{it}) : 0 \leq t \leq 2\pi\}. \quad (8.2)$$

Proof. By [equation \(4.5\)](#),

$$\pi_X(\mathcal{N}) = \pi_X(a(S)) = a(\pi_X(S)). \quad (8.3)$$

By Atkinson's theorem and polynomial spectral mapping,

$$\sigma_{\text{ess}}(\mathcal{N}) = \sigma_{\mathcal{C}(X)}(a(\pi_X(S))) = a\left(\sigma_{\mathcal{C}(X)}(\pi_X(S))\right). \quad (8.4)$$

The shift theorems give

$$\sigma_{\mathcal{C}(X)}(\pi_X(S)) = \sigma_{\text{ess}}(S) = \mathbb{T} \quad (8.5)$$

on both c_0 and c . Therefore

$$\sigma_{\text{ess}}(\mathcal{N}) = a(\mathbb{T}). \quad (8.6)$$

$\square$

Corollary 8.2 (Essential spectral radius). *Under the hypotheses of the preceding theorem,*

$$r_{\text{ess}}(\mathcal{N}) = \max_{|z|=1} |a(z)|. \quad (8.7)$$

Proof. The essential spectral radius is the maximum modulus of the compact set $\sigma_{\text{ess}}(\mathcal{N}) = a(\mathbb{T})$. $\square$

Corollary 8.3 (Normalised averaging symbols). *Suppose the limiting coefficients are real and nonnegative and*

$$\sum_{j=0}^m a_j = 1. \quad (8.8)$$

Then

$$1 = a(1) \in \sigma_{\text{ess}}(\mathcal{N}) \quad (8.9)$$

and

$$r_{\text{ess}}(\mathcal{N}) = 1. \quad (8.10)$$

Proof. For $|z| = 1$,

$$|a(z)| \leq \sum_{j=0}^m a_j |z|^j = 1. \quad (8.11)$$

Equality occurs at $z = 1$. □

Corollary 8.4 (Essential-spectrum stability). *Two asymptotically finite-band operators with the same limiting coefficients have the same Fredholm essential spectrum on c_0 and on c .*

Remark 8.5. *The essential spectrum depends only on the tail limits and is insensitive to finitely many initial rows and, more generally, to any coefficient perturbation that vanishes along each band. This is exactly the information encoded by the Calkin algebra.*

9 Fredholm index and winding number

Let $d = \deg a$ and assume $d \geq 1$. For $\lambda \in \mathbb{C}$, define

$$N_a(\lambda) = \#\{z \in \mathbb{D} : a(z) = \lambda\}, \quad (9.1)$$

where zeros are counted with algebraic multiplicity.

Theorem 9.1 (Fredholm index theorem). *Let $\lambda \notin a(\mathbb{T})$. Then $\mathcal{N} - \lambda I$ is Fredholm and*

$$\text{ind}(\mathcal{N} - \lambda I) = -N_a(\lambda). \quad (9.2)$$

Equivalently,

$$\text{ind}(\mathcal{N} - \lambda I) = -\text{wind}(a(\mathbb{T}), \lambda). \quad (9.3)$$

Proof. Factor

$$a(z) - \lambda = a_d \prod_{k=1}^d (z - r_k), \quad (9.4)$$

with roots repeated according to multiplicity. Since $\lambda \notin a(\mathbb{T})$, no r_k lies on $\mathbb{T}$. Hence

$$a(S) - \lambda I = a_d \prod_{k=1}^d (S - r_k I) \quad (9.5)$$

is Fredholm. By index additivity and the shift theorems,

$$\text{ind}(a(S) - \lambda I) = \sum_{k=1}^d \text{ind}(S - r_k I) = -\#\{k : |r_k| < 1\} = -N_a(\lambda). \quad (9.6)$$

Because $\mathcal{N} - \lambda I$ is a compact perturbation of $a(S) - \lambda I$, the same index holds for $\mathcal{N} - \lambda I$. For the winding-number expression, the argument principle gives

$$N_a(\lambda) = \frac{1}{2\pi i} \int_{|z|=1} \frac{a'(z)}{a(z) - \lambda} dz = \text{wind}(a(\mathbb{T}), \lambda), \quad (9.7)$$

because $a(z) - \lambda$ has no zeros on the unit circle. □

Corollary 9.2 (Possible index values). *If $d = \deg a$, then outside the essential spectrum*

$$\text{ind}(\mathcal{N} - \lambda I) \in \{0, -1, \dots, -d\}. \quad (9.8)$$

Corollary 9.3 (Local constancy). *The function*

$$\lambda \mapsto \text{ind}(\mathcal{N} - \lambda I) \quad (9.9)$$

is constant on every connected component of $\mathbb{C} \setminus a(\mathbb{T})$.

Proof. The integer $N_a(\lambda)$ can change only when a zero of $a(z) - \lambda$ crosses the unit circle, which occurs exactly when $\lambda \in a(\mathbb{T})$. $\square$

Remark 9.4. *The root-counting formula gives a practical computational procedure. For any prescribed $\lambda \notin a(\mathbb{T})$, solve $a(z) = \lambda$, count roots in $\mathbb{D}$, and change the sign of that count. For higher-degree symbols, the Jury criteria or Schur-Cohn may also be used to count zeros in the unit disk without explicitly solving the polynomial.*

10 Point spectrum and localisation of the full spectrum

The essential spectrum and index are determined entirely by the limiting symbol, but the full spectrum can also be influenced by the non-asymptotic part of the matrix. The lower-triangular form gives a useful restriction.

Proposition 10.1 (Diagonal candidate principle). *Let $\mathcal{N}$ be any lower-triangular operator of the form [equation \(2.8\)](#) on c_0 or c . If λ is an eigenvalue, then*

$$\lambda \in \{a_{0,n} : n \geq 0\}. \quad (10.1)$$

Proof. Let $\mathcal{N}x = \lambda x$ with $x \neq 0$, and let

$$r = \min\{n : x_n \neq 0\}. \quad (10.2)$$

The r th coordinate equation contains no contribution from $x_0, \dots, x_{r-1}$ because those entries vanish. Hence

$$a_{0,r}x_r = \lambda x_r. \quad (10.3)$$

Since $x_r \neq 0$, $\lambda = a_{0,r}$. $\square$

Remark 10.2. *The converse does not necessarily hold. A diagonal entry is only a potential eigenvalue as the recurrence generated by subsequent rows may fail to produce a bounded, null, or convergent eigenvector.*

Theorem 10.3 (Spectral localisation). *Let $\mathcal{N}$ satisfy [equation \(2.11\)](#) with symbol a . Then, on $X = c_0$ or $X = c$,*

$$a(\overline{\mathbb{D}}) \subseteq \sigma(\mathcal{N}) \subseteq a(\overline{\mathbb{D}}) \cup \{a_{0,n} : n \geq 0\}. \quad (10.4)$$

Proof. First let $\lambda \in a(\overline{\mathbb{D}})$. If $\lambda \in a(\mathbb{T})$, then $\lambda \in \sigma_{\text{ess}}(\mathcal{N}) \subseteq \sigma(\mathcal{N})$. If $\lambda \in a(\mathbb{D}) \setminus a(\mathbb{T})$, then $N_a(\lambda) \geq 1$, so

$$\text{ind}(\mathcal{N} - \lambda I) = -N_a(\lambda) < 0. \quad (10.5)$$

An invertible operator has index zero, hence $\mathcal{N} - \lambda I$ is not invertible and $\lambda \in \sigma(\mathcal{N})$. This proves the left inclusion. Now suppose $\lambda \notin a(\overline{\mathbb{D}})$. Then $a(S) - \lambda I$ is invertible because $\sigma(a(S)) = a(\overline{\mathbb{D}})$ by polynomial spectral mapping and $\sigma(S) = \overline{\mathbb{D}}$ on both spaces. Since

$$\mathcal{N} - \lambda I = (a(S) - \lambda I) + K \quad (10.6)$$

with K compact, $\mathcal{N} - \lambda I$ is Fredholm of index zero. If it is not invertible, then its kernel cannot be zero: otherwise index zero would force codimension of the closed range to be zero, making

the operator bijective. Therefore every spectral point outside $a(\overline{\mathbb{D}})$ is an eigenvalue. By the diagonal candidate principle, such a point must belong to $\{a_{0,n} : n \geq 0\}$. $\square$

Corollary 10.4 (Exact full spectrum under diagonal containment). *If*

$$a_{0,n} \in a(\overline{\mathbb{D}}) \quad \text{for every } n \geq 0, \quad (10.7)$$

then

$$\sigma(\mathcal{N}) = a(\overline{\mathbb{D}}). \quad (10.8)$$

Proof. Under [equation \(10.7\)](#), the right side of [equation \(10.4\)](#) reduces to $a(\overline{\mathbb{D}})$. $\square$

Corollary 10.5 (Spectral radius under exact localisation). *Under (10.7),*

$$r(\mathcal{N}) = \max_{|z| \leq 1} |a(z)| = \max_{|z|=1} |a(z)|. \quad (10.9)$$

If the limiting coefficients are nonnegative and sum to one, then $r(\mathcal{N}) = 1$.

Proof. The first equality follows from [equation \(10.8\)](#) and the second follows from the maximum modulus principle. In the **normalised** nonnegative case, $|a(z)| \leq 1$ on $\overline{\mathbb{D}}$ and $a(1) = 1$. $\square$

Proposition 10.6 (The eigenvalue 1 on c for **normalised** averaging operators). **If $\mathbf{N}$ is normalised row by row, then on c**

$$1 \in \sigma_p(\mathcal{N}), \quad \mathbf{1} \in \text{Ker}(\mathcal{N} - I). \quad (10.10)$$

If, in addition, the limiting coefficients sum to one, then

$$1 \in \sigma_p(\mathcal{N}) \cap \sigma_{\text{ess}}(\mathcal{N}), \quad (10.11)$$

so 1 is an embedded eigenvalue.

Proof. Row normalization gives $\mathcal{N}\mathbf{1} = \mathbf{1}$. The limiting symbol satisfies $a(1) = 1$, so $1 \in a(\mathbb{T}) = \sigma_{\text{ess}}(\mathcal{N})$. $\square$

11 A non-constant three-term family

An example is now constructed in which the coefficients vary in every tail row. The difference from the limiting Toeplitz operator is therefore compact but not finite rank.

Example 11.1 (Variable three-term averaging family). *Define $\mathcal{N}_3$ by*

$$(\mathcal{N}_3 x)_0 = x_0, \quad (\mathcal{N}_3 x)_1 = \frac{x_0 + x_1}{2}, \quad (11.1)$$

and, for $n \geq 2$,

$$(\mathcal{N}_3 x)_n = \left(\frac{1}{3} + \varepsilon_n\right) x_n + \left(\frac{1}{3} - \frac{\varepsilon_n}{2}\right) x_{n-1} + \left(\frac{1}{3} - \frac{\varepsilon_n}{2}\right) x_{n-2}, \quad (11.2)$$

where

$$\varepsilon_n = \frac{1}{n+2}. \quad (11.3)$$

Every row is nonnegative and sums to one. The coefficient limits are

$$a_0 = a_1 = a_2 = \frac{1}{3}, \quad (11.4)$$

so the asymptotic symbol is

$$p(z) = \frac{1 + z + z^2}{3}. \quad (11.5)$$

Proposition 11.2 (Compact but infinite-rank tail). *For the operator $\mathcal{N}_3$ in [equation \(11.2\)](#),*

$$\mathcal{N}_3 = p(S) + K, \quad (11.6)$$

where K is compact on c_0 and c , but $\text{rank } K = \infty$.

Proof. Compactness follows from $\varepsilon_n \rightarrow 0$ and the compact asymptotic reduction theorem. For $n \geq 2$, the diagonal deviation is

$$a_{0,n} - \frac{1}{3} = \varepsilon_n \neq 0.$$

Thus the variable tail never becomes exactly equal to its limiting band. By the finite-rank characterisation, K cannot have finite rank. $\square$

Theorem 11.3 (Essential spectrum and index for the variable three-term family). *On both c_0 and c ,*

$$\sigma_{\text{ess}}(\mathcal{N}_3) = \left\{ \frac{1 + e^{it} + e^{2it}}{3} : 0 \leq t \leq 2\pi \right\}. \quad (11.7)$$

Equivalently,

$$p(e^{it}) = \frac{1 + 2 \cos t}{3} e^{it}. \quad (11.8)$$

For

$$\lambda \notin \sigma_{\text{ess}}(\mathcal{N}_3), \quad (11.9)$$

let $N_3(\lambda)$ be the number of roots in $\mathbb{D}$ of

$$z^2 + z + 1 - 3\lambda = 0. \quad (11.10)$$

Then

$$\text{ind}(\mathcal{N}_3 - \lambda I) = -N_3(\lambda). \quad (11.11)$$

The possible indices are $0, -1, -2$.

Proof. Apply the essential spectrum and index theorems to [equation \(11.5\)](#). $\square$

The essential spectral curve is plotted in [figure 2](#). The variable coefficients in [equation \(11.2\)](#) alter the compact part of the operator but do not change this curve.

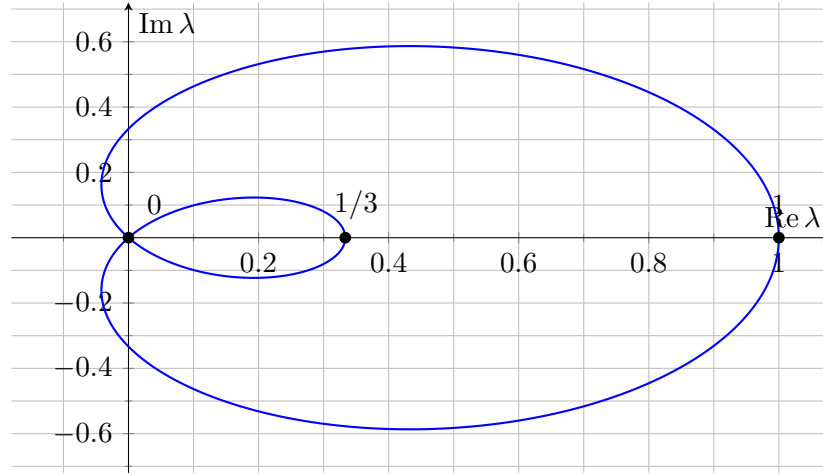


Figure 2: Fredholm essential spectrum $p(\mathbb{T})$ of the limiting three-term symbol $p(z) = (1+z+z^2)/3$. The same curve applies to all three-band operator whose coefficients converge to $(1/3, 1/3, 1/3)$.

Corollary 11.4 (Real-axis index). *For real $\lambda \notin \{0, \frac{1}{3}, 1\}$,*

$$\text{ind}(\mathcal{N}_3 - \lambda I) = \begin{cases} 0, & \lambda < 0, \\ -2, & 0 < \lambda < \frac{1}{3}, \\ -1, & \frac{1}{3} < \lambda < 1, \\ 0, & \lambda > 1. \end{cases} \quad (11.12)$$

The points $0, \frac{1}{3}, 1$ lie on the essential spectrum.

Proof. The roots are

$$r_{\pm}(\lambda) = \frac{-1 \pm \sqrt{12\lambda - 3}}{2}. \quad (11.13)$$

On each real component outside the essential curve, count the roots with modulus less than one and apply the index formula. The count is 0, 2, 1, 0 on the four intervals listed above. $\square$

Corollary 11.5 (Full spectrum of the variable three-term example). *For $\mathcal{N}_3$,*

$$\sigma(\mathcal{N}_3) = p(\overline{\mathbb{D}}) \quad (11.14)$$

on both c_0 and c .

Proof. The diagonal entries are

$$1, \quad \frac{1}{2}, \quad \frac{1}{3} + \frac{1}{4}, \quad \frac{1}{3} + \frac{1}{5}, \quad \dots \quad (11.15)$$

and all lie in $[0, 1] \subset p(\overline{\mathbb{D}})$. Hence the diagonal-containment corollary applies. $\square$

Remark 11.6. *This example is an important model of the new setting. Its limiting symbol is the same polynomial that arises for a stationary three-term average, but the operator itself is not eventually Toeplitz. The compact perturbation has infinite rank. Thus the essential spectrum and index are stable under a nontrivial tail deformation extending through infinitely many rows.*

12 A variable four-term averaging family

The framework is not restricted to quadratic symbols.

Example 12.1 (Variable four-term family). *Let*

$$\delta_n = \frac{1}{10(n+2)} \quad (n \geq 3), \quad (12.1)$$

and define, for $n \geq 3$,

$$\begin{aligned} (\mathcal{N}_4 x)_n &= \left(\frac{1}{4} + 3\delta_n\right) x_n + \left(\frac{1}{4} - \delta_n\right) x_{n-1} \\ &\quad + \left(\frac{1}{4} - \delta_n\right) x_{n-2} + \left(\frac{1}{4} - \delta_n\right) x_{n-3}. \end{aligned} \quad (12.2)$$

Choose the first three rows to be ordinary **normalised** averages:

$$(\mathcal{N}_4 x)_0 = x_0, \quad (\mathcal{N}_4 x)_1 = \frac{x_0 + x_1}{2}, \quad (\mathcal{N}_4 x)_2 = \frac{x_0 + x_1 + x_2}{3}. \quad (12.3)$$

Then $\mathcal{N}_4$ is positive, **row-normalised**, and its limiting symbol is

$$b(z) = \frac{1 + z + z^2 + z^3}{4}. \quad (12.4)$$

Corollary 12.2 (Four-term essential spectrum). *On c_0 and c ,*

$$\sigma_{\text{ess}}(\mathcal{N}_4) = \left\{ \frac{1 + e^{it} + e^{2it} + e^{3it}}{4} : 0 \leq t \leq 2\pi \right\}. \quad (12.5)$$

For λ outside this curve,

$$\text{ind}(\mathcal{N}_4 - \lambda I) = -\# \left\{ z \in \mathbb{D} : z^3 + z^2 + z + 1 - 4\lambda = 0 \right\}. \quad (12.6)$$

Thus the possible indices are

$$0, -1, -2, -3. \quad (12.7)$$

Proof. Apply the main theorems to the cubic symbol [equation \(12.4\)](#). □

Remark 12.3. *Since*

$$1 + z + z^2 + z^3 = (1 + z)(1 + z^2), \quad (12.8)$$

the symbol vanishes at $z = -1, i, -i$, all on the unit circle. Hence

$$0 \in \sigma_{\text{ess}}(\mathcal{N}_4). \quad (12.9)$$

Also $b(1) = 1$, so $1 \in \sigma_{\text{ess}}(\mathcal{N}_4)$ and, on c , it is an eigenvalue because the operator is row-normalised.

13 Consequences for families with a common asymptotic band

The compactness theorem gives a stability result for entire families of variable matrices.

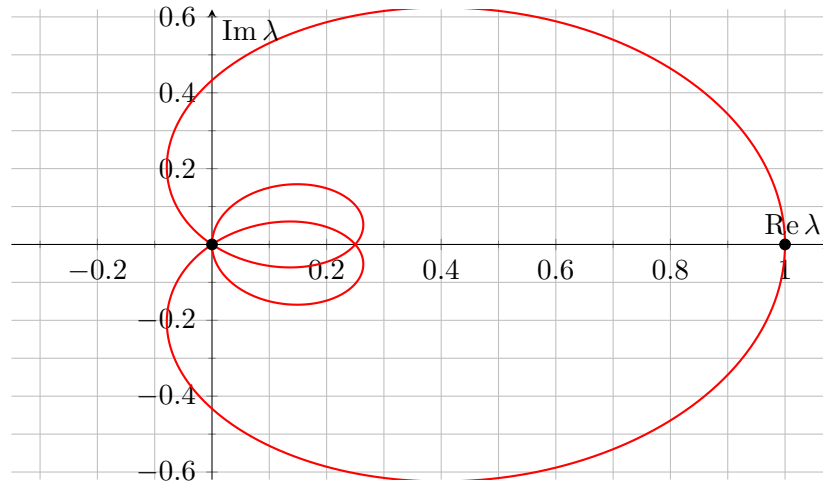


Figure 3: Essential spectral curve $b(\mathbb{T})$ for the four-term limiting average $b(z) = (1+z+z^2+z^3)/4$.

Theorem 13.1 (Common-symbol stability). *Let $\{\mathcal{N}_\theta : \theta \in \Theta\}$ be a family of bandwidth- m operators such that, for every θ and every j ,*

$$a_{j,n}^{(\theta)} \longrightarrow a_j \quad (13.1)$$

with the same limits a_j . Then all members of the family satisfy

$$\pi_X(\mathcal{N}_\theta) = a(\pi_X(S)), \quad (13.2)$$

and hence

$$\sigma_{\text{ess}}(\mathcal{N}_\theta) = a(\mathbb{T}) \quad (13.3)$$

and

$$\text{ind}(\mathcal{N}_\theta - \lambda I) = -N_a(\lambda) \quad (\lambda \notin a(\mathbb{T})). \quad (13.4)$$

Proof. For every θ ,

$$\mathcal{N}_\theta - a(S) \in \mathcal{K}(X). \quad (13.5)$$

The result follows from compact stability of the essential spectrum and Fredholm index. $\square$

Remark 13.2 (What can change under the compact tail deformation). *The essential spectrum and the index function cannot change when the limiting symbol is fixed. The point spectrum can change, because compact perturbations may introduce or remove isolated eigenvalues in index-zero regions. The diagonal-candidate principle and [equation \(10.4\)](#) quantify this remaining freedom for the present lower-triangular class.*

Corollary 13.3 (Robustness under rates of convergence). *No specific rate in [equation \(2.11\)](#) is required for the essential spectrum and index formulas. Whether*

$$a_{j,n} - a_j = O(n^{-1}), \quad O(n^{-2}), \quad \text{or merely } o(1), \quad (13.6)$$

the Calkin-algebra class is the same, provided the coefficients converge.

Remark 13.4 (Quantitative approximation). *Although the Fredholm conclusions require only convergence, the estimate [equation \(4.6\)](#) is useful numerically. It gives an explicit operator-norm*

bound for replacing the variable tail by its limiting constant band after row N . Faster coefficient convergence therefore yields faster finite-section approximation of the compact remainder.

14 Comparison with established operator frameworks

The results fit naturally into several established strands of operator theory.

14.1 Summability and Nörlund matrices

Hardy’s (1949) classical treatment and Wilansky (2000) functional-analytic development demonstrate how matrix summability methods can be studied through Banach-space techniques. This paper retains the lower-triangular averaging mechanism but isolates a finite moving window and allows its weights to vary with the row index. For **normalised** weights, the operators remain regular on convergent sequences because limits are preserved. Recent spectral studies of specific Nörlund matrices include Ochieng et al. (2025) on c_0 and Yeboah and Iddrisu (2026) on c and c_0 . The present formulation moves from one prescribed coefficient pattern to a class whose coefficients may vary indefinitely while approaching a limiting band.

14.2 Toeplitz and shift-polynomial structure

The limiting operator

$$a(S) = a_0I + a_1S + \cdots + a_mS^m \quad (14.1)$$

is an analytic one-sided Toeplitz operator generated by the unilateral shift. Symbol methods for Toeplitz operators are classical and are developed systematically by (Böttcher & Silbermann, 2006). The present contribution is an explicit coefficient-level compactness criterion that tells precisely when a variable finite-band averaging matrix belongs to the same Calkin class as such a shift polynomial.

14.3 Band-dominated and limit-operator viewpoints

Band-dominated operator theory studies operators that can be approximated in operator norm by band operators, with limit operators encoding ^{behaviour} at infinity (Rabinovich et al., 2004). Hagger et al. (2016) further **emphasise** that Calkin-algebra information controls Fredholmness, essential spectra, essential norms and essential pseudospectra in broad band-dominated settings. The operators considered here constitute a simple but explicit one-sided class with a unique polynomial limit symbol. This permits direct formulas on c_0 and c , avoids the full machinery of general limit-operator theory, and yields the additional lower-triangular full-spectrum **localisation** in equation (10.4).

14.4 Fredholm theory

The Fredholm analysis relies on Atkinson’s (1951) theorem, compact perturbation invariance, and standard Banach-space spectral theory (Conway, 1990; Kato, 1995; Gohberg & Kreĭn, 1969). Contemporary variants of Fredholm spectral theory continue to be developed; see, for example, El Ouali et al. (2022) and Kong et al. (2024). This paper uses the classical Fredholm essential spectrum because it interacts directly with compact perturbations and the Calkin algebra.

15 Further deductions and extensions

The framework suggests several directions that can be pursued without changing its central mechanism.

15.1 Longer finite-memory averages

For any fixed m , **normalised** limits

$$a_j \geq 0, \quad \sum_{j=0}^m a_j = 1 \quad (15.1)$$

give

$$a(z) = \sum_{j=0}^m a_j z^j, \quad \sigma_{\text{ess}}(\mathcal{N}) = a(\mathbb{T}), \quad r_{\text{ess}}(\mathcal{N}) = 1. \quad (15.2)$$

The Fredholm index takes values from 0 down to $-d$, where $d = \deg a$.

15.2 Non-normalised complex coefficients

Positivity and row normalization are not required for the compactness, essential-spectrum, or index theorems. Complex asymptotic coefficients are allowed, with

$$r_{\text{ess}}(\mathcal{N}) = \max_{|z|=1} \left| \sum_{j=0}^m a_j z^j \right|. \quad (15.3)$$

This broadens the framework from averaging methods to general lower-triangular finite-band operators with convergent subdiagonals.

15.3 Asymptotically periodic bands

A natural next step is to replace convergence to constants by convergence to a periodic cycle. The limiting object would then be a block Laurent or block Toeplitz symbol rather than a scalar polynomial. One expects the essential spectrum to be controlled by the union of spectra of the corresponding matrix-valued symbol. Such an extension would connect the present coefficient-level analysis more directly with periodic limit operators and band-dominated theory ([Rabinovich et al., 2004](#); [Seidel, 2014](#)).

15.4 Several limit points

If a coefficient band does not converge but possesses several asymptotic limit patterns, there need not be a single polynomial symbol. In that case a family of limit symbols is more appropriate, and the essential spectrum is expected to reflect the union of spectral sets associated with all limit operators. This lies beyond the single-symbol class treated here.

15.5 Other sequence spaces

The same formulas suggest investigation on ℓ^p spaces, $1 \leq p < \infty$, and on weighted sequence spaces. The shift index remains a central ingredient, but compactness criteria and boundary

behaviour may depend on the ambient space. General band-dominated results provide a natural comparison framework (Seidel, 2014; Hagger et al., 2016).

16 Conclusion

This paper has developed a unified Fredholm and spectral framework for variable-coefficient finite-band Nörlund-type operators on c_0 and c . The defining feature is that the coefficients along each of finitely many bands are allowed to vary through infinitely many rows while converging to limiting values. The resulting asymptotic symbol

$$a(z) = a_0 + a_1 z + \cdots + a_m z^m \quad (16.1)$$

encodes the operator at infinity. The **compactness characterisation** shows that bandwise coefficient convergence is exactly what is required for

$$\mathcal{N} - a(S) \quad (16.2)$$

to be compact. This compact remainder is finite rank only in the eventually constant case, otherwise it may have infinite rank. Therefore, the theory extends to nonstationary tails. Passing to the Calkin algebra gives the exact Fredholm essential spectrum

$$\sigma_{\text{ess}}(\mathcal{N}) = a(\mathbb{T}) \quad (16.3)$$

on both sequence spaces. Outside this curve, factorization of $a(z) - \lambda$ and the Fredholm theory of the unilateral shift yield

$$\text{ind}(\mathcal{N} - \lambda I) = -\#\{z \in \mathbb{D} : a(z) = \lambda\} = -\text{wind}(a(\mathbb{T}), \lambda). \quad (16.4)$$

The ordinary spectrum satisfies the **localisation**

$$a(\overline{\mathbb{D}}) \subseteq \sigma(\mathcal{N}) \subseteq a(\overline{\mathbb{D}}) \cup \{a_{0,n} : n \geq 0\}, \quad (16.5)$$

which becomes the exact identity

$$\sigma(\mathcal{N}) = a(\overline{\mathbb{D}}) \quad (16.6)$$

when the diagonal coefficients remain inside the polynomial image of the closed disk. For **normalised** nonnegative averaging matrices, the symbol satisfies $a(1) = 1$ and the essential spectral radius is one. On c , row normalization also gives the eigenvector $\mathbf{1}$ at $\lambda = 1$, so the eigenvalue is embedded in the essential spectrum. The three-term and four-term examples demonstrate that the framework is not tied to one specific matrix. In particular, the three-term variable family has a compact perturbation of infinite rank while retaining an exactly computable essential spectrum, Fredholm index, and full spectrum. This establishes a natural route from fixed Nörlund matrices to asymptotically structured families of operators.

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Conflict of interest.

The author declares no competing interests.

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