

An application of absolute matrix summability to infinite series

Original Research Article

ABSTRACT

This paper investigates sufficient conditions for absolute matrix summability factors of an infinite series. Starting from a known theorem on absolute Riesz summability factors, the study extends the result to the more general φ - $|B; \delta|_k$ summability framework, where $B = (b_{\eta\omega})$ is a positive normal matrix. The development uses an almost increasing sequence (X_η) together with conditions imposed on the matrix entries, the factor sequence (ξ_η) , the sequence (z_η) of $(C, 1)$ means associated with (ηd_η) , and the parameters k and δ . Two lower semimatrices generated by B are introduced to express the relevant sequence-to-sequence and series-to-series transformations. Under the stated hypotheses, the main theorem establishes that the factored series $\sum d_\eta \xi_\eta$ is φ - $|B; \delta|_k$ summable for $k \geq 1$ and $0 \leq \delta < 1/k$. The proof applies Abel's transformation and decomposes the transformed expression into four components. Each component is then estimated using the matrix assumptions, properties of almost increasing sequences, and Hölder's inequality. The resulting bounds establish the required convergence without introducing assumptions beyond those stated in the theorem. The argument therefore connects the matrix conditions directly with the summability conclusion. Appropriate choices of δ , φ_η , and the matrix entries reduce the theorem to previously known absolute matrix and absolute Riesz summability results, showing that the established result provides a common framework for these special cases.

Keywords: absolute matrix summability; almost increasing sequence; infinite series; summability factors; Riesz mean; normal matrix; Hölder's inequality; Minkowski's inequality; matrix transformation; absolute Riesz summability.

Mathematics Subject Classification (2020): 26D15, 40D15, 40F05, 40G99.

1 Introduction

A positive sequence (h_η) is called almost increasing if there exist a positive increasing sequence (a_η) together with positive constants K and L such that (Bari & Stechkin, 1956)

$$Ka_\eta \leq h_\eta \leq La_\eta.$$

It is clear that every increasing sequence is also almost increasing, whereas the converse does not necessarily hold. For example, the sequence

$$h_\eta = \eta e^{(-1)^\eta}$$

is almost increasing but is not increasing.

For the infinite series $\sum d_\eta$, let (s_η) denote the corresponding sequence of partial sums. Suppose that $B = (b_{\eta\omega})$ is a normal matrix, that is, a lower triangular matrix whose diagonal entries are all nonzero. The matrix B induces the sequence transformation $s = (s_\eta) \mapsto Bs = (B_\eta(s))$, where

$$B_\eta(s) = \sum_{\omega=0}^{\eta} b_{\eta\omega} s_\omega, \quad \eta = 0, 1, \dots$$

Consider the positive sequence (φ_η) . The infinite series $\sum d_\eta$ is φ - $|B; \delta|_k$ summable, provided that **(Özarslan, 2018)**

$$\sum_{\eta=1}^{\infty} \varphi_\eta^{\delta k + k - 1} |B_\eta(s) - B_{\eta-1}(s)|^k < \infty, \quad k \geq 1 \quad \text{and} \quad \delta \geq 0.$$

The infinite series $\sum d_\eta$ is said to be $|\bar{N}, q_\eta|_k$ summable if

$$\sum_{\eta=1}^{\infty} \left(\frac{Q_\eta}{q_\eta} \right)^{k-1} |\alpha_\eta - \alpha_{\eta-1}|^k < \infty, \quad k \geq 1 \quad (\text{Bor, 1985}),$$

where

$$\alpha_\eta = \frac{1}{Q_\eta} \sum_{\omega=0}^{\eta} q_\omega s_\omega$$

and (q_η) is a positive sequence satisfying

$$Q_\eta = \sum_{\omega=0}^{\eta} q_\omega \longrightarrow \infty, \quad \eta \longrightarrow \infty \quad (Q_{-\omega} = q_{-\omega} = 0, \omega \geq 1).$$

If $\varphi_\eta = Q_\eta/q_\eta$ and $\delta = 0$, then the φ - $|B; \delta|_k$ summability method coincides with the $|B, q_\eta|_k$ summability method **(Sulaiman, 2003)**. Furthermore, by taking $\varphi_\eta = Q_\eta/q_\eta$, $\delta = 0$, and $b_{\eta\omega} = q_\omega/Q_\eta$, one obtains the $|\bar{N}, q_\eta|_k$ summability method.

Recent studies have continued to develop absolute summability factors and absolute matrix summability for infinite series **(Łenski & Szal, 2024; Göğüş & Gölbol, 2026)**, while matrix-generated sequence spaces provide related context for sequence-transform methods **(Candan & Kayaduman, 2015)**. However, within the framework considered in this manuscript, a gap remains between the known absolute Riesz summability factor result stated in Theorem 1 and its formulation for the more general φ - $|B; \delta|_k$ absolute matrix summability setting. Accordingly, the objective of this study is to extend Theorem 1 to the φ - $|B; \delta|_k$ summability method under the conditions developed in the subsequent sections.

2 Known Result

The result below, established by **Mazhar (1999)**, characterises the $|\bar{N}, q_\eta|_k$ summability factors of an infinite series.

Theorem 1. *Suppose that (X_η) is an almost increasing sequence and that the following conditions are satisfied:*

$$|\xi_\mu| X_\mu = O(1) \quad \text{as } \mu \rightarrow \infty, \quad (1)$$

$$\sum_{\eta=1}^{\mu} \eta X_\eta |\Delta^2 \xi_\eta| = O(1) \quad \text{as } \mu \rightarrow \infty, \quad (2)$$

$$\sum_{\eta=1}^{\mu} \frac{Q_\eta}{\eta} = O(P_\mu) \quad \text{as } \mu \rightarrow \infty, \quad (3)$$

$$\sum_{\eta=1}^{\mu} \frac{|z_\eta|^k}{\eta} = O(X_\mu) \quad \text{as } \mu \rightarrow \infty, \quad (4)$$

$$\sum_{\eta=1}^{\mu} \frac{q_\eta}{Q_\eta} |z_\eta|^k = O(X_\mu) \quad \text{as } \mu \rightarrow \infty. \quad (5)$$

Here $\Delta^2 \xi_\eta = \Delta(\Delta \xi_\eta)$, $\Delta \xi_\eta = \xi_\eta - \xi_{\eta+1}$, and (z_η) is the sequence of $(C, 1)$ means associated with (ηd_η) . Under these assumptions, the series $\sum d_\eta \xi_\eta$ is $|\bar{N}, q_\eta|_k$ summable for $k \geq 1$.

3 Main Result

Several studies have investigated absolute matrix summability factors for infinite series using different classes of sequences, with particular emphasis on their **characterisation** and applications to various summability methods. For related results, the reader is referred to (**Karakas, 2018; Kartal, 2019; Ögdük, 2009; Özarslan & Şakar, 2015; Özarslan, 2019a, 2019b, 2021, 2023; Özarslan & Kartal, 2020; Özarslan & Kartal Erdoğan, 2025; Özarslan, 2026**). The main objective of the present paper is to extend Theorem 1 to the more general setting of φ - $|B; \delta|_k$ summability.

To formulate our main theorem, we first introduce some additional notation. For a given normal matrix $B = (b_{\eta\omega})$, define the two lower semimatrices $\bar{B} = (\bar{b}_{\eta\omega})$ and $\hat{B} = (\hat{b}_{\eta\omega})$ by

$$\bar{b}_{\eta\omega} = \sum_{v=\omega}^{\eta} b_{\eta v}, \quad \eta, \omega = 0, 1, \dots \quad (6)$$

and

$$\hat{b}_{00} = \bar{b}_{00} = b_{00}, \quad \hat{b}_{\eta\omega} = \bar{b}_{\eta\omega} - \bar{b}_{\eta-1, \omega}, \quad \eta = 1, 2, \dots \quad (7)$$

The transformations from series to sequences and from series to series are given by the matrices $\bar{B}$ and $\hat{B}$, respectively. Hence,

$$B_\eta(s) = \sum_{\omega=0}^{\eta} b_{\eta\omega} s_\omega = \sum_{\omega=0}^{\eta} \bar{b}_{\eta\omega} a_\omega \quad (8)$$

and

$$\bar{\Delta} B_\eta(s) = \sum_{\omega=0}^{\eta} \hat{b}_{\eta\omega} a_\omega. \quad (9)$$

Theorem 2. Consider a positive normal matrix $B = (b_{\eta\omega})$ satisfying

$$\bar{b}_{\eta 0} = 1, \quad \eta = 0, 1, \dots, \quad (10)$$

$$b_{\eta-1, \omega} \geq b_{\eta\omega}, \quad \text{for } \eta \geq \omega + 1, \quad (11)$$

$$b_{\eta\eta} = O\left(\frac{q_\eta}{Q_\eta}\right), \quad (12)$$

$$\sum_{\omega=1}^{\eta-1} \frac{|\hat{b}_{\eta, \omega+1}|}{\omega} = O(b_{\eta\eta}), \quad (13)$$

$$\sum_{\eta=\omega+1}^{\mu+1} \varphi_\eta^{\delta k} |\hat{b}_{\eta, \omega+1}| = O\left(\varphi_\omega^{\delta k}\right) \quad \text{as } \mu \rightarrow \infty, \quad (14)$$

$$\sum_{\eta=\omega+1}^{\mu+1} \varphi_\eta^{\delta k} |\Delta_\omega \hat{b}_{\eta\omega}| = O\left(\varphi_\omega^{\delta k-1}\right) \quad \text{as } \mu \rightarrow \infty. \quad (15)$$

Here $\Delta_\omega(\hat{b}_{\eta\omega}) = \hat{b}_{\eta\omega} - \hat{b}_{\eta, \omega+1}$. Assume that (X_η) is an almost increasing sequence satisfying $\varphi_\eta q_\eta = O(Q_\eta)$. Furthermore, suppose that conditions (1) and (2) of Theorem 1, together with

$$\sum_{\eta=1}^{\mu} \varphi_\eta^{\delta k} \frac{1}{\eta} |z_\eta|^k = O(X_\mu) \quad \text{as } \mu \rightarrow \infty, \quad (16)$$

$$\sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k-1} |z_{\eta}|^k = O(X_{\mu}) \quad \text{as } \mu \rightarrow \infty, \quad (17)$$

hold. Under these assumptions, the series $\sum d_{\eta} \xi_{\eta}$ is φ - $|B; \delta|_k$ summable for $k \geq 1$ and $0 \leq \delta < 1/k$.

Before proving Theorem 2, we state the following lemma.

Lemma 1. (Mazhar, 1999) *Whenever the conditions of Theorem 2 hold, it follows that*

$$\eta X_{\eta} |\Delta \xi_{\eta}| = O(1) \quad \text{as } \eta \rightarrow \infty, \quad (18)$$

$$\sum_{\eta=1}^{\infty} X_{\eta} |\Delta \xi_{\eta}| < \infty. \quad (19)$$

4 Proof of Theorem 2

Let the B -transform of the series $\sum d_{\eta} \xi_{\eta}$ be (θ_{η}) . Using (8) and (9), we obtain

$$\bar{\Delta} \theta_{\eta} = \sum_{\omega=1}^{\eta} \frac{\hat{b}_{\eta\omega} \xi_{\omega}}{\omega} \omega d_{\omega}.$$

Using Abel's formula, we get

$$\begin{aligned} \bar{\Delta} \theta_{\eta} &= \sum_{\omega=1}^{\eta-1} \Delta_{\omega} \left(\frac{\hat{b}_{\eta\omega} \xi_{\omega}}{\omega} \right) \sum_{r=1}^{\omega} r d_r + \frac{\hat{b}_{\eta\eta} \xi_{\eta}}{\eta} \sum_{r=1}^{\eta} r d_r \\ &= \frac{\eta+1}{\eta} b_{\eta\eta} \xi_{\eta} z_{\eta} + \sum_{\omega=1}^{\eta-1} \frac{\omega+1}{\omega} \Delta_{\omega} (\hat{b}_{\eta\omega}) \xi_{\omega} z_{\omega} \\ &\quad + \sum_{\omega=1}^{\eta-1} \frac{\omega+1}{\omega} \hat{b}_{\eta,\omega+1} \Delta \xi_{\omega} z_{\omega} + \sum_{\omega=1}^{\eta-1} \frac{1}{\omega} \hat{b}_{\eta,\omega+1} \xi_{\omega+1} z_{\omega} \\ &= \theta_{\eta,1} + \theta_{\eta,2} + \theta_{\eta,3} + \theta_{\eta,4}. \end{aligned}$$

To prove Theorem 2, we will show $\sum_{\eta=1}^{\infty} \varphi_{\eta}^{\delta k+k-1} |\theta_{\eta,r}|^k < \infty$ for $r = 1, 2, 3, 4$. First, using (12), we obtain

$$\begin{aligned} \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k+k-1} |\theta_{\eta,1}|^k &= \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k+k-1} \left| \frac{\eta+1}{\eta} b_{\eta\eta} \xi_{\eta} z_{\eta} \right|^k \\ &= O(1) \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k+k-1} b_{\eta\eta}^k |\xi_{\eta}|^k |z_{\eta}|^k \\ &= O(1) \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k-1} \left(\frac{\varphi_{\eta} q_{\eta}}{Q_{\eta}} \right)^k |\xi_{\eta}|^{k-1} |\xi_{\eta}| |z_{\eta}|^k. \end{aligned}$$

Since (X_{η}) is an almost increasing sequence, condition (1) implies that $|\xi_{\eta}|^{k-1} = O(1)$. Hence,

$$\begin{aligned} \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k+k-1} |\theta_{\eta,1}|^k &= O(1) \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k-1} |\xi_{\eta}| |z_{\eta}|^k \\ &= O(1) \sum_{\eta=1}^{\mu-1} \Delta |\xi_{\eta}| \sum_{\omega=1}^{\eta} \varphi_{\omega}^{\delta k-1} |z_{\omega}|^k + O(1) |\xi_{\mu}| \sum_{\eta=1}^{\mu} \varphi_{\eta}^{\delta k-1} |z_{\eta}|^k \end{aligned}$$

$$= O(1) \sum_{\eta=1}^{\mu-1} |\Delta \xi_\eta| X_\eta + O(1) |\xi_\mu| X_\mu = O(1) \quad \text{as } \mu \rightarrow \infty,$$

by means of Abel's transformation together with conditions (17), (19), and (1).

Next, by an application of Hölder's inequality,

$$\begin{aligned} \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} |\theta_{\eta,2}|^k &= \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} \left| \sum_{\omega=1}^{\eta-1} \frac{\omega+1}{\omega} \Delta_\omega(\widehat{b}_{\eta\omega}) \xi_\omega z_\omega \right|^k \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} \left(\sum_{\omega=1}^{\eta-1} |\Delta_\omega(\widehat{b}_{\eta\omega})| |\xi_\omega| |z_\omega| \right)^k \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} \left(\sum_{\omega=1}^{\eta-1} |\Delta_\omega(\widehat{b}_{\eta\omega})| |\xi_\omega|^k |z_\omega|^k \right) \left(\sum_{\omega=1}^{\eta-1} |\Delta_\omega(\widehat{b}_{\eta\omega})| \right)^{k-1}. \end{aligned}$$

Using (6), (7), (10), and (11), it is clear that

$$\sum_{\omega=1}^{\eta-1} |\Delta_\omega(\widehat{b}_{\eta\omega})| = \sum_{\omega=1}^{\eta-1} (b_{\eta-1,\omega} - b_{\eta\omega}) \leq b_{\eta\eta}.$$

Now, using (12), (15), and (1),

$$\begin{aligned} \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} |\theta_{\eta,2}|^k &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} b_{\eta\eta}^{k-1} \left(\sum_{\omega=1}^{\eta-1} |\Delta_\omega(\widehat{b}_{\eta\omega})| |\xi_\omega|^k |z_\omega|^k \right) \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k} \left(\frac{\varphi_\eta q_\eta}{Q_\eta} \right)^{k-1} \left(\sum_{\omega=1}^{\eta-1} |\Delta_\omega(\widehat{b}_{\eta\omega})| |\xi_\omega|^k |z_\omega|^k \right) \\ &= O(1) \sum_{\omega=1}^{\mu} |\xi_\omega|^k |z_\omega|^k \sum_{\eta=\omega+1}^{\mu+1} \varphi_\eta^{\delta k} |\Delta_\omega(\widehat{b}_{\eta\omega})| \\ &= O(1) \sum_{\omega=1}^{\mu} \varphi_\omega^{\delta k-1} |\xi_\omega| |z_\omega|^k. \end{aligned}$$

Proceeding as in the estimate for $\theta_{\eta,1}$,

$$\sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} |\theta_{\eta,2}|^k = O(1) \quad \text{as } \mu \rightarrow \infty.$$

A further application of Hölder's inequality gives

$$\begin{aligned} \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} |\theta_{\eta,3}|^k &= \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} \left| \sum_{\omega=1}^{\eta-1} \frac{\omega+1}{\omega} \widehat{b}_{\eta,\omega+1} \Delta \xi_\omega z_\omega \right|^k \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\Delta \xi_\omega| |z_\omega| \right)^k \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_\eta^{\delta k+k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\Delta \xi_\omega| |z_\omega|^k \right) \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\Delta \xi_\omega| \right)^{k-1}. \end{aligned}$$

Here

$$\begin{aligned}\widehat{b}_{\eta,\omega+1} &= \bar{b}_{\eta,\omega+1} - \bar{b}_{\eta-1,\omega+1} \\ &= b_{\eta\eta} + \sum_{v=\omega+1}^{\eta-1} (b_{\eta v} - b_{\eta-1,v}) \leq b_{\eta\eta}\end{aligned}$$

by (7), (6), and (11).

Furthermore, using condition (19), we get

$$\sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} |\theta_{\eta,3}|^k = O(1) \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} b_{\eta\eta}^{k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\Delta \xi_{\omega}| |z_{\omega}|^k \right).$$

Then, using (12), (14), (16), (2), (19), and (18), we obtain

$$\begin{aligned}\sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} |\theta_{\eta,3}|^k &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k} \left(\frac{\varphi_{\eta} q_{\eta}}{Q_{\eta}} \right)^{k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\Delta \xi_{\omega}| |z_{\omega}|^k \right) \\ &= O(1) \sum_{\omega=1}^{\mu} |\Delta \xi_{\omega}| |z_{\omega}|^k \sum_{\eta=\omega+1}^{\mu+1} \varphi_{\eta}^{\delta k} |\widehat{b}_{\eta,\omega+1}| \\ &= O(1) \sum_{\omega=1}^{\mu} \varphi_{\omega}^{\delta k} \omega |\Delta \xi_{\omega}| \frac{|z_{\omega}|^k}{\omega} \\ &= O(1) \sum_{\omega=1}^{\mu-1} \Delta(\omega |\Delta \xi_{\omega}|) \sum_{r=1}^{\omega} \varphi_r^{\delta k} \frac{1}{r} |z_r|^k + O(1) \mu |\Delta \xi_{\mu}| \sum_{\omega=1}^{\mu} \varphi_{\omega}^{\delta k} \frac{1}{\omega} |z_{\omega}|^k \\ &= O(1) \sum_{\omega=1}^{\mu-1} \omega X_{\omega} |\Delta^2 \xi_{\omega}| + O(1) \sum_{\omega=1}^{\mu-1} |\Delta \xi_{\omega}| X_{\omega} + O(1) \mu |\Delta \xi_{\mu}| X_{\mu} \\ &= O(1) \quad \text{as } \mu \rightarrow \infty.\end{aligned}$$

As a final step, we derive

$$\begin{aligned}\sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} |\theta_{\eta,4}|^k &= \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} \left| \sum_{\omega=1}^{\eta-1} \frac{1}{\omega} \widehat{b}_{\eta,\omega+1} \xi_{\omega+1} z_{\omega} \right|^k \\ &\leq \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\xi_{\omega+1}| \frac{|z_{\omega}|}{\omega} \right)^k \\ &\leq \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\xi_{\omega+1}|^k \frac{|z_{\omega}|^k}{\omega} \right) \left(\sum_{\omega=1}^{\eta-1} \frac{|\widehat{b}_{\eta,\omega+1}|}{\omega} \right)^{k-1} \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k+k-1} b_{\eta\eta}^{k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\xi_{\omega+1}| \frac{|z_{\omega}|^k}{\omega} \right) \\ &= O(1) \sum_{\eta=2}^{\mu+1} \varphi_{\eta}^{\delta k} \left(\frac{\varphi_{\eta} q_{\eta}}{Q_{\eta}} \right)^{k-1} \left(\sum_{\omega=1}^{\eta-1} |\widehat{b}_{\eta,\omega+1}| |\xi_{\omega+1}| \frac{|z_{\omega}|^k}{\omega} \right) \\ &= O(1) \sum_{\omega=1}^{\mu} |\xi_{\omega+1}| \frac{|z_{\omega}|^k}{\omega} \sum_{\eta=\omega+1}^{\mu+1} \varphi_{\eta}^{\delta k} |\widehat{b}_{\eta,\omega+1}| \\ &= O(1) \sum_{\omega=1}^{\mu} \varphi_{\omega}^{\delta k} \frac{1}{\omega} |\xi_{\omega+1}| |z_{\omega}|^k\end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{\omega=1}^{\mu-1} |\Delta_{\xi_{\omega+1}}| \sum_{r=1}^{\omega} \varphi_r^{\delta k} \frac{1}{r} |z_r|^k + O(1) |\xi_{\mu+1}| \sum_{\omega=1}^{\mu} \varphi_{\omega}^{\delta k} \frac{1}{\omega} |z_{\omega}|^k \\
&= O(1) \sum_{\omega=1}^{\mu-1} |\Delta_{\xi_{\omega+1}}| X_{\omega+1} + O(1) |\xi_{\mu+1}| X_{\mu+1} \\
&= O(1) \quad \text{as } \mu \rightarrow \infty,
\end{aligned}$$

by using (13), (12), (14), (16), (19), and (1). Accordingly, Theorem 2 is proved.

5 Conclusions

This study extends a known absolute Riesz summability factor theorem to the broader $\varphi\text{-}|B; \delta|_k$ absolute matrix summability setting. The main result is obtained for a positive normal matrix $B = (b_{\eta\omega})$ under the stated structural conditions on its entries, together with the prescribed assumptions on the almost increasing sequence (X_{η}) , the factor sequence (ξ_{η}) , the sequence (z_{η}) , and the parameters k and δ . By introducing the associated lower semimatrices and applying Abel's transformation, the proof separates the transformed series into four terms and establishes the required convergence estimate for each term. Consequently, $\sum d_{\eta} \xi_{\eta}$ is $\varphi\text{-}|B; \delta|_k$ summable for $k \geq 1$ and $0 \leq \delta < 1/k$. The theorem also contains the stated earlier summability results as special cases. In particular, suitable choices of δ , φ_{η} , and $b_{\eta\omega}$ recover the corresponding $|B, q_{\eta}|_k$ and $|\bar{N}, q_{\eta}|_k$ conclusions. Thus, the result unifies these cases within the matrix framework considered in the manuscript without altering their underlying assumptions and clarifies how the general theorem reduces under the specified parameter choices.

6 Limitations

The result is established under specific assumptions on the positive normal matrix B , the almost increasing sequence (X_{η}) , the factor sequence (ξ_{η}) , the sequence (z_{η}) , and the parameters k and δ . These conditions restrict the immediate scope of the theorem to the stated $\varphi\text{-}|B; \delta|_k$ framework. Further work may examine whether some matrix or sequence conditions can be weakened and whether analogous factor theorems remain valid for broader classes of absolute matrix summability methods.

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