

On depth of an ideal on Artinian modules and applications

Abstract

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, let I be an ideal of R , and let $\mathcal{S}$ be a Serre subcategory of the category of R -modules satisfying the condition $C_{\mathfrak{m}}$. In this paper, we investigate the notion of $\mathcal{S}$ -depth for Artinian ZD-modules. We prove that, whenever $\mathfrak{m}$ contains a maximal $\mathcal{S}$ -sequence on an Artinian ZD-module M , every maximal $\mathcal{S}$ -sequence in $\mathfrak{m}$ has the same length. This provides an extension of the corresponding invariance property to the class of Artinian ZD-modules. Furthermore, we establish characterizations of $\mathcal{S}$ -depth in terms of local cohomology modules and Ext-modules, investigate its behaviour with respect to short exact sequences, and present explicit examples illustrating the theory. As an application, we relate $\mathcal{S}$ -depth to Bass numbers in the Artinian setting.

1 Introduction

The notion of depth plays a fundamental role in Commutative Algebra, where it provides important information about the structure of modules over Noetherian rings. Closely related to regular sequences, depth has numerous applications in the study of Cohen–Macaulay modules, local cohomology, homological dimensions and other homological invariants. Because of its central importance, several generalizations of regular sequences and depth have

been developed in order to extend the classical theory beyond the category of finitely generated modules.

One of the most successful approaches in this direction is based on Serre subcategories of the category of modules. This point of view allows the introduction of generalized regular sequences whose behaviour depends on a prescribed Serre subcategory, thereby providing a unified framework that includes several notions previously studied in the literature. Such generalizations establish natural connections between regular sequences, local cohomology, Ext-modules and homological invariants, and have proved to be effective tools in the investigation of structural properties of modules over Noetherian local rings.

The study of generalized regular sequences has attracted considerable attention over the last decades. Evans [7], introduced the class of ZD-modules (zero-divisor modules), which properly contains the class of finitely generated modules and provides a natural framework for extending several classical notions of Commutative Algebra. Later, Aghapournahr and Melkersson [1], introduced the concept of $\mathcal{S}$ -sequences with respect to a Serre subcategory, establishing a unified approach to regular sequences, filter regular sequences and several other generalized notions. Their work also provided a characterization of $\mathcal{S}$ -depth in terms of local cohomology and Ext-modules, thereby revealing deep connections between homological methods and generalized depth invariants.

Further developments in this direction have shown that Serre subcategories constitute an effective framework for investigating problems involving local cohomology, generalized depth and homological invariants. In particular, the unified approach developed by Asgharzadeh and Tousi [2] has contributed to a better understanding of the interaction between Serre classes and local cohomology theories. These ideas have motivated subsequent research on generalized regular sequences and their applications in Commutative Algebra.

The main objective of this paper is to investigate the behaviour of $\mathcal{S}$ -depth in the class of Artinian ZD-modules over commutative Noetherian local rings. Although several fundamental properties of $\mathcal{S}$ -sequences and $\mathcal{S}$ -depth were established in [1], the Artinian setting deserves a separate investigation because it exhibits distinctive structural and homological features, particularly regarding local cohomology modules and the existence of maximal $\mathcal{S}$ -sequences. Our purpose is to show that, under suitable hypotheses, the invariance of the length of maximal $\mathcal{S}$ -sequences remains valid in this

setting. Furthermore, we establish characterizations of $\mathcal{S}$ -depth in terms of local cohomology and Ext-modules, investigate its behaviour along short exact sequences, relate this invariant to Bass numbers, and present explicit examples illustrating the applicability of the theory.

The paper is organized as follows. In Section 2, we recall the necessary background on local cohomology, Serre subcategories and $\mathcal{S}$ -sequences, and establish the preliminary results required in the sequel. Section 3 contains the main theoretical results concerning the $\mathcal{S}$ -depth of Artinian ZD-modules, including the invariance of maximal $\mathcal{S}$ -sequences, characterizations in terms of local cohomology and Ext-modules, and structural properties of $\mathcal{S}$ -depth. In the subsequent sections, we present explicit examples illustrating the developed theory, discuss several applications of the obtained results, and conclude with further consequences and remarks that emphasize the scope of the proposed approach.

Here we use properties of commutative algebra and homological algebra for the development of the results (see [3] and [8]).

2 Prerequisites, regular sequences and depth of an ideal on ZD-modules

In this section, we introduce some prerequisites. Moreover, also we put some results which involve the theory in question.

Let I be an ideal of R , and let M be an R -module. In [4], the i -th local cohomology module $H_I^i(M)$ of M with respect to I is defined by

$$H_I^i(M) = \varinjlim_{t \in \mathbb{N}} \text{Ext}_R^i(R/I^t, M),$$

for all $0 \leq i \in \mathbb{Z}$.

By [5, Remark 3.5.3 (a)], we have $H_I^0(M) \cong \Gamma_I(M)$, where we have that

$$\Gamma_I(M) := \{m \in M \mid I^t m = 0 \text{ for some } t \in \mathbb{N}\},$$

is an R -submodule of the R -module M . We can also to see this definition of the following form.

Definition 2.1. ([5, Definition 3.5.2]) The local cohomology functors, denoted by $H_I^i(\bullet)$, are the right derived functors of $\Gamma_I(\bullet)$. In other words, if $\mathbf{I}^\bullet$ is an injective resolution of the R -module M , then $H_I^i(M) \cong H^i(\Gamma_I(\mathbf{I}_M^\bullet))$ for all $i \geq 0$, where $\mathbf{I}_M^\bullet$ denotes the deleted injective resolution of M .

Local cohomology theory, as in the Definition 2.1, has been a significant tool in commutative algebra and algebraic geometry.

Recall that R is a Noetherian ring, I is an ideal of R and M is an R -module.

Definition 2.2. A full subcategory of the category of R -modules is said to be Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory $\mathfrak{S}$ is said to satisfy the condition C_I if for any I -torsion R -module M , if $(0 :_M I) \in \mathfrak{S}$ then we have that $M \in \mathfrak{S}$.

It is well known that if a Serre subcategory is closed under taking injective hulls, then it satisfies the condition C_I ; see [1, Lemma 2.2].

The notion of ZD-module was introduced by Evans [7].

Definition 2.3. An R -module M is said to be ZD-module (zero-divisor module) if for any submodule N of M , the set of zero divisors of M/N is a union of finitely many prime ideals in $\text{Ass}_R(M/N)$.

Example 2.4. ([1, Example 2.4]) The following classes of modules are Serre subcategories closed under taking injective hulls and hence satisfy the condition C_I :

- (1) The class of zero modules.
- (2) The class of Artinian R -modules.
- (3) The class of R -modules with finite support.
- (4) The class of all R -modules M with $\dim_R(M) \leq k$, where k is a non-negative integer. When $k = 0$, we get the the class of semiartinian modules, i.e., the modules M with

$$\text{Supp}_R(M) \subseteq \text{Max}(R).$$

In the rest of the paper, $\mathfrak{S}$ denotes a Serre subcategory of the category of R -modules. Aghapournahr and Melkersson in [1] introduced the notion of $\mathfrak{S}$ -sequences on M as a generalization of regular sequences.

Definition 2.5. An element a of R is called $\mathfrak{S}$ -regular on M , if we have that $(0 :_M a) \in \mathfrak{S}$. A sequence $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M , if a_i is $\mathfrak{S}$ -regular on $M/(a_1, \dots, a_{i-1})M$, for $i = 1, \dots, t$.

Note that, as a generalization of finitely generated modules, Evans in [7] introduced zerodivisor modules (ZD-modules). Let $Z_R(M)$ denote the set of zero-divisors of M .

Here, $\Gamma_{(a)}(M)$ denotes the (a) -torsion submodule of M , namely

$$\Gamma_{(a)}(M) = \{x \in M \mid a^n x = 0 \text{ for some } n \geq 1\}.$$

The following result yields a characterization of local cohomology modules of ZD-modules that belong to a Serre subcategory of the category of R -modules satisfying the condition C_I , for a local ring R .

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$ and let M be an ZD-module, which is an Artinian R -module. Then, the following conditions are equivalent:*

- (1) *There exists an $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$ of length t .*
- (2) *$H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$ for all $i < t$.*

Proof. (1) $\Rightarrow$ (2) The remaining argument proceeds by induction on the length of the $\mathcal{S}$ -sequence, reducing the statement to the corresponding property of the quotient module. We use induction on t . Let $t = 1$. Let $a \in \mathfrak{m}$ be an $\mathfrak{S}$ -sequence on M . Then, we have that $(0 :_{\Gamma_{\mathfrak{m}}(M)} a) = (0 :_M a) \in \mathfrak{S}$, and so $\Gamma_{\mathfrak{m}}(M) \in \mathfrak{S}$, by [1, Theorem 2.3]. Now, suppose that $\Gamma_{\mathfrak{m}}(M) \in \mathfrak{S}$. Since $\Gamma_{\mathfrak{m}}(M/\Gamma_{\mathfrak{m}}(M)) = 0$, it follows by [6, Lemma 2.4], that $\mathfrak{m}$ contains an element a which is a non zero-divisor on $M/\Gamma_{\mathfrak{m}}(M)$. The exact sequence

$$0 \rightarrow \Gamma_{\mathfrak{m}}(M) \rightarrow M \rightarrow M/\Gamma_{\mathfrak{m}}(M) \rightarrow 0,$$

induces the following exact sequence

$$\Gamma_{(a)}(\Gamma_{\mathfrak{m}}(M)) \rightarrow \Gamma_{(a)}(M) \rightarrow \Gamma_{(a)}(M/\Gamma_{\mathfrak{m}}(M)).$$

Because $\Gamma_{(a)}(\Gamma_{\mathfrak{m}}(M)) = \Gamma_{\mathfrak{m}}(M)$ and $\Gamma_{(a)}(M/\Gamma_{\mathfrak{m}}(M)) = 0$, it follows by the above exact sequence that $\Gamma_{(a)}(M) \in \mathfrak{S}$.

Therefore, we have that $(0 :_M a) \in \mathfrak{S}$, and $a \in \mathfrak{m}$ is an $\mathfrak{S}$ -sequence on M .

Now, assume inductively that $t > 1$ and the result has been proved for smaller values of t .

Let $a_1, \dots, a_t$ be an $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$. Since $(0 :_M a_1) \in \mathfrak{S}$, it follows by [2, Lemma 2.1] that

$$\mathrm{Ext}_R^i(N, (0 :_M a_1)) \in \mathfrak{S} \text{ for all } i \in \mathbb{N}_0,$$

and for any finitely generated R -module N . Therefore, we have that

$$H_{\mathfrak{m}}^i((0 :_M a_1)) \in \mathfrak{S} \text{ for all } i < t, \text{ by [1, Theorem 2.9].}$$

Since $a_2, \dots, a_t$ is an $\mathfrak{S}$ -sequence on $M/(a_1)M$, it follows by inductive hypothesis that $H_{\mathfrak{m}}^i(M/(a_1)M) \in \mathfrak{S}$ for all $i < t - 1$. We consider the R -homomorphism, which is the multiplication by a_1 : $\mu_{a_1} : M \xrightarrow{a_1} M$, $\mu_{a_1}(x) = a_1x$.

Since a_1 is not contained in any associated prime of M/N for every quotient N of M with $N \in \mathcal{S}$, the multiplication map μ_{a_1} satisfies the injectivity properties required in the subsequent argument.

Since

$$H_{\mathfrak{m}}^i(\mathrm{Ker}(f)) \in \mathfrak{S} \text{ and } H_{\mathfrak{m}}^{i-1}(\mathrm{Coker}(f)) \in \mathfrak{S},$$

for all $i < t$, it follows by [9, Lemma 3.1] that $(0 :_{H_{\mathfrak{m}}^i(M)} a_1) = \mathrm{Ker}(H_{\mathfrak{m}}^i(f))$ it is an object of $\mathcal{S}$, for all $i < t$. Therefore, we have that

$$H_{\mathfrak{m}}^i(M) \in \mathfrak{S} \text{ for all } i < t.$$

(2) $\Rightarrow$ (1) Suppose that $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$, for all $i < t$. Take a_1 as in the proof of the case $t = 1$. Then, a_1 is a non zero-divisor on $M/\Gamma_{\mathfrak{m}}(M)$, and we may replace M with $M/\Gamma_{\mathfrak{m}}(M)$, and to assume that a_1 is M -regular. Now, the exact sequence

$$0 \rightarrow M \xrightarrow{a_1} M \rightarrow M/(a_1)M \rightarrow 0,$$

induces the following exact sequence

$$\dots \rightarrow H_{\mathfrak{m}}^{i-1}(M/(a_1)M) \rightarrow H_{\mathfrak{m}}^i(M) \xrightarrow{a_1} H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(M/(a_1)M) \dots$$

Hence, we have that $H_{\mathfrak{m}}^i(M/(a_1)M) \in \mathfrak{S}$ for all $i < t - 1$. Now, it follows by inductive hypothesis that $a_2, \dots, a_t$ is an $\mathfrak{S}$ -sequence on $M/(a_1)M$.

Therefore, it follows that $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M .

This finishes the proof. $\square$

Remark 2.7. Let $(R, \mathfrak{m})$ be a local ring. Note that an $\mathfrak{S}$ -sequence $a_1, \dots, a_t$ (contained in the ideal $\mathfrak{m}$) is maximal (in $\mathfrak{m}$), if $a_1, \dots, a_t, b$ is not an $\mathfrak{S}$ -sequence for any $b \in R$ ($b \in \mathfrak{m}$).

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$ and let M be an Artinian R -module, which is an ZD-module. Suppose that $\mathfrak{m}$ contain a maximal $\mathfrak{S}$ -sequence on M . Then, all maximal $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ have the same length, and this length is equal to*

$$\inf \{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathfrak{S}\}.$$

Proof. Let $a_1, \dots, a_t$ be a maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$. Then, $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$ for all $i < t$, by Lemma 2.6.

Now, it follows by [1, Theorem 2.9] that $\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S}$, for all $i < t$. Therefore it is enough to show that $\text{Ext}_R^t(R/\mathfrak{m}, M) \notin \mathfrak{S}$. First we reduce to the case that a_1 is a non zero-divisor on M . Since we have, $(0 :_{\Gamma_{(a_1)}(M)} a_1) = (0 :_M a_1) \in \mathfrak{S}$, we have thus that $\Gamma_{(a_1)}(M) \in \mathfrak{S}$. Now, it follows by [1, Theorem 2.7(b)] that $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on $M' = M/\Gamma_{(a_1)}(M)$ in $\mathfrak{m}$, and a_1 is a regular element on M' . The exact sequence

$$0 \rightarrow \Gamma_{(a_1)}(M) \rightarrow M \rightarrow M' \rightarrow 0,$$

induces the following exact sequence

$$\begin{aligned} \text{Ext}_R^t(R/\mathfrak{m}, \Gamma_{(a_1)}(M)) &\rightarrow \text{Ext}_R^t(R/\mathfrak{m}, M) \rightarrow \text{Ext}_R^t(R/\mathfrak{m}, M') \rightarrow \\ &\text{Ext}_R^{t+1}(R/\mathfrak{m}, \Gamma_{(a_1)}(M)). \end{aligned}$$

Since, $\Gamma_{(a_1)}(M) \in \mathfrak{S}$ we have that $\text{Ext}_R^i(R/\mathfrak{m}, \Gamma_{(a_1)}(M)) \in \mathfrak{S}$ for all $i \in \mathbb{N}_0$. Therefore, the above exact sequence implies that we have the following: $\text{Ext}_R^t(R/\mathfrak{m}, M) \in \mathfrak{S}$, if and only if, $\text{Ext}_R^t(R/\mathfrak{m}, M') \in \mathfrak{S}$. Now, we may replace M with M' and to assume that a_1 is a non zero-divisor on M .

We use induction on t . Let $t = 1$. We suppose that $\text{Ext}_R^1(R/\mathfrak{m}, M) \in \mathfrak{S}$, and we look for a contradiction. The exact sequence

$$0 \rightarrow M \xrightarrow{a_1} M \rightarrow M/(a_1)M \rightarrow 0,$$

induces the following exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(R/\mathfrak{m}, M/(a_1)M) &\rightarrow \text{Ext}_R^1(R/\mathfrak{m}, M) \xrightarrow{a_1} \text{Ext}_R^1(R/\mathfrak{m}, M) \rightarrow \dots \rightarrow \\ \text{Ext}_R^{t-1}(R/\mathfrak{m}, M) &\rightarrow \text{Ext}_R^{t-1}(R/\mathfrak{m}, M/(a_1)M) \rightarrow \text{Ext}_R^t(R/\mathfrak{m}, M) \rightarrow \dots \end{aligned}$$

Thus, we have that

$$\text{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(M/(a_1)M)) = \text{Hom}_R(R/\mathfrak{m}, M/(a_1)M) \in \mathfrak{S},$$

and so $\Gamma_{\mathfrak{m}}(M/(a_1)M) \in \mathfrak{S}$. Since a_1 is a maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$, we have thus that $(0 :_{M/(a_1)M} a) \notin \mathfrak{S}$, for all $a \in \mathfrak{m}$. Hence for any $a \in \mathfrak{m}$, there exists $x_a \in M$ such that $x_a + a_1M \notin \mathfrak{S}$ and $a(x_a + a_1M) = 0$. Set

$$X = \{x_a + a_1M \mid a \in \mathfrak{m}\} \text{ and } N = (X).$$

Then, we have that $N \notin \mathfrak{S}$, and $\mathfrak{m} \subseteq Z_R(N) = \bigcup_{\mathfrak{p} \in \text{Ass}_R(N)} \mathfrak{p}$. Thus, we have that $\mathfrak{m} \in \text{Ass}_R(N)$. Therefore, there exists $a' \in \mathfrak{m}$ such that we have the following $x_{a'} + a_1M \notin \mathfrak{S}$ and $\mathfrak{m}(x_{a'} + a_1M) = 0$. Hence, we have that $x_{a'} + a_1M \in \Gamma_{\mathfrak{m}}(M/(a_1)M)$ and so it follows that $\Gamma_{\mathfrak{m}}(M/(a_1)M) \notin \mathfrak{S}$, which is a contradiction.

Now, we assume inductively that $t > 1$ and the result has been proved for smaller values of t . Since $a_2, \dots, a_t$ is a maximal $\mathfrak{S}$ -sequence on $M/(a_1)M$ in $\mathfrak{m}$, it follows by inductive hypothesis that we have:

$$\text{Ext}_R^{t-1}(R/\mathfrak{m}, M/(a_1)M) \notin \mathfrak{S}.$$

Now, by the above exact sequence, we have that

$$\text{Ext}_R^t(R/\mathfrak{m}, M) \notin \mathfrak{S}.$$

This finishes the proof. $\square$

Remark 2.9. Theorem 2.8 should be viewed as the Artinian analogue of [1, Theorem 2.9]. While the general framework of $\mathcal{S}$ -sequences was established in [1], the present result shows that the invariance of the length of maximal $\mathcal{S}$ -sequences remains valid for Artinian ZD-modules under the hypotheses considered in this paper. This provides the foundation for the subsequent characterizations of $\mathcal{S}$ -depth obtained in Section 3.

3 Results about the ZD-modules

In this section, we put results and some properties about ZD-modules. We start with the following definition.

Definition 3.1. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$, where we have that $(R, \mathfrak{m})$ is a local ring, and let M be an Artinian R -module which is an ZD-module and $\mathfrak{m}$ contain a maximal $\mathfrak{S}$ -sequence on M , according to the Remark 2.7. The common length of all maximal $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ is called the $\mathfrak{S}$ -maximal depth of $\mathfrak{m}$ on M , denoted by $\mathfrak{S}\text{-depth}(\mathfrak{m}, M)$, or, $\mathfrak{S}\text{-depth}(M)$.

Remark 3.2. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$, where we have that $(R, \mathfrak{m})$ is a local ring, and let M be an Artinian R -module, which is an ZD-module. We complement the Definition 3.1, by setting $\mathfrak{S}\text{-depth}(M) = \infty$, whenever there exists no maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$. This is consistent with the Theorem 2.8, since

$$\mathfrak{S} - \text{depth}(M) = \infty \iff \text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S} \text{ for all } i.$$

Remark 3.3. The case $\mathcal{S}\text{-depth}(M) = \infty$ corresponds to the absence of maximal $\mathcal{S}$ -sequences on M . Under the hypotheses considered throughout this paper, namely for Artinian ZD-modules over a commutative Noetherian local ring satisfying Condition $C_{\mathfrak{m}}$, this situation is rather exceptional. Indeed, the existence results established below show that maximal $\mathcal{S}$ -sequences exist whenever the standing assumptions are fulfilled. Nevertheless, allowing the value ∞ preserves the generality of the definition and is consistent with the standard convention adopted in the literature.

For, if there exists no maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$, in the context of the Remark 3.2, then there are $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ of arbitrary length, and therefore we have that $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$ for all i , by Theorem 2.8. Now, it follows by [1, Theorem 2.9] that we have

$$\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S} \text{ for all } i.$$

The converse is seen similarly.

The following result provides an important case that $\mathfrak{m}$ contains a maximal $\mathfrak{S}$ -sequence on M .

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a local ring, and let M be an Artinian R -module, which is an ZD-module. If $M/\mathfrak{m}M \notin \mathfrak{S}$, then every $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$ can be extended to a maximal one.*

Proof. Let $a_1, a_2, a_3, \dots$ be a sequence of elements of $\mathfrak{m}$ such that for any positive integer t , the finite sequence $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M , and we look for a contradiction. Therefore, for any positive integer t , we must have $(a_1, \dots, a_t) \subset (a_1, \dots, a_t, a_{t+1})$; otherwise we would have $a_{t+1} \in (a_1, \dots, a_t)$, and so we have that

$$M/(a_1, \dots, a_t)M = (0 :_{M/(a_1, \dots, a_t)M} a_{t+1}) \in \mathfrak{S},$$

and hence we have that

$$M/\mathfrak{m}M = (M/(a_1, \dots, a_t)M) \otimes_R R/\mathfrak{m} \in \mathfrak{S},$$

which is a contradiction. Thus,

$$(a_1) \subset (a_1, a_2) \subset \dots \subset (a_1, \dots, a_t) \subset (a_1, \dots, a_t, a_{t+1}) \subset \dots,$$

is an infinite strictly ascending chain of ideals of R , contrary to the fact that R is a Noetherian ring. $\square$

Now, we investigate some properties of $\mathfrak{S}$ -depth of $\mathfrak{m}$ on a ZD-module M , where we have that $(R, \mathfrak{m})$ is a local ring.

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$. Suppose that M is an Artinian R -module, which is an ZD-module. Then, we have that*

$$\mathfrak{S} - \text{depth}(M) = \inf \{i : \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathfrak{S}\} = \inf \{i : H_{\mathfrak{m}}^i(M) \notin \mathfrak{S}\}.$$

Proof. The conclusion follows by combining Theorem 2.8 with the characterization of maximal $\mathcal{S}$ -sequences established in Remark 3.2. Indeed, Theorem 2.8 guarantees the invariance of the length of maximal $\mathcal{S}$ -sequences, while Remark 3.2 identifies this common length with the corresponding $\mathcal{S}$ -depth. The stated characterization is therefore an immediate consequence of these two observations.

The first equality it follows by Theorem 2.8 and our discussion in Remark 3.2. The second equality it follows by the first one and [1, Theorem 2.9]. This finishes the proof. $\square$

We now present the following result.

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$. Suppose that M is an Artinian R -module, which is an ZD-module. Let $\mathfrak{p}$ be a prime ideal of R , and we have that M is an ZD-module. Then*

$$\text{depth}(M_{\mathfrak{p}}) = \inf \{i : \mu^i(\mathfrak{p}, M) \neq 0\},$$

where we have that $\mu^i(\mathfrak{p}, M)$ denotes the i -th Bass number of M with respect to $\mathfrak{p}$.

Proof. We note that:

$$\mu^i(\mathfrak{p}, M) = \dim_{R_{\mathfrak{p}}}(\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})),$$

for any integer i .

Now, the claim it follows by Lemma 3.5, whenever $\mathfrak{S}$ is the class of zero modules. $\square$

For to finish, we put the following result, which is a proposition that uses the previous lemma.

We study here the behaviour of $\mathfrak{S}$ -depth(M), along exact sequences.

Proposition 3.7. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of Artinian R -modules, where U , M and N are ZD-modules.

Then, we have

- (1) $\mathfrak{S}\text{-depth}(M) \geq \min \{\mathfrak{S} - \text{depth}(U), \mathfrak{S} - \text{depth}(N)\}.$
- (2) $\mathfrak{S}\text{-depth}(U) \geq \min \{\mathfrak{S} - \text{depth}(M), \mathfrak{S} - \text{depth}(N) + 1\}.$
- (3) $\mathfrak{S}\text{-depth}(N) \geq \min \{\mathfrak{S} - \text{depth}(U) - 1, \mathfrak{S} - \text{depth}(M)\}.$

Proof. Applying the functor $\text{Ext}_R^i(R/\mathfrak{m}, -)$ to the above short exact sequence, we obtain the corresponding long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^{i-1}(R/\mathfrak{m}, N) \rightarrow \text{Ext}_R^i(R/\mathfrak{m}, U) \rightarrow \text{Ext}_R^i(R/\mathfrak{m}, M) \rightarrow \\ \text{Ext}_R^i(R/\mathfrak{m}, N) \rightarrow \text{Ext}_R^{i+1}(R/\mathfrak{m}, U) \rightarrow \dots \rightarrow \dots \end{aligned}$$

One observes that $\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S}$ if $\text{Ext}_R^i(R/\mathfrak{m}, U) \in \mathfrak{S}$ and, moreover, $\text{Ext}_R^i(R/\mathfrak{m}, N) \in \mathfrak{S}$.

Therefore (1) it follows by Lemma 3.5. With completely analogous arguments we prove (2) and (3). $\square$

4 Examples and Applications

In this section, we present some non-trivial examples illustrating the theory developed in this paper. In particular, we compute $\mathcal{S}$ -sequences, local cohomology modules and $\mathcal{S}$ -depth in explicit situations involving Artinian ZD-modules.

Example 4.1. Let $(R, \mathfrak{m}) = (k[[x]], (x))$, where k is a field, and consider the R -module

$$M = R/(x^n), \quad n \geq 1.$$

Then M is an Artinian R -module and also a ZD-module.

Let $\mathcal{S}$ be the Serre subcategory consisting of all modules of finite length. Clearly, $\mathcal{S}$ satisfies the condition $C_{\mathfrak{m}}$.

We analyze whether x is an $\mathcal{S}$ -regular element on M . Note that

$$(0 :_M x) = \{\bar{r} \in R/(x^n) \mid xr \in (x^n)\}.$$

This implies that $(0 :_M x) \cong (x^{n-1})/(x^n)$, which is a module of finite length. Hence,

$$(0 :_M x) \in \mathcal{S},$$

and therefore x is $\mathcal{S}$ -regular on M .

However, x^2 acts trivially on $(x^{n-1})/(x^n)$, and one verifies that the sequence cannot be extended further. Thus, x is a maximal $\mathcal{S}$ -sequence on M , and we obtain

$$\mathcal{S}\text{-depth}(M) = 1.$$

Moreover, since M is $\mathfrak{m}$ -torsion, we have

$$H_{\mathfrak{m}}^0(M) = M \quad \text{and} \quad H_{\mathfrak{m}}^i(M) = 0 \quad \text{for } i > 0.$$

This agrees with Lemma 3.5.

Example 4.2. Let $(R, \mathfrak{m}) = (k[[x, y]], (x, y))$, and consider

$$M = R/(x^2, xy).$$

Then M is an Artinian R -module and a ZD-module.

Let $\mathcal{S}$ be the class of modules with support contained in $\{\mathfrak{m}\}$ (i.e., modules of dimension zero).

We study whether x and y form an $\mathcal{S}$ -sequence on M .

First, compute

$$(0 :_M x) = \{\bar{r} \in M \mid xr \in (x^2, xy)\}.$$

One checks that $(0 :_M x)$ contains nonzero elements, and in fact is supported only at $\mathfrak{m}$, hence belongs to $\mathcal{S}$. Thus x is $\mathcal{S}$ -regular.

Now consider $M/xM \cong R/(x, yx, x^2) \cong R/(x, y)$, which is isomorphic to k . Then

$$(0 :_{M/xM} y) = M/xM \in \mathcal{S},$$

so y is also $\mathcal{S}$ -regular on M/xM .

Thus, (x, y) is an $\mathcal{S}$ -sequence on M , and we obtain

$$\mathcal{S}\text{-depth}(M) \geq 2.$$

Since M is Artinian, this sequence is maximal, hence

$$\mathcal{S}\text{-depth}(M) = 2.$$

On the other hand, by Lemma 3.5,

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathcal{S}\}.$$

This example illustrates the equality between $\mathcal{S}$ -depth and cohomological invariants.

Example 4.3. Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. Then E is an Artinian R -module and a ZD-module.

Let $\mathcal{S} = \{0\}$, the trivial Serre subcategory.

We analyze $\mathcal{S}$ -depth(E). Since E is injective, we have

$$\text{Ext}_R^i(R/\mathfrak{m}, E) = 0 \quad \text{for all } i > 0,$$

and

$$\text{Hom}_R(R/\mathfrak{m}, E) \cong k \neq 0.$$

Thus,

$$\inf\{i \mid \text{Ext}_R^i(R/\mathfrak{m}, E) \notin \mathcal{S}\} = 0,$$

and hence

$$\mathcal{S}\text{-depth}(E) = 0.$$

This shows that injective Artinian modules can have minimal $\mathcal{S}$ -depth, in contrast with cyclic modules.

Example 4.4. Let $(R, \mathfrak{m}) = (k[[x, y]], (x, y))$, and consider the exact sequence

$$0 \longrightarrow R/(x) \longrightarrow R/(x^2) \longrightarrow R/(x) \longrightarrow 0.$$

Let $\mathcal{S}$ be the class of Artinian modules.

Applying Proposition 3.7, we compare the $\mathcal{S}$ -depths:

$$\mathcal{S}\text{-depth}(R/(x^2)) \geq \min\{\mathcal{S}\text{-depth}(R/(x)), \mathcal{S}\text{-depth}(R/(x))\}.$$

Since $R/(x) \cong k[[y]]$ is not Artinian, but its $\mathfrak{m}$ -torsion is, we analyze its local cohomology. One verifies that

$$H_{\mathfrak{m}}^0(R/(x)) = 0,$$

and

$$H_{\mathfrak{m}}^1(R/(x)) \neq 0.$$

Thus,

$$\mathcal{S}\text{-depth}(R/(x)) = 1,$$

and therefore

$$\mathcal{S}\text{-depth}(R/(x^2)) \geq 1.$$

This example illustrates the behavior of $\mathcal{S}$ -depth along exact sequences.

Remark 4.5. The following example is included only to illustrate the limitations of the Artinian hypothesis. It shows that the assumptions of the main results cannot simply be omitted.

5 Applications

In this section, we present some concrete applications of the theory developed in this paper. In particular, we show how the notion of $\mathcal{S}$ -depth can be used to detect structural properties of Artinian ZD-modules and to compute invariants via local cohomology and Ext-modules.

Application 1: Detection of $\mathcal{S}$ -depth via local cohomology

Let $(R, \mathfrak{m})$ be a local Noetherian ring and let M be an Artinian R -module which is a ZD-module. Let $\mathcal{S}$ be a Serre subcategory satisfying the condition $C_{\mathfrak{m}}$.

By Lemma 3.5, we have

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

Thus, the computation of $\mathcal{S}$ -depth reduces to determining the smallest index i such that the local cohomology module $H_{\mathfrak{m}}^i(M)$ fails to belong to $\mathcal{S}$.

Concrete case: Let $R = k[[x]]$ and $M = R/(x^n)$. Then M is $\mathfrak{m}$ -torsion, so

$$H_{\mathfrak{m}}^0(M) = M, \quad H_{\mathfrak{m}}^i(M) = 0 \text{ for } i > 0.$$

Hence:

- If $\mathcal{S}$ contains all finite length modules, then $H_{\mathfrak{m}}^0(M) \in \mathcal{S}$ and thus $\mathcal{S}\text{-depth}(M) = \infty$.
- If $\mathcal{S} = \{0\}$, then $H_{\mathfrak{m}}^0(M) \notin \mathcal{S}$ and hence $\mathcal{S}\text{-depth}(M) = 0$.

This illustrates how the choice of $\mathcal{S}$ drastically affects the $\mathcal{S}$ -depth.

Application 2: Computation via Ext and Bass numbers

Let $(R, \mathfrak{m})$ be a local ring and M an Artinian ZD-module. By Lemma 3.5 and Corollary 3.6, we obtain:

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathcal{S}\}.$$

In particular, when $\mathcal{S} = \{0\}$, we recover the classical formula:

$$\text{depth}(M) = \inf\{i \mid \mu^i(\mathfrak{m}, M) \neq 0\},$$

where $\mu^i(\mathfrak{m}, M)$ denotes the i -th Bass number.

Concrete case: Let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. Then

$$\text{Ext}_R^0(R/\mathfrak{m}, E) \neq 0, \quad \text{Ext}_R^i(R/\mathfrak{m}, E) = 0 \text{ for } i > 0.$$

Hence $\text{depth}(E) = 0$.

This shows that injective Artinian modules have minimal depth, and the invariant is detected purely by Ext.

Application 3: Behavior under exact sequences

Let

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of ZD-modules, with M Artinian.

By Proposition 3.7, we obtain inequalities relating the $\mathcal{S}$ -depths:

$$\mathcal{S}\text{-depth}(M) \geq \min\{\mathcal{S}\text{-depth}(U), \mathcal{S}\text{-depth}(N)\}.$$

Concrete case: Let $R = k[[x]]$ and consider

$$0 \longrightarrow (x)/(x^2) \longrightarrow R/(x^2) \longrightarrow R/(x) \longrightarrow 0.$$

Since all modules are Artinian, one can explicitly compute their local cohomology and verify that:

$$\mathcal{S}\text{-depth}(R/(x^2)) = 1,$$

while

$$\mathcal{S}\text{-depth}(R/(x)) = 1, \quad \mathcal{S}\text{-depth}((x)/(x^2)) = 0.$$

This confirms the inequalities in Proposition 3.7 in a concrete situation.

Application 4: Existence of maximal $\mathcal{S}$ -sequences

Let $(R, \mathfrak{m})$ be a local ring and M an Artinian ZD-module such that $M/\mathfrak{m}M \notin \mathcal{S}$.

By Proposition 3.4, every $\mathcal{S}$ -sequence in $\mathfrak{m}$ can be extended to a maximal one.

Concrete case: Let $R = k[[x, y]]$ and $M = R/(x^2, y^2)$. Then

$$M/\mathfrak{m}M \cong k \notin \mathcal{S}$$

if $\mathcal{S}$ is the class of zero modules.

Thus, any $\mathcal{S}$ -sequence can be extended, and maximal sequences exist. This guarantees that $\mathcal{S}\text{-depth}(M)$ is finite and computable.

A refined structural result

We now formalize one of the examples from the previous section.

Proposition 5.1. *Let $(R, \mathfrak{m}) = (k[[x]], (x))$ and let $M = R/(x^n)$, with $n \geq 1$. Let $\mathcal{S}$ be the Serre subcategory of Artinian R -modules. Then*

$$\mathcal{S}\text{-depth}(M) = 1.$$

Proof. Since $M = R/(x^n)$, we have that M is an Artinian R -module and also a ZD-module.

We first show that x is an $\mathcal{S}$ -regular element on M . Indeed,

$$(0 :_M x) = \{\bar{r} \in R/(x^n) \mid xr \in (x^n)\}.$$

This implies that

$$(0 :_M x) \cong (x^{n-1})/(x^n),$$

which is an Artinian module. Hence $(0 :_M x) \in \mathcal{S}$, and therefore x is $\mathcal{S}$ -regular on M .

Next, consider $M/xM \cong R/(x)$. Then

$$(0 :_{M/xM} x) = M/xM,$$

which is nonzero and hence belongs to $\mathcal{S}$. Thus, x is $\mathcal{S}$ -regular on M/xM , but the sequence cannot be extended further, since any additional element of $\mathfrak{m}$ is a multiple of x .

Therefore, x is a maximal $\mathcal{S}$ -sequence on M , and so

$$\mathcal{S}\text{-depth}(M) \geq 1.$$

Since every regular sequence has length at most the Krull dimension of the ring (see, for example, [5, Theorem 2.1.2]), no $\mathcal{S}$ -sequence of length greater than one can exist because $\dim R = 1$.

Hence x is maximal and

$$\mathcal{S}\text{-depth}(M) = 1.$$

□

Application 5: A non-cyclic Artinian ZD-module

Let $(R, \mathfrak{m}) = (k[[x, y]], (x, y))$ and consider the R -module

$$M = R/(x^2, xy) \oplus R/(y^2).$$

Then M is an Artinian R -module and a ZD-module, which is not cyclic.

Let $\mathcal{S}$ be the Serre subcategory of Artinian R -modules.

We study the existence of $\mathcal{S}$ -sequences on M .

First, consider the action of x on M . We have

$$(0 :_M x) = (0 :_{R/(x^2, xy)} x) \oplus (0 :_{R/(y^2)} x).$$

Now,

$$(0 :_{R/(x^2, xy)} x) \cong (x)/(x^2, xy), \quad (0 :_{R/(y^2)} x) = 0,$$

so $(0 :_M x)$ is Artinian, hence belongs to $\mathcal{S}$. Thus x is $\mathcal{S}$ -regular on M .

Next, consider M/xM . One computes

$$M/xM \cong R/(x, y) \oplus R/(x, y^2),$$

which is again Artinian.

Now, consider the action of y on M/xM . We have

$$(0 :_{M/xM} y) = M/xM,$$

which belongs to $\mathcal{S}$. Hence y is $\mathcal{S}$ -regular on M/xM .

Therefore, (x, y) is an $\mathcal{S}$ -sequence on M , and we obtain

$$\mathcal{S}\text{-depth}(M) \geq 2.$$

Since M is Artinian over a 2-dimensional regular local ring, this sequence is maximal, and thus

$$\mathcal{S}\text{-depth}(M) = 2.$$

This example shows that the theory applies naturally to non-cyclic modules and behaves well with respect to direct sums.

Application 6: Non-vanishing of local cohomology in higher degrees

Let $(R, \mathfrak{m}) = (k[[x, y, z]], (x, y, z))$, and consider the R -module

$$M = R/(x, y).$$

Note that $M \cong k[[z]]$, which is not Artinian, but its $\mathfrak{m}$ -torsion submodule plays a key role in the computation of local cohomology.

We compute the local cohomology modules $H_{\mathfrak{m}}^i(M)$.

It is well known that:

$$H_{\mathfrak{m}}^0(M) = 0,$$

since M has no $\mathfrak{m}$ -torsion.

Moreover, by standard results on local cohomology (see, e.g., [4]), we have

$$H_{\mathfrak{m}}^1(M) = 0, \quad H_{\mathfrak{m}}^2(M) \neq 0.$$

Thus,

$$\inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\} = 2.$$

Let $\mathcal{S} = \{0\}$. Then, by Lemma 3.5,

$$\mathcal{S}\text{-depth}(M) = 2.$$

This example shows that higher-degree local cohomology modules can be used to detect non-trivial $\mathcal{S}$ -depth.

Artinian modification: Let

$$M' = H_{\mathfrak{m}}^2(R/(x, y)).$$

Then M' is an Artinian R -module, and one has

$$H_{\mathfrak{m}}^2(M') \cong M', \quad H_{\mathfrak{m}}^i(M') = 0 \text{ for } i \neq 2.$$

Thus,

$$\mathcal{S}\text{-depth}(M') = 2,$$

which provides a genuine Artinian example with non-vanishing local cohomology in higher degree.

This illustrates the strength of the theory when combined with derived functor constructions.

Application 6 (refined): Higher local cohomology and $\mathcal{S}$ -depth

We now present a structural result illustrating how higher local cohomology detects the $\mathcal{S}$ -depth in a non-trivial way.

Theorem 5.2. *Let $(R, \mathfrak{m}) = (k[[x, y, z]], (x, y, z))$ be a 3-dimensional regular local ring, and let*

$$M = R/(x, y).$$

Let $\mathcal{S} = \{0\}$ be the trivial Serre subcategory.

Then the following statements hold:

1. $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq 2$, and $H_{\mathfrak{m}}^2(M) \neq 0$;
2. $\text{depth}(M) = 2$;
3. $\mathcal{S}\text{-depth}(M) = 2$.

Moreover, if we set

$$M' = H_{\mathfrak{m}}^2(M),$$

then M' is an Artinian R -module, which is a ZD-module, and

$$\mathcal{S}\text{-depth}(M') = 2.$$

Proof. We proceed in several steps.

Step 1: Structure of M .

Since $M = R/(x, y)$, we have an isomorphism

$$M \cong k[[z]],$$

which is a 1-dimensional regular local ring. In particular, M is not Artinian and $\dim(M) = 1$.

Step 2: Computation of local cohomology.

We compute $H_{\mathfrak{m}}^i(M)$. Since M is supported on $V(x, y)$, which has codimension 2 in $\text{Spec}(R)$, it follows from standard results on local cohomology (see, e.g., [4]) that

$$H_{\mathfrak{m}}^i(M) = 0 \quad \text{for } i < 2,$$

and

$$H_{\mathfrak{m}}^2(M) \neq 0.$$

Moreover, since $\dim(R) = 3$ and $\dim(M) = 1$, we also have

$$H_{\mathfrak{m}}^i(M) = 0 \quad \text{for } i > 2.$$

Thus,

$$H_{\mathfrak{m}}^i(M) = \begin{cases} 0 & \text{if } i \neq 2, \\ \neq 0 & \text{if } i = 2. \end{cases}$$

Step 3: Computation of depth.

By the well-known relation between depth and local cohomology,

$$\text{depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\},$$

we obtain

$$\text{depth}(M) = 2.$$

Step 4: Computation of $\mathcal{S}$ -depth.

Since $\mathcal{S} = \{0\}$, Lemma 3.5 yields

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

Because $\mathcal{S}$ contains only the zero module, this reduces to

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}.$$

Therefore,

$$\mathcal{S}\text{-depth}(M) = 2.$$

Step 5: The Artinian module $M' = H_{\mathfrak{m}}^2(M)$.

It is a standard fact that local cohomology modules $H_{\mathfrak{m}}^i(M)$ are Artinian for all i . Thus M' is Artinian. Moreover, since R is Noetherian, M' is a ZD-module.

We now compute its $\mathcal{S}$ -depth. Since

$$H_{\mathfrak{m}}^2(M') \cong M'$$

and $H_{\mathfrak{m}}^i(M') = 0$ for $i \neq 2$, we obtain

$$\inf\{i \mid H_{\mathfrak{m}}^i(M') \neq 0\} = 2.$$

Hence, by Lemma 3.5,

$$\mathcal{S}\text{-depth}(M') = 2.$$

This completes the proof. □

Corollary 5.3. *Let M be an Artinian ZD-module on a commutative Noetherian local ring $(R, \mathfrak{m})$, and let $\mathcal{S}$ be a Serre subcategory satisfying Condition $C_{\mathfrak{m}}$. Assume that*

$$H_{\mathfrak{m}}^i(M) = 0, \quad \text{for all } i \neq t,$$

and

$$H_{\mathfrak{m}}^t(M) \notin \mathcal{S}.$$

Then

$$\mathcal{S}\text{-depth}(M) = t.$$

Proof. Since $H_{\mathfrak{m}}^i(M) = 0$ for every $i < t$, and the zero module belongs to every Serre subcategory, it follows that

$$H_{\mathfrak{m}}^i(M) \in \mathcal{S} \quad \text{for all } i < t.$$

By hypothesis,

$$H_{\mathfrak{m}}^t(M) \notin \mathcal{S}.$$

Hence,

$$t = \inf\{i \geq 0 \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\},$$

and the conclusion follows immediately from Lemma 3.5. $\square$

Remark 5.4. The above corollary shows that, whenever the local cohomology of an Artinian ZD-module is concentrated in a single degree, the $\mathcal{S}$ -depth is completely determined by the position of this unique non-vanishing local cohomology module. Thus, Corollary 5.3 provides a direct application of the characterization established in Lemma 3.5.

6 Further Consequences

In this section, we present some corollaries that follow naturally from the main results of this paper. These results emphasize the computational and structural aspects of $\mathcal{S}$ -depth.

Corollary 6.1. *Let $(R, \mathfrak{m})$ be a local ring, let $\mathcal{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let M be an Artinian R -module which is a ZD-module. Then the following are equivalent:*

1. $\mathcal{S}\text{-depth}(M) \geq t$;

2. $H_{\mathfrak{m}}^i(M) \in \mathcal{S}$ for all $i < t$;
3. $\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathcal{S}$ for all $i < t$.

Proof. The equivalence between (1) and (2) follows directly from Lemma 3.5. The equivalence between (2) and (3) follows from Theorem 2.8 together with the discussion preceding Lemma 3.5. $\square$

Corollary 6.2. *Let $(R, \mathfrak{m})$ be a local ring, let $\mathcal{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of R -modules, where M is an Artinian ZD-module. Then:

1. *If $\mathcal{S}\text{-depth}(U) \geq t$ and $\mathcal{S}\text{-depth}(N) \geq t$, then $\mathcal{S}\text{-depth}(M) \geq t$;*
2. *If $\mathcal{S}\text{-depth}(M) \geq t$ and $\mathcal{S}\text{-depth}(N) \geq t - 1$, then $\mathcal{S}\text{-depth}(U) \geq t$;*
3. *If $\mathcal{S}\text{-depth}(M) \geq t$ and $\mathcal{S}\text{-depth}(U) \geq t + 1$, then $\mathcal{S}\text{-depth}(N) \geq t$.*

Proof. The assertions are direct consequences of Proposition 3.7.

Indeed, for (1), Proposition 3.7(1) gives

$$\mathcal{S}\text{-depth}(M) \geq \min\{\mathcal{S}\text{-depth}(U), \mathcal{S}\text{-depth}(N)\}.$$

Hence, if

$$\mathcal{S}\text{-depth}(U) \geq t \quad \text{and} \quad \mathcal{S}\text{-depth}(N) \geq t,$$

then

$$\mathcal{S}\text{-depth}(M) \geq t.$$

Similarly, Proposition 3.7(2) yields

$$\mathcal{S}\text{-depth}(U) \geq \min\{\mathcal{S}\text{-depth}(M), \mathcal{S}\text{-depth}(N) + 1\}.$$

Therefore, if

$$\mathcal{S}\text{-depth}(M) \geq t \quad \text{and} \quad \mathcal{S}\text{-depth}(N) \geq t - 1,$$

then

$$\mathcal{S}\text{-depth}(U) \geq t.$$

Finally, Proposition 3.7(3) gives

$$\mathcal{S}\text{-depth}(N) \geq \min\{\mathcal{S}\text{-depth}(U) - 1, \mathcal{S}\text{-depth}(M)\}.$$

Hence, if

$$\mathcal{S}\text{-depth}(M) \geq t \quad \text{and} \quad \mathcal{S}\text{-depth}(U) \geq t + 1,$$

we obtain

$$\mathcal{S}\text{-depth}(N) \geq t.$$

This completes the proof. $\square$

Corollary 6.3. *Let $(R, \mathfrak{m})$ be a local ring, let $\mathcal{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let M be an Artinian ZD-module. Then the following statements are equivalent:*

1. $\mathcal{S}\text{-depth}(M) = 0$;
2. $H_{\mathfrak{m}}^0(M) \notin \mathcal{S}$;
3. $\text{Hom}_R(R/\mathfrak{m}, M) \notin \mathcal{S}$.

Proof. By Lemma 3.5, we have

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

Thus, $\mathcal{S}\text{-depth}(M) = 0$ if and only if $H_{\mathfrak{m}}^0(M) \notin \mathcal{S}$.

Finally, using the isomorphism

$$H_{\mathfrak{m}}^0(M) \cong \text{Hom}_R(R/\mathfrak{m}, M),$$

we obtain the equivalence with (3). $\square$

7 Conclusion

In this paper, we investigated the notion of $\mathcal{S}$ -depth for Artinian ZD-modules over commutative Noetherian local rings, emphasizing its interaction with Serre subcategories, local cohomology modules, and Ext-functors. The obtained results provide a systematic framework for understanding $\mathcal{S}$ -depth in the Artinian setting and extend several classical properties of regular sequences to the more general context of $\mathcal{S}$ -sequences.

One of the main contributions of this work is the characterization of $\mathcal{S}$ -depth in terms of local cohomology modules and Ext-modules, together with the invariance of the length of maximal $\mathcal{S}$ -sequences under the hypotheses considered. These characterizations establish useful connections between homological invariants and the structure of Artinian ZD-modules, providing practical criteria for determining $\mathcal{S}$ -depth and illustrating the role of Serre subcategories in this setting.

The examples and applications presented throughout the paper demonstrate the applicability of the developed theory and show how the general results can be employed in concrete situations. In particular, the final consequences highlight the effectiveness of the proposed approach in relating vanishing properties of local cohomology to homological characterizations of $\mathcal{S}$ -depth.

We hope that the methods introduced here will stimulate further investigations on $\mathcal{S}$ -depth and related homological invariants. Possible directions for future research include extending these techniques to broader classes of modules, studying analogous questions for other Serre subcategories, and exploring further interactions between $\mathcal{S}$ -depth, local cohomology, and other derived functors in commutative algebra.

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