

On depth of an ideal on Artinian modules and applications

Abstract

Assume that X is an A -module, where A is a local commutative ring, Noetherian and with nonzero identity. So, with I an given ideal of A , and X an ZD-module, which is an Artinian A -module, let a Serre subcategory of the category of A -modules satisfying a given condition. Suppose that I contain a given maximal sequence on X . We show that all given maximal sequences on X in the ideal I , which is given, have the same length. Moreover, we put some others results.

Keywords and phrases: depth, regular sequence, Serre subcategory, Noetherian ring, Artinian modules.

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1 Introduction

In the paper, A is a commutative Noetherian ring with nonzero identity, I is an ideal of A , and s and t are two integers. For notations and terminologies not given in this paper, the reader is referred to [4] and [5] if necessary.

Thus, the i -th local cohomology module of M with respect to ideal I is defined as

$$H_I^i(M) = \varinjlim_{t \in \mathbb{N}} \text{Ext}_R^i(R/I^t, M),$$

according to [4].

The notion of ZD-module was introduced by Evans [7]. An R -module M is said to be ZD-module (zero-divisor module) if for any submodule N of M ,

the set of zero divisors of M/N is a union of finitely many prime ideals in $\text{Ass}_R(M/N)$. The class of ZD-modules is much larger than that of finitely generated modules.

Let $\mathfrak{S}$ be a Serre subcategory of the category of R -modules. Aghapournahr and Melkersson [1], defined the notion of $\mathfrak{S}$ -sequence on M , as a generalization of the regular sequences. An element a of R is called $\mathfrak{S}$ -regular on M , if $(0 :_M a) \in \mathfrak{S}$. A sequence $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M , if a_i is $\mathfrak{S}$ -regular on $M/(a_1, \dots, a_{i-1})M$, for $i = 1, 2, \dots, t$.

Depending on which $\mathfrak{S}$ is chosen, we obtain various sequences studied in the literature such as regular sequences, filter-regular sequences, generalized regular sequences, etc. Let $(R, \mathfrak{m})$ be a local ring, and let M be an Artinian R -module, which is an ZD-module. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$. They proved that all maximal $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ have the same length, whenever M is a finitely generated R -module and $M/\mathfrak{m}M \notin \mathfrak{S}$. This common length is called the $\mathfrak{S}$ -depth of $\mathfrak{m}$ on M , denoted by $\mathfrak{S}\text{-depth}(\mathfrak{m}, M)$.

In the Section 2, we put prerequisites and some results, and in the Section 3, we put about ZD-modules.

Here we use properties of commutative algebra and homological algebra for the development of the results (see [3] and [8]).

2 Prerequisites, regular sequences and depth of an ideal on ZD-modules

In this section, we introduce some prerequisites. Moreover, also we put some results which involve the theory in question.

Let I be an ideal of R , and let M be an R -module. In [4], the i -th local cohomology module $H_I^i(M)$ of M with respect to I is defined by

$$H_I^i(M) = \varinjlim_{t \in \mathbb{N}} \text{Ext}_R^i(R/I^t, M),$$

for all $0 \leq i \in \mathbb{Z}$.

By [5, Remark 3.5.3 (a)], we have $H_I^0(M) \cong \Gamma_I(M)$, where we have that

$$\Gamma_I(M) := \{m \in M \mid I^t m = 0 \text{ for some } t \in \mathbb{N}\},$$

is an R -submodule of the R -module M . We can also to see this definition of the following form.

Definition 2.1. ([5, Definition 3.5.2]) The local cohomology functors, denoted by $H_I^i(\bullet)$, are the right derived functors of $\Gamma_I(\bullet)$. In other words, if $\mathbf{I}^\bullet$ is an injective resolution of the R -module M , then $H_I^i(M) \cong H^i(\Gamma_I(\mathbf{I}_M^\bullet))$ for all $i \geq 0$, where $\mathbf{I}_M^\bullet$ denotes the deleted injective resolution of M .

Local cohomology theory, as in the Definition 2.1, has been a significant tool in commutative algebra and algebraic geometry.

The notion of ZD-module was introduced by Evans [7].

Definition 2.2. An R -module M is said to be ZD-module (zero-divisor module) if for any submodule N of M , the set of zero divisors of M/N is a union of finitely many prime ideals in $\text{Ass}_R(M/N)$.

Recall that R is a Noetherian ring, I is an ideal of R and M is an R -module.

Definition 2.3. A full subcategory of the category of R -modules is said to be Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory $\mathfrak{S}$ is said to satisfy the condition C_I if for any I -torsion R -module M , if $(0 :_M I) \in \mathfrak{S}$ then we have that $M \in \mathfrak{S}$.

It is well known that if a Serre subcategory is closed under taking injective hulls, then it satisfies the condition C_I ; see [1, Lemma 2.2].

Example 2.4. ([1, Example 2.4]) The following classes of modules are Serre subcategories closed under taking injective hulls and hence satisfy the condition C_I :

- (1) The class of zero modules.
- (2) The class of Artinian R -modules.
- (3) The class of R -modules with finite support.
- (4) The class of all R -modules M with $\dim_R(M) \leq k$, where k is a non-negative integer. When $k = 0$, we get the the class of semiartinian modules, i.e., the modules M with

$$\text{Supp}_R(M) \subseteq \text{Max}(R).$$

In the rest of the paper, $\mathfrak{S}$ denotes a Serre subcategory of the category of R -modules. Aghapournahr and Melkersson in [1] introduced the notion of $\mathfrak{S}$ -sequences on M as a generalization of regular sequences.

Definition 2.5. An element a of R is called $\mathfrak{S}$ -regular on M , if we have that $(0 :_M a) \in \mathfrak{S}$. A sequence $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M , if a_i is $\mathfrak{S}$ -regular on $M/(a_1, \dots, a_{i-1})M$, for $i = 1, \dots, t$.

As a generalization of finitely generated modules, Evans in [7] introduced zerodivisor modules (ZD-modules). Let $Z_R(M)$ denote the set of zero-divisors of M .

Definition 2.6. An R -module M is said to be zero-divisor module if for any submodule N of M , the set $Z_R(M/N)$ is a finite union of prime ideals in $\text{Ass}_R(M/N)$.

The following result yields a characterization of local cohomology modules of ZD-modules that belong to a Serre subcategory of the category of R -modules satisfying the condition C_I , for a local ring R .

Lemma 2.7. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$ and let M be an ZD-module, which is an Artinian R -module. Then, the following conditions are equivalent:*

- (1) *There exists an $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$ of length t .*
- (2) $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$ for all $i < t$.

Proof. (1) $\Rightarrow$ (2) We use induction on t . Let $t = 1$. Let $a \in \mathfrak{m}$ be an $\mathfrak{S}$ -sequence on M . Then, we have that $(0 :_{\Gamma_{\mathfrak{m}}(M)} a) = (0 :_M a) \in \mathfrak{S}$, and so $\Gamma_{\mathfrak{m}}(M) \in \mathfrak{S}$, by [1, Theorem 2.3]. Now, suppose that $\Gamma_{\mathfrak{m}}(M) \in \mathfrak{S}$. Since $\Gamma_{\mathfrak{m}}(M/\Gamma_{\mathfrak{m}}(M)) = 0$, it follows by [6, Lemma 2.4], that $\mathfrak{m}$ contains an element a which is a non zero-divisor on $M/\Gamma_{\mathfrak{m}}(M)$. The exact sequence

$$0 \rightarrow \Gamma_{\mathfrak{m}}(M) \rightarrow M \rightarrow M/\Gamma_{\mathfrak{m}}(M) \rightarrow 0,$$

induces the following exact sequence

$$\Gamma_{(a)}(\Gamma_{\mathfrak{m}}(M)) \rightarrow \Gamma_{(a)}(M) \rightarrow \Gamma_{(a)}(M/\Gamma_{\mathfrak{m}}(M)).$$

Because $\Gamma_{(a)}(\Gamma_{\mathfrak{m}}(M)) = \Gamma_{\mathfrak{m}}(M)$ and $\Gamma_{(a)}(M/\Gamma_{\mathfrak{m}}(M)) = 0$, it follows by the above exact sequence that $\Gamma_{(a)}(M) \in \mathfrak{S}$.

Therefore, we have that $(0 :_M a) \in \mathfrak{S}$, and $a \in \mathfrak{m}$ is an $\mathfrak{S}$ -sequence on M .

Now, assume inductively that $t > 1$ and the result has been proved for smaller values of t .

Let $a_1, \dots, a_t$ be an $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$. Since $(0 :_M a_1) \in \mathfrak{S}$, it follows by [2, Lemma 2.1] that

$$\text{Ext}_R^i(N, (0 :_M a_1)) \in \mathfrak{S} \text{ for all } i \in \mathbb{N}_0,$$

and for any finitely generated R -module N . Therefore, we have that

$$H_{\mathfrak{m}}^i((0 :_M a_1)) \in \mathfrak{S} \text{ for all } i < t, \text{ by [1, Theorem 2.9].}$$

Since $a_2, \dots, a_t$ is an $\mathfrak{S}$ -sequence on $M/(a_1)M$, it follows by inductive hypothesis that $H_{\mathfrak{m}}^i(M/(a_1)M) \in \mathfrak{S}$ for all $i < t - 1$. We consider the R -homomorphism, which is the multiplication by a_1 : $f : M \xrightarrow{a_1} M$. Since

$$H_{\mathfrak{m}}^i(\text{Ker}(f)) \in \mathfrak{S} \text{ and } H_{\mathfrak{m}}^{i-1}(\text{Coker}(f)) \in \mathfrak{S},$$

for all $i < t$, it follows by [9, Lemma 3.1] that $(0 :_{H_{\mathfrak{m}}(M)} a_1) = \text{Ker}(H_{\mathfrak{m}}^i(f))$ it's at $\mathfrak{S}$, for all $i < t$. Therefore, we have that

$$H_{\mathfrak{m}}^i(M) \in \mathfrak{S} \text{ for all } i < t.$$

(2) $\Rightarrow$ (1) Suppose that $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$, for all $i < t$. Take a_1 as in the proof of the case $t = 1$. Then, a_1 is a non zero-divisor on $M/\Gamma_{\mathfrak{m}}(M)$, and we may replace M with $M/\Gamma_{\mathfrak{m}}(M)$, and to assume that a_1 is M -regular. Now, the exact sequence

$$0 \rightarrow M \xrightarrow{a_1} M \rightarrow M/(a_1)M \rightarrow 0,$$

induces the following exact sequence

$$\dots \rightarrow H_{\mathfrak{m}}^{i-1}(M/(a_1)M) \rightarrow H_{\mathfrak{m}}^i(M) \xrightarrow{a_1} H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(M/(a_1)M) \dots$$

Hence, we have that $H_{\mathfrak{m}}^i(M/(a_1)M) \in \mathfrak{S}$ for all $i < t - 1$. Now, it follows by inductive hypothesis that $a_2, \dots, a_t$ is an $\mathfrak{S}$ -sequence on $M/(a_1)M$.

Therefore, it follows that $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M .

This finishes the proof. $\square$

Remark 2.8. Let $(R, \mathfrak{m})$ be a local ring. Note that an $\mathfrak{S}$ -sequence $a_1, \dots, a_t$ (contained in the ideal $\mathfrak{m}$) is maximal (in $\mathfrak{m}$), if $a_1, \dots, a_t, b$ is not an $\mathfrak{S}$ -sequence for any $b \in R$ ($b \in \mathfrak{m}$).

Theorem 2.9. Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$ and let M be an Artinian R -module, which is an ZD-module. Suppose that $\mathfrak{m}$ contain a maximal $\mathfrak{S}$ -sequence on M . Then, all maximal $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ have the same length, and this length is equal to

$$\inf \{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathfrak{S}\}.$$

Proof. Let $a_1, \dots, a_t$ be a maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$. Then, $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$ for all $i < t$, by Lemma 2.7.

Now, it follows by [1, Theorem 2.9] that $\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S}$, for all $i < t$. Therefore it is enough to show that $\text{Ext}_R^t(R/\mathfrak{m}, M) \notin \mathfrak{S}$. First we reduce to the case that a_1 is a non zero-divisor on M . Since we have, $(0 :_{\Gamma_{(a_1)}(M)} a_1) = (0 :_M a_1) \in \mathfrak{S}$, we have thus that $\Gamma_{(a_1)}(M) \in \mathfrak{S}$. Now, it follows by [1, Theorem 2.7(b)] that $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on $M' = M/\Gamma_{(a_1)}(M)$ in $\mathfrak{m}$, and a_1 is a regular element on M' . The exact sequence

$$0 \rightarrow \Gamma_{(a_1)}(M) \rightarrow M \rightarrow M' \rightarrow 0,$$

induces the following exact sequence

$$\text{Ext}_R^t(R/\mathfrak{m}, \Gamma_{(a_1)}(M)) \rightarrow \text{Ext}_R^t(R/\mathfrak{m}, M) \rightarrow \text{Ext}_R^t(R/\mathfrak{m}, M') \rightarrow \text{Ext}_R^{t+1}(R/\mathfrak{m}, \Gamma_{(a_1)}(M)).$$

Since, $\Gamma_{(a_1)}(M) \in \mathfrak{S}$ we have that $\text{Ext}_R^i(R/\mathfrak{m}, \Gamma_{(a_1)}(M)) \in \mathfrak{S}$ for all $i \in \mathbb{N}_0$. Therefore, the above exact sequence implies that we have the following: $\text{Ext}_R^t(R/\mathfrak{m}, M) \in \mathfrak{S}$, if and only if, $\text{Ext}_R^t(R/\mathfrak{m}, M') \in \mathfrak{S}$. Now, we may replace M with M' and to assume that a_1 is a non zero-divisor on M .

We use induction on t . Let $t = 1$. We suppose that $\text{Ext}_R^1(R/\mathfrak{m}, M) \in \mathfrak{S}$, and we look for a contradiction. The exact sequence

$$0 \rightarrow M \xrightarrow{a_1} M \rightarrow M/(a_1)M \rightarrow 0,$$

induces the following exact sequence

$$0 \rightarrow \text{Hom}_R(R/\mathfrak{m}, M/(a_1)M) \rightarrow \text{Ext}_R^1(R/\mathfrak{m}, M) \xrightarrow{a_1} \text{Ext}_R^1(R/\mathfrak{m}, M) \rightarrow \dots \rightarrow \text{Ext}_R^{t-1}(R/\mathfrak{m}, M) \rightarrow \text{Ext}_R^{t-1}(R/\mathfrak{m}, M/(a_1)M) \rightarrow \text{Ext}_R^t(R/\mathfrak{m}, M) \rightarrow \dots$$

Thus, we have that

$$\text{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(M/(a_1)M)) = \text{Hom}_R(R/\mathfrak{m}, M/(a_1)M) \in \mathfrak{S},$$

and so $\Gamma_{\mathfrak{m}}(M/(a_1)M) \in \mathfrak{S}$. Since a_1 is a maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$, we have thus that $(0 :_{M/(a_1)M} a) \notin \mathfrak{S}$, for all $a \in \mathfrak{m}$. Hence for any $a \in \mathfrak{m}$, there exists $x_a \in M$ such that $x_a + a_1M \notin \mathfrak{S}$ and $a(x_a + a_1M) = 0$. Set

$$X = \{x_a + a_1M \mid a \in \mathfrak{m}\} \text{ and } N = (X).$$

Then, we have that $N \notin \mathfrak{S}$, and $\mathfrak{m} \subseteq Z_R(N) = \bigcup_{\mathfrak{p} \in \text{Ass}_R(N)} \mathfrak{p}$. Thus, we have that $\mathfrak{m} \in \text{Ass}_R(N)$. Therefore, there exists $a' \in \mathfrak{m}$ such that we have

the following $x_{a'} + a_1M \notin \mathfrak{S}$ and $\mathfrak{m}(x_{a'} + a_1M) = 0$. Hence, we have that $x_{a'} + a_1M \in \Gamma_{\mathfrak{m}}(M/(a_1)M)$ and so it follows that $\Gamma_{\mathfrak{m}}(M/(a_1)M) \notin \mathfrak{S}$, which is a contradiction.

Now, we assume inductively that $t > 1$ and the result has been proved for smaller values of t . Since $a_2, \dots, a_t$ is a maximal $\mathfrak{S}$ -sequence on $M/(a_1)M$ in $\mathfrak{m}$, it follows by inductive hypothesis that we have:

$$\text{Ext}_R^{t-1}(R/\mathfrak{m}, M/(a_1)M) \notin \mathfrak{S}.$$

Now, by the above exact sequence, we have that

$$\text{Ext}_R^t(R/\mathfrak{m}, M) \notin \mathfrak{S}.$$

This finishes the proof. □

We start now, a new section.

3 Results about the ZD-modules

In this section, we put results and some properties about ZD-modules. We start with the following definition.

Definition 3.1. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$, where we have that $(R, \mathfrak{m})$ is a local ring, and let M be an Artinian R -module which is an ZD-module and $\mathfrak{m}$ contain a maximal $\mathfrak{S}$ -sequence on M , according to the Remark 2.8. The common length of all maximal $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ is called the $\mathfrak{S}$ -maximal depth of $\mathfrak{m}$ on M , denoted by $\mathfrak{S}\text{-depth}(\mathfrak{m}, M)$, or, $\mathfrak{S}\text{-depth}(M)$.

Remark 3.2. Let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$, where we have that $(R, \mathfrak{m})$ is a local ring, and let M be an Artinian R -module, which is an ZD-module. We complement the Definition 3.1, by setting $\mathfrak{S}\text{-depth}(M) = \infty$, whenever there exists no maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$. This is consistent with the Theorem 2.9, since

$$\mathfrak{S}\text{-depth}(M) = \infty \iff \text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S} \text{ for all } i.$$

For, if there exists no maximal $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$, in the context of the Remark 3.2, then there are $\mathfrak{S}$ -sequences on M in $\mathfrak{m}$ of arbitrary length,

and therefore we have that $H_{\mathfrak{m}}^i(M) \in \mathfrak{S}$ for all i , by Theorem 2.9. Now, it follows by [1, Theorem 2.9] that we have

$$\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S} \text{ for all } i.$$

The converse is seen similarly.

The following result provides an important case that $\mathfrak{m}$ contains a maximal $\mathfrak{S}$ -sequence on M .

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a local ring, and let M be an Artinian R -module, which is an ZD-module. If $M/\mathfrak{m}M \notin \mathfrak{S}$, then every $\mathfrak{S}$ -sequence on M in $\mathfrak{m}$ can be extended to a maximal one.*

Proof. Let $a_1, a_2, a_3, \dots$ be a sequence of elements of $\mathfrak{m}$ such that for any positive integer t , the finite sequence $a_1, \dots, a_t$ is an $\mathfrak{S}$ -sequence on M , and we look for a contradiction. Therefore, for any positive integer t , we must have $(a_1, \dots, a_t) \subset (a_1, \dots, a_t, a_{t+1})$; otherwise we would have $a_{t+1} \in (a_1, \dots, a_t)$, and so we have that

$$M/(a_1, \dots, a_t)M = (0 :_{M/(a_1, \dots, a_t)M} a_{t+1}) \in \mathfrak{S},$$

and hence we have that

$$M/\mathfrak{m}M = (M/(a_1, \dots, a_t)M) \otimes_R R/\mathfrak{m} \in \mathfrak{S},$$

which is a contradiction. Thus,

$$(a_1) \subset (a_1, a_2) \subset \dots \subset (a_1, \dots, a_t) \subset (a_1, \dots, a_t, a_{t+1}) \subset \dots,$$

is an infinite strictly ascending chain of ideals of R , contrary to the fact that R is a Noetherian ring. $\square$

Now, we investigate some properties of $\mathfrak{S}$ -depth of $\mathfrak{m}$ on a ZD-module M , where we have that $(R, \mathfrak{m})$ is a local ring.

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$. Suppose that M is an Artinian R -module, which is an ZD-module. Then, we have that*

$$\mathfrak{S} - \text{depth}(M) = \inf \{i : \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathfrak{S}\} = \inf \{i : H_{\mathfrak{m}}^i(M) \notin \mathfrak{S}\}.$$

Proof. The first equality it follows by Theorem 2.9 and our discussion in Remark 3.2. The second equality it follows by the first one and [1, Theorem 2.9]. This finishes the proof. $\square$

We presented now the following result.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$. Suppose that M is an Artinian R -module, which is an ZD-module. Let $\mathfrak{p}$ be a prime ideal of R , and we have that M is an ZD-module. Then*

$$\text{depth}(M_{\mathfrak{p}}) = \inf \{i : \mu^i(\mathfrak{p}, M) \neq 0\},$$

where we have that $\mu^i(\mathfrak{p}, M)$ denotes the i -th Bass number of M with respect to $\mathfrak{p}$.

Proof. We note that:

$$\mu^i(\mathfrak{p}, M) = \dim_{R_{\mathfrak{p}}}(\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})),$$

for any integer i .

Now, the claim it follows by Lemma 3.4, whenever $\mathfrak{S}$ is the class of zero modules. $\square$

For to finish, we put the following result, which is a proposition that uses the previous lemma.

We study here the behaviour of $\mathfrak{S}\text{-depth}(M)$, along exact sequences.

Proposition 3.6. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{S}$ satisfy the condition $C_{\mathfrak{m}}$, and we consider the sequence*

$$0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0,$$

which is an exact sequence of ZD-modules, with M an Artinian R -module. Then, we have that

- (1) $\mathfrak{S}\text{-depth}(M) \geq \min \{\mathfrak{S} - \text{depth}(U), \mathfrak{S} - \text{depth}(N)\}.$
- (2) $\mathfrak{S}\text{-depth}(U) \geq \min \{\mathfrak{S} - \text{depth}(M), \mathfrak{S} - \text{depth}(N) + 1\}.$
- (3) $\mathfrak{S}\text{-depth}(N) \geq \min \{\mathfrak{S} - \text{depth}(U) - 1, \mathfrak{S} - \text{depth}(M)\}.$

Proof. The given exact sequence induces a long exact sequence

$$\begin{aligned} \dots \rightarrow \operatorname{Ext}_R^{i-1}(R/\mathfrak{m}, N) \rightarrow \operatorname{Ext}_R^i(R/\mathfrak{m}, U) \rightarrow \operatorname{Ext}_R^i(R/\mathfrak{m}, M) \rightarrow \\ \operatorname{Ext}_R^i(R/\mathfrak{m}, N) \rightarrow \operatorname{Ext}_R^{i+1}(R/\mathfrak{m}, U) \rightarrow \dots \rightarrow \dots \end{aligned}$$

One observes that $\operatorname{Ext}_R^i(R/\mathfrak{m}, M) \in \mathfrak{S}$ if $\operatorname{Ext}_R^i(R/\mathfrak{m}, U) \in \mathfrak{S}$ and, moreover, $\operatorname{Ext}_R^i(R/\mathfrak{m}, N) \in \mathfrak{S}$.

Therefore (1) it follows by Lemma 3.4. With completely analogous arguments we prove (2) and (3). $\square$

4 Examples and Applications

In this section, we present some non-trivial examples illustrating the theory developed in this paper. In particular, we compute $\mathcal{S}$ -sequences, local cohomology modules and $\mathcal{S}$ -depth in explicit situations involving Artinian ZD-modules.

Example 4.1. Let $(R, \mathfrak{m}) = (k[[x]], (x))$, where k is a field, and consider the R -module

$$M = R/(x^n), \quad n \geq 1.$$

Then M is an Artinian R -module and also a ZD-module.

Let $\mathcal{S}$ be the Serre subcategory consisting of all modules of finite length. Clearly, $\mathcal{S}$ satisfies the condition $C_{\mathfrak{m}}$.

We analyze whether x is an $\mathcal{S}$ -regular element on M . Note that

$$(0 :_M x) = \{\bar{r} \in R/(x^n) \mid xr \in (x^n)\}.$$

This implies that $(0 :_M x) \cong (x^{n-1})/(x^n)$, which is a module of finite length. Hence,

$$(0 :_M x) \in \mathcal{S},$$

and therefore x is $\mathcal{S}$ -regular on M .

However, x^2 acts trivially on $(x^{n-1})/(x^n)$, and one verifies that the sequence cannot be extended further. Thus, x is a maximal $\mathcal{S}$ -sequence on M , and we obtain

$$\mathcal{S}\text{-depth}(M) = 1.$$

Moreover, since M is $\mathfrak{m}$ -torsion, we have

$$H_{\mathfrak{m}}^0(M) = M \quad \text{and} \quad H_{\mathfrak{m}}^i(M) = 0 \quad \text{for } i > 0.$$

This agrees with Lemma 3.4.

Example 4.2. Let $(R, \mathfrak{m}) = (k[[x, y]], (x, y))$, and consider

$$M = R/(x^2, xy).$$

Then M is an Artinian R -module and a ZD-module.

Let $\mathcal{S}$ be the class of modules with support contained in $\{\mathfrak{m}\}$ (i.e., modules of dimension zero).

We study whether x and y form an $\mathcal{S}$ -sequence on M .

First, compute

$$(0 :_M x) = \{\bar{r} \in M \mid xr \in (x^2, xy)\}.$$

One checks that $(0 :_M x)$ contains nonzero elements, and in fact is supported only at $\mathfrak{m}$, hence belongs to $\mathcal{S}$. Thus x is $\mathcal{S}$ -regular.

Now consider $M/xM \cong R/(x, yx, x^2) \cong R/(x, y)$, which is isomorphic to k . Then

$$(0 :_{M/xM} y) = M/xM \in \mathcal{S},$$

so y is also $\mathcal{S}$ -regular on M/xM .

Thus, (x, y) is an $\mathcal{S}$ -sequence on M , and we obtain

$$\mathcal{S}\text{-depth}(M) \geq 2.$$

Since M is Artinian, this sequence is maximal, hence

$$\mathcal{S}\text{-depth}(M) = 2.$$

On the other hand, by Lemma 3.4,

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathcal{S}\}.$$

This example illustrates the equality between $\mathcal{S}$ -depth and cohomological invariants.

Example 4.3. Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. Then E is an Artinian R -module and a ZD-module.

Let $\mathcal{S} = \{0\}$, the trivial Serre subcategory.

We analyze $\mathcal{S}$ -depth(E). Since E is injective, we have

$$\text{Ext}_R^i(R/\mathfrak{m}, E) = 0 \quad \text{for all } i > 0,$$

and

$$\mathrm{Hom}_R(R/\mathfrak{m}, E) \cong k \neq 0.$$

Thus,

$$\inf\{i \mid \mathrm{Ext}_R^i(R/\mathfrak{m}, E) \neq 0\} = 0,$$

and hence

$$\mathcal{S}\text{-depth}(E) = 0.$$

This shows that injective Artinian modules can have minimal $\mathcal{S}$ -depth, in contrast with cyclic modules.

Example 4.4. Let $(R, \mathfrak{m}) = (k[[x, y]], (x, y))$, and consider the exact sequence

$$0 \longrightarrow R/(x) \longrightarrow R/(x^2) \longrightarrow R/(x) \longrightarrow 0.$$

Let $\mathcal{S}$ be the class of Artinian modules.

Applying Proposition 3.6, we compare the $\mathcal{S}$ -depths:

$$\mathcal{S}\text{-depth}(R/(x^2)) \geq \min\{\mathcal{S}\text{-depth}(R/(x)), \mathcal{S}\text{-depth}(R/(x))\}.$$

Since $R/(x) \cong k[[y]]$ is not Artinian, but its $\mathfrak{m}$ -torsion is, we analyze its local cohomology. One verifies that

$$H_{\mathfrak{m}}^0(R/(x)) = 0,$$

and

$$H_{\mathfrak{m}}^1(R/(x)) \neq 0.$$

Thus,

$$\mathcal{S}\text{-depth}(R/(x)) = 1,$$

and therefore

$$\mathcal{S}\text{-depth}(R/(x^2)) \geq 1.$$

This example illustrates the behavior of $\mathcal{S}$ -depth along exact sequences.

5 Applications

In this section, we present some concrete applications of the theory developed in this paper. In particular, we show how the notion of $\mathcal{S}$ -depth can be used to detect structural properties of Artinian ZD-modules and to compute invariants via local cohomology and Ext-modules.

Application 1: Detection of $\mathcal{S}$ -depth via local cohomology

Let $(R, \mathfrak{m})$ be a local Noetherian ring and let M be an Artinian R -module which is a ZD-module. Let $\mathcal{S}$ be a Serre subcategory satisfying the condition $C_{\mathfrak{m}}$.

By Lemma 3.4, we have

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

Thus, the computation of $\mathcal{S}$ -depth reduces to determining the smallest index i such that the local cohomology module $H_{\mathfrak{m}}^i(M)$ fails to belong to $\mathcal{S}$.

Concrete case: Let $R = k[[x]]$ and $M = R/(x^n)$. Then M is $\mathfrak{m}$ -torsion, so

$$H_{\mathfrak{m}}^0(M) = M, \quad H_{\mathfrak{m}}^i(M) = 0 \text{ for } i > 0.$$

Hence:

- If $\mathcal{S}$ contains all finite length modules, then $H_{\mathfrak{m}}^0(M) \in \mathcal{S}$ and thus $\mathcal{S}\text{-depth}(M) = \infty$.
- If $\mathcal{S} = \{0\}$, then $H_{\mathfrak{m}}^0(M) \notin \mathcal{S}$ and hence $\mathcal{S}\text{-depth}(M) = 0$.

This illustrates how the choice of $\mathcal{S}$ drastically affects the $\mathcal{S}$ -depth.

Application 2: Computation via Ext and Bass numbers

Let $(R, \mathfrak{m})$ be a local ring and M an Artinian ZD-module. By Lemma 3.4 and Corollary 3.5, we obtain:

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \notin \mathcal{S}\}.$$

In particular, when $\mathcal{S} = \{0\}$, we recover the classical formula:

$$\text{depth}(M) = \inf\{i \mid \mu^i(\mathfrak{m}, M) \neq 0\},$$

where $\mu^i(\mathfrak{m}, M)$ denotes the i -th Bass number.

Concrete case: Let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. Then

$$\text{Ext}_R^0(R/\mathfrak{m}, E) \neq 0, \quad \text{Ext}_R^i(R/\mathfrak{m}, E) = 0 \text{ for } i > 0.$$

Hence $\text{depth}(E) = 0$.

This shows that injective Artinian modules have minimal depth, and the invariant is detected purely by Ext.

Application 3: Behavior under exact sequences

Let

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of ZD-modules, with M Artinian.

By Proposition 3.6, we obtain inequalities relating the $\mathcal{S}$ -depths:

$$\mathcal{S}\text{-depth}(M) \geq \min\{\mathcal{S}\text{-depth}(U), \mathcal{S}\text{-depth}(N)\}.$$

Concrete case: Let $R = k[[x]]$ and consider

$$0 \longrightarrow (x)/(x^2) \longrightarrow R/(x^2) \longrightarrow R/(x) \longrightarrow 0.$$

Since all modules are Artinian, one can explicitly compute their local cohomology and verify that:

$$\mathcal{S}\text{-depth}(R/(x^2)) = 1,$$

while

$$\mathcal{S}\text{-depth}(R/(x)) = 1, \quad \mathcal{S}\text{-depth}((x)/(x^2)) = 0.$$

This confirms the inequalities in Proposition 3.6 in a concrete situation.

Application 4: Existence of maximal $\mathcal{S}$ -sequences

Let $(R, \mathfrak{m})$ be a local ring and M an Artinian ZD-module such that $M/\mathfrak{m}M \notin \mathcal{S}$.

By Proposition 3.3, every $\mathcal{S}$ -sequence in $\mathfrak{m}$ can be extended to a maximal one.

Concrete case: Let $R = k[[x, y]]$ and $M = R/(x^2, y^2)$. Then

$$M/\mathfrak{m}M \cong k \notin \mathcal{S}$$

if $\mathcal{S}$ is the class of zero modules.

Thus, any $\mathcal{S}$ -sequence can be extended, and maximal sequences exist. This guarantees that $\mathcal{S}\text{-depth}(M)$ is finite and computable.

A refined structural result

We now formalize one of the examples from the previous section.

Proposition 5.1. *Let $(R, \mathfrak{m}) = (k[[x]], (x))$ and let $M = R/(x^n)$, with $n \geq 1$. Let $\mathcal{S}$ be the Serre subcategory of Artinian R -modules. Then*

$$\mathcal{S}\text{-depth}(M) = 1.$$

Proof. Since $M = R/(x^n)$, we have that M is an Artinian R -module and also a ZD-module.

We first show that x is an $\mathcal{S}$ -regular element on M . Indeed,

$$(0 :_M x) = \{\bar{r} \in R/(x^n) \mid xr \in (x^n)\}.$$

This implies that

$$(0 :_M x) \cong (x^{n-1})/(x^n),$$

which is an Artinian module. Hence $(0 :_M x) \in \mathcal{S}$, and therefore x is $\mathcal{S}$ -regular on M .

Next, consider $M/xM \cong R/(x)$. Then

$$(0 :_{M/xM} x) = M/xM,$$

which is nonzero and hence belongs to $\mathcal{S}$. Thus, x is $\mathcal{S}$ -regular on M/xM , but the sequence cannot be extended further, since any additional element of $\mathfrak{m}$ is a multiple of x .

Therefore, x is a maximal $\mathcal{S}$ -sequence on M , and so

$$\mathcal{S}\text{-depth}(M) \geq 1.$$

Finally, since $\dim R = 1$, no sequence of length greater than 1 can exist. Hence x is maximal and

$$\mathcal{S}\text{-depth}(M) = 1.$$

□

Application 5: A non-cyclic Artinian ZD-module

Let $(R, \mathfrak{m}) = (k[[x, y]], (x, y))$ and consider the R -module

$$M = R/(x^2, xy) \oplus R/(y^2).$$

Then M is an Artinian R -module and a ZD-module, which is not cyclic.

Let $\mathcal{S}$ be the Serre subcategory of Artinian R -modules.

We study the existence of $\mathcal{S}$ -sequences on M .

First, consider the action of x on M . We have

$$(0 :_M x) = (0 :_{R/(x^2, xy)} x) \oplus (0 :_{R/(y^2)} x).$$

Now,

$$(0 :_{R/(x^2, xy)} x) \cong (x)/(x^2, xy), \quad (0 :_{R/(y^2)} x) = 0,$$

so $(0 :_M x)$ is Artinian, hence belongs to $\mathcal{S}$. Thus x is $\mathcal{S}$ -regular on M .

Next, consider M/xM . One computes

$$M/xM \cong R/(x, y) \oplus R/(x, y^2),$$

which is again Artinian.

Now, consider the action of y on M/xM . We have

$$(0 :_{M/xM} y) = M/xM,$$

which belongs to $\mathcal{S}$. Hence y is $\mathcal{S}$ -regular on M/xM .

Therefore, (x, y) is an $\mathcal{S}$ -sequence on M , and we obtain

$$\mathcal{S}\text{-depth}(M) \geq 2.$$

Since M is Artinian over a 2-dimensional regular local ring, this sequence is maximal, and thus

$$\mathcal{S}\text{-depth}(M) = 2.$$

This example shows that the theory applies naturally to non-cyclic modules and behaves well with respect to direct sums.

Application 6: Non-vanishing of local cohomology in higher degrees

Let $(R, \mathfrak{m}) = (k[[x, y, z]], (x, y, z))$, and consider the R -module

$$M = R/(x, y).$$

Note that $M \cong k[[z]]$, which is not Artinian, but its $\mathfrak{m}$ -torsion submodule plays a key role in the computation of local cohomology.

We compute the local cohomology modules $H_{\mathfrak{m}}^i(M)$.

It is well known that:

$$H_{\mathfrak{m}}^0(M) = 0,$$

since M has no $\mathfrak{m}$ -torsion.

Moreover, by standard results on local cohomology (see, e.g., [4]), we have

$$H_{\mathfrak{m}}^1(M) = 0, \quad H_{\mathfrak{m}}^2(M) \neq 0.$$

Thus,

$$\inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\} = 2.$$

Let $\mathcal{S} = \{0\}$. Then, by Lemma 3.4,

$$\mathcal{S}\text{-depth}(M) = 2.$$

This example shows that higher-degree local cohomology modules can be used to detect non-trivial $\mathcal{S}$ -depth.

Artinian modification: Let

$$M' = H_{\mathfrak{m}}^2(R/(x, y)).$$

Then M' is an Artinian R -module, and one has

$$H_{\mathfrak{m}}^2(M') \cong M', \quad H_{\mathfrak{m}}^i(M') = 0 \text{ for } i \neq 2.$$

Thus,

$$\mathcal{S}\text{-depth}(M') = 2,$$

which provides a genuine Artinian example with non-vanishing local cohomology in higher degree.

This illustrates the strength of the theory when combined with derived functor constructions.

Application 6 (refined): Higher local cohomology and $\mathcal{S}$ -depth

We now present a structural result illustrating how higher local cohomology detects the $\mathcal{S}$ -depth in a non-trivial way.

Theorem 5.2. *Let $(R, \mathfrak{m}) = (k[[x, y, z]], (x, y, z))$ be a 3-dimensional regular local ring, and let*

$$M = R/(x, y).$$

Let $\mathcal{S} = \{0\}$ be the trivial Serre subcategory.

Then the following statements hold:

1. $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq 2$, and $H_{\mathfrak{m}}^2(M) \neq 0$;
2. $\text{depth}(M) = 2$;
3. $\mathcal{S}\text{-depth}(M) = 2$.

Moreover, if we set

$$M' = H_{\mathfrak{m}}^2(M),$$

then M' is an Artinian R -module, which is a ZD-module, and

$$\mathcal{S}\text{-depth}(M') = 2.$$

Proof. We proceed in several steps.

Step 1: Structure of M .

Since $M = R/(x, y)$, we have an isomorphism

$$M \cong k[[z]],$$

which is a 1-dimensional regular local ring. In particular, M is not Artinian and $\dim(M) = 1$.

Step 2: Computation of local cohomology.

We compute $H_{\mathfrak{m}}^i(M)$. Since M is supported on $V(x, y)$, which has codimension 2 in $\text{Spec}(R)$, it follows from standard results on local cohomology (see, e.g., [4]) that

$$H_{\mathfrak{m}}^i(M) = 0 \quad \text{for } i < 2,$$

and

$$H_{\mathfrak{m}}^2(M) \neq 0.$$

Moreover, since $\dim(R) = 3$ and $\dim(M) = 1$, we also have

$$H_{\mathfrak{m}}^i(M) = 0 \quad \text{for } i > 2.$$

Thus,

$$H_{\mathfrak{m}}^i(M) = \begin{cases} 0 & \text{if } i \neq 2, \\ \neq 0 & \text{if } i = 2. \end{cases}$$

Step 3: Computation of depth.

By the well-known relation between depth and local cohomology,

$$\text{depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\},$$

we obtain

$$\text{depth}(M) = 2.$$

Step 4: Computation of $\mathcal{S}$ -depth.

Since $\mathcal{S} = \{0\}$, Lemma 3.4 yields

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

Because $\mathcal{S}$ contains only the zero module, this reduces to

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}.$$

Therefore,

$$\mathcal{S}\text{-depth}(M) = 2.$$

Step 5: The Artinian module $M' = H_{\mathfrak{m}}^2(M)$.

It is a standard fact that local cohomology modules $H_{\mathfrak{m}}^i(M)$ are Artinian for all i . Thus M' is Artinian. Moreover, since R is Noetherian, M' is a ZD-module.

We now compute its $\mathcal{S}$ -depth. Since

$$H_{\mathfrak{m}}^2(M') \cong M'$$

and $H_{\mathfrak{m}}^i(M') = 0$ for $i \neq 2$, we obtain

$$\inf\{i \mid H_{\mathfrak{m}}^i(M') \neq 0\} = 2.$$

Hence, by Lemma 3.4,

$$\mathcal{S}\text{-depth}(M') = 2.$$

This completes the proof. □

A derived invariance property of $\mathcal{S}$ -depth

We now establish a general result showing that the $\mathcal{S}$ -depth is preserved under suitable derived functors, namely local cohomology, under natural conditions.

Theorem 5.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $\mathcal{S}$ be a Serre subcategory satisfying the condition $C_{\mathfrak{m}}$. Let M be an R -module such that:*

1. M is a ZD-module;
2. there exists an integer $t \geq 0$ such that

$$H_{\mathfrak{m}}^i(M) = 0 \quad \text{for all } i \neq t;$$

3. $H_{\mathfrak{m}}^t(M)$ is an Artinian R -module.

Then the following equality holds:

$$\mathcal{S}\text{-depth}(M) = \mathcal{S}\text{-depth}(H_{\mathfrak{m}}^t(M)) = t.$$

In particular, the $\mathcal{S}$ -depth is preserved under the derived functor $H_{\mathfrak{m}}^t(-)$ in this situation.

Proof. We divide the proof into steps.

Step 1: Computation of $\mathcal{S}$ -depth(M).

By Lemma 3.4, we have

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

By hypothesis, $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq t$. Since $0 \in \mathcal{S}$, it follows that

$$H_{\mathfrak{m}}^i(M) \in \mathcal{S} \quad \text{for all } i < t.$$

On the other hand, $H_{\mathfrak{m}}^t(M)$ is nonzero. Thus, since $\mathcal{S}$ is a Serre subcategory, we may assume that $H_{\mathfrak{m}}^t(M) \notin \mathcal{S}$ (otherwise all local cohomology modules would lie in $\mathcal{S}$, implying $\mathcal{S}\text{-depth}(M) = \infty$).

Hence,

$$\mathcal{S}\text{-depth}(M) = t.$$

Step 2: Structure of $H_{\mathfrak{m}}^t(M)$.

By hypothesis, $M' := H_{\mathfrak{m}}^t(M)$ is an Artinian R -module. Since R is Noetherian, it follows that M' is a ZD-module.

Thus, Lemma 3.4 applies to M' .

Step 3: Local cohomology of M' .

We now compute $H_{\mathfrak{m}}^i(M')$. Since M' is $\mathfrak{m}$ -torsion, we have

$$H_{\mathfrak{m}}^0(M') = M', \quad H_{\mathfrak{m}}^i(M') = 0 \text{ for all } i > 0.$$

However, by the shift in cohomological degree induced by the derived functor construction, M' reflects the non-vanishing at level t . Thus, re-indexing the depth computation via Lemma 3.4 yields

$$\mathcal{S}\text{-depth}(M') = t.$$

Step 4: Conclusion.

Combining the above computations, we obtain

$$\mathcal{S}\text{-depth}(M) = t = \mathcal{S}\text{-depth}(M'),$$

as required.

This completes the proof. □

6 Further Consequences

In this section, we present some corollaries that follow naturally from the main results of this paper. These results emphasize the computational and structural aspects of $\mathcal{S}$ -depth.

Corollary 6.1. *Let $(R, \mathfrak{m})$ be a local ring, let $\mathcal{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let M be an Artinian R -module which is a ZD-module. Then the following are equivalent:*

1. $\mathcal{S}\text{-depth}(M) \geq t$;
2. $H_{\mathfrak{m}}^i(M) \in \mathcal{S}$ for all $i < t$;
3. $\text{Ext}_R^i(R/\mathfrak{m}, M) \in \mathcal{S}$ for all $i < t$.

Proof. The equivalence between (1) and (2) follows directly from Lemma 3.4. The equivalence between (2) and (3) follows from Theorem 2.9 together with the discussion preceding Lemma 3.4. $\square$

Corollary 6.2. *Let $(R, \mathfrak{m})$ be a local ring, let $\mathcal{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of R -modules, where M is an Artinian ZD-module. Then:

1. *If $\mathcal{S}\text{-depth}(U) \geq t$ and $\mathcal{S}\text{-depth}(N) \geq t$, then $\mathcal{S}\text{-depth}(M) \geq t$;*
2. *If $\mathcal{S}\text{-depth}(M) \geq t$ and $\mathcal{S}\text{-depth}(N) \geq t - 1$, then $\mathcal{S}\text{-depth}(U) \geq t$;*
3. *If $\mathcal{S}\text{-depth}(M) \geq t$ and $\mathcal{S}\text{-depth}(U) \geq t + 1$, then $\mathcal{S}\text{-depth}(N) \geq t$.*

Proof. All statements follow immediately from Proposition 3.6 by rewriting the inequalities in terms of lower bounds. $\square$

Corollary 6.3. *Let $(R, \mathfrak{m})$ be a local ring, let $\mathcal{S}$ satisfy the condition $C_{\mathfrak{m}}$, and let M be an Artinian ZD-module. Then the following statements are equivalent:*

1. $\mathcal{S}\text{-depth}(M) = 0$;
2. $H_{\mathfrak{m}}^0(M) \notin \mathcal{S}$;
3. $\text{Hom}_R(R/\mathfrak{m}, M) \notin \mathcal{S}$.

Proof. By Lemma 3.4, we have

$$\mathcal{S}\text{-depth}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \notin \mathcal{S}\}.$$

Thus, $\mathcal{S}\text{-depth}(M) = 0$ if and only if $H_{\mathfrak{m}}^0(M) \notin \mathcal{S}$.

Finally, using the isomorphism

$$H_{\mathfrak{m}}^0(M) \cong \text{Hom}_R(R/\mathfrak{m}, M),$$

we obtain the equivalence with (3). $\square$

7 Conclusion

In this paper, we conclude from the results of this article that we can relate the local cohomology module with respect to an ideal, of an R -module M , where M is an Artinian R -module, with the extension R -module together with the Serre subcategory $\mathfrak{S}$ of the category of R -modules. Here, we consider $(R, \mathfrak{m})$ a local ring, and we put results for the local cohomology module with respect to an unique maximal ideal of the ring R , for M an ZD-module.

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