

On Contra-continuous Functions in Ideal Topological Space

Abstract

Aims/objectives: The aim of this paper is to introduce a class of functions using the concepts in ideal topological space called the contra $R - \mathcal{I}$ -continuous functions. The paper also introduced almost contra $R - \mathcal{I}$ -continuous functions and investigated certain properties and several characterizations of such concepts. Further, contra $R - \mathcal{I}$ -closed graphs and strongly contra $R - \mathcal{I}$ -closed graphs are dealt with.

Study design: Theoretical

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Methodology: The application of logical reasoning to deduce new theorems from established principles.

Conclusion: This paper introduced and systematically investigated the concepts of contra $R - \mathcal{I}$ -continuous and almost contra $R - \mathcal{I}$ -continuous functions in ideal topological spaces. It provides analysis of their fundamental properties, characterizations, and relationships with other existing continuous functions. Furthermore, it examined the behaviors of contra $R - \mathcal{I}$ -closed graphs and strongly contra $R - \mathcal{I}$ -closed graphs.

Keywords: contra $R - \mathcal{I}$ - continuous functions, almost contra $R - \mathcal{I}$ - continuous functions, contra $R - \mathcal{I}$ - closed graphs, strongly contra $R - \mathcal{I}$ - closed graphs.

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1 Introduction

In topology many research had taken place in the area of continuity especially on generalized continuity. one strong version of continuity was contra-continuity. The concept of contra-continuity was first investigated by Dontchev in 1996 [1]. Later S. Jafari and T. Noiri [2] and E. Ekici [3] further developed the weak forms of contra-continuity. In the section 3. of this paper, using the notion of $R - \mathcal{I}$ -open sets, the contra $R - \mathcal{I}$ -continuous functions are introduced and studied. In section 4.,

almost contra $R - \mathcal{I}$ -continuous functions are investigated. Section 5. deals with $R - \mathcal{I}$ -closed, contra $R - \mathcal{I}$ -closed and strongly contra $R - \mathcal{I}$ -closed graphs.

2 Preliminaries

By a space (X, τ) , we mean a topological space with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X, τ) , the system of open neighborhoods of x is denoted by $N(x) = \{U \in \tau : x \in U\}$. For a given subset A of a space (X, τ) , $Cl(A)$ and $Int(A)$ are used to denote the closure of A and interior of A , respectively, with respect to the topology.

A nonempty collection of subsets of a set X is said to be an ideal $\mathcal{I}$ on X , if it satisfies the following two conditions: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal topological space (or ideal space) $(X, \tau, \mathcal{I})$ means a topological space (X, τ) with an ideal $\mathcal{I}$ defined on X . Let (X, τ) be a topological space with an ideal $\mathcal{I}$ defined on X . Then for any subset A of X , $A^*(\mathcal{I}, \tau) = \{x \in X/A \cap U \notin \mathcal{I} \text{ for every } U \in N(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ . If there is no ambiguity, we will write $A^*(\mathcal{I})$ or simply A^* for $A^*(\mathcal{I}, \tau)$. Also, $Cl^*(A) = A \cup A^*$. It defines a Kuratowski closure operator for the topology $\tau^*(\mathcal{I})$ (or simply τ^*) which is finer than τ [4].

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $R - \mathcal{I}$ -open (resp. regular open) if $Int(Cl^*(A)) = A$ (resp. $Int(Cl(A))=A$). We call a subset A of $(X, \tau, \mathcal{I})$ is $R - \mathcal{I}$ -closed if its complement is $R - \mathcal{I}$ -open. The intersection of all $R - \mathcal{I}$ -closed sets containing A is called the $R - \mathcal{I}$ -closure of A and is denoted by $R - \mathcal{I} - Cl(A)$. The $R - \mathcal{I}$ -interior of A is defined by the union of all $R - \mathcal{I}$ -open sets contained in A and is denoted by $R - \mathcal{I} - Int(A)$. The family of all $R - \mathcal{I}$ -open (resp. $R - \mathcal{I}$ -closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $RIO(X, x)$ (resp. $RIC(X, x)$). A subset N of X is called an $R - \mathcal{I}$ -nbd of a point $x \in X$ if there exists an $R - \mathcal{I}$ -open set U of $(X, \tau, \mathcal{I})$ such that $x \in U \subseteq N$ [5].

Definition 2.1. [6] A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $R - \mathcal{I}$ -continuous if for each $x \in X$ and each open set $V \in \sigma$ containing $f(x)$, there exists an $R - \mathcal{I}$ -open set $U \subset X$ such that $x \in U$ and $f(U) \subset V$.

Definition 2.2. [7] Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then the set $\cap\{U \in \tau : A \subset U\}$. is called the kernel of A and denoted by $Ker(A)$.

Lemma 2.1. [2] Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$.

1. $x \in Ker(A)$ if and only if $A \cap F \neq \phi$ for any closed subset F of X containing x
2. $A \subset Ker(A)$ and $A = Ker(A)$ if A is open in X
3. if $A \subset B$, then $Ker(A) \subset Ker(B)$

Lemma 2.2. The following properties hold for a subset A of an ideal topological space $(X, \tau, \mathcal{I})$.

1. $R - \mathcal{I} - Int(A) = X \setminus R - \mathcal{I} - Cl(X - A)$.
2. $x \in R - \mathcal{I} - Cl(A)$ if and only if $A \cap U \neq \phi$ for each $U \in RIO(X, x)$

3 On contra $R - \mathcal{I}$ -continuous Functions

Definition 3.1. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called a contra $R - \mathcal{I}$ -continuous function if $f^{-1}(V)$ is $R - \mathcal{I}$ -closed in X for every open set V in Y .

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$. Let $f : (X, \tau, \mathcal{I}) \rightarrow (X, \sigma)$ be such that $f(a) = c, f(b) = a, f(c) = a$. Then f is contra $R - \mathcal{I}$ -continuous.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$. Let $g : (X, \tau, \mathcal{I}) \rightarrow (X, \sigma)$ be such that $g(a) = a, g(b) = b, g(c) = c$. Then g is contra continuous but not contra $R - \mathcal{I}$ -continuous.

Theorem 3.3. For a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ the following conditions are equivalent.

1. f is contra $R - \mathcal{I}$ -continuous
2. for each $x \in X$ and each closed set F in Y containing $f(x)$ there exist a $R - \mathcal{I}$ -open set U containing x such that $f(U) \subset F$
3. for each $x \in X$ and each closed set F in Y containing $f(x)$, $f^{-1}(F)$ is $R - \mathcal{I}$ -open in X
4. $f(R - \mathcal{I} - Cl(A)) \subset Ker(f(A))$ for every $A \subset X$
5. $(R - \mathcal{I} - Cl(f^{-1}(B))) \subset f^{-1}(Ker(B))$ for every $B \subset Y$

Proof.

1. $\Rightarrow$ 2.

Let $x \in X$ and F be any closed subset of Y containing $f(x)$. Then by the contra $R - \mathcal{I}$ -continuity of f , $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is $R - \mathcal{I}$ -closed in X . Hence by taking $U = f^{-1}(F)$, we get there exists a $R - \mathcal{I}$ -open set U containing x such that $f(U) \subset F$.

2. $\Rightarrow$ 3.

Let F be any closed subset of Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$ and for each $x \in f^{-1}(F)$ there exists a $R - \mathcal{I}$ -open set U_x containing x such that $f(U_x) \subset F$. Hence $f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}$. So $f^{-1}(F)$ is $R - \mathcal{I}$ -open in X .

We use Lemma 1. in all the following implications.

3. $\Rightarrow$ 4.

Let $A \subset X$ and assume that $y \notin Ker(f(A))$. Then there exists a closed set F in Y with $y \in F$ such that $f(A) \cap F = \phi$. This implies $A \cap f^{-1}(F) = \phi$ and so $R - \mathcal{I} - Cl(A) \subset X \setminus f^{-1}(F)$. Thus $R - \mathcal{I} - Cl(A) \cap f^{-1}(F) = \phi$ and we obtain $f(R - \mathcal{I} - Cl(A)) \cap F = \phi$. So $y \notin f(R - \mathcal{I} - Cl(A))$. Hence $f(R - \mathcal{I} - Cl(A)) \subset Ker(f(A))$ for every $A \subset X$.

4. $\Rightarrow$ 5.

Let $B \subset Y$. Then $f(R - \mathcal{I} - Cl(f^{-1}(B))) \subset Ker(f(f^{-1}(B))) \subset Ker(B)$. Thus $(R - \mathcal{I} - Cl(f^{-1}(B))) \subset f^{-1}(Ker(B))$ for every $B \subset Y$.

5. $\Rightarrow$ 1.

Let V be a open set in Y . Then $(R - \mathcal{I} - Cl(f^{-1}(V))) \subset f^{-1}(Ker(V)) = f^{-1}(V)$ and so $(R - \mathcal{I} - Cl(f^{-1}(V))) = f^{-1}(V)$. Thus $f^{-1}(V)$ is $R - \mathcal{I}$ -closed in X and hence f is contra $R - \mathcal{I}$ -continuous.

Theorem 3.4. Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a function which is contra $R - \mathcal{I}$ -continuous where Y is a regular space, then f is $R - \mathcal{I}$ -continuous.

Proof.

Let $x \in X$ and V be an open set in Y with $f(x) \in V$. By the regularity of Y , there exists an open set H in Y containing $f(x)$ such that $Cl(H) \subset V$. Also there exists a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset Cl(H)$, since f is contra $R - \mathcal{I}$ -continuous. Thus $f(U) \subset V$ and f is $R - \mathcal{I}$ -continuous.

Definition 3.2. [5] A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is called $R - \mathcal{I}$ -open if image of each $R - \mathcal{I}$ -open set in X is $R - \mathcal{J}$ -open in Y .

Definition 3.3. [5] A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is $R^* - \mathcal{I}$ -irresolute if the inverse image of every $R - \mathcal{J}$ -open set in Y is $R - \mathcal{I}$ -open in X .

Definition 3.4. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $R - \mathcal{I}$ -connected if X cannot be written as the union of two non-empty disjoint $R - \mathcal{I}$ -open subsets of X .

Definition 3.5. The $R - \mathcal{I}$ -frontier of a subset A of a space X , denoted by $R - \mathcal{I} - \partial(A)$ and is defined as $R - \mathcal{I} - \partial(A) = R - \mathcal{I} - Cl(A) \cap R - \mathcal{I} - Cl(X \setminus A)$.

Definition 3.6. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is called almost- $R - \mathcal{I}$ -continuous if, for each $x \in X$ and for each open set $V \in \mathcal{J}$ containing $f(x)$, there exist a $R - \mathcal{I}$ -open set $U \in \mathcal{I}$ containing x such that $f(U) \subset R - \mathcal{J} - Int(Cl(V))$.

Definition 3.7. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called almost weakly- $R - \mathcal{I}$ -continuous if, for each $x \in X$ and for each open set $V \in \mathcal{J}$ containing $f(x)$, there exist a $R - \mathcal{I}$ -open set $U \in \mathcal{I}$ containing x such that $f(U) \subset Cl(V)$.

Theorem 3.5. Let X be a $R - \mathcal{I}$ -connected space. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a contra $R - \mathcal{I}$ -continuous function, then Y is not a discrete space.

Proof.

On the contrary assume that Y is discrete. Then let S be a proper non-empty clopen set in Y . Since f is contra $R - \mathcal{I}$ -continuous, $f^{-1}(S)$ is a proper non-empty $R - \mathcal{I}$ -clopen set in X . This contradicts X is $R - \mathcal{I}$ -connected.

Theorem 3.6. Let X be a $R - \mathcal{I}$ -connected space. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is contra $R - \mathcal{I}$ -continuous and surjective, then Y is connected.

Proof.

Assume that Y is disconnected. Then $Y = A \cup B$, where A and B are disjoint non-empty clopen subsets of Y . Since f is contra $R - \mathcal{I}$ -continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty $R - \mathcal{I}$ -clopen sets in X . Also $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$. This contradicts X is $R - \mathcal{I}$ -connected and so Y is connected.

Theorem 3.7. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is $R - \mathcal{I}$ -open and contra- $R - \mathcal{I}$ -continuous then f is almost- $R - \mathcal{I}$ -continuous.

Proof.

Let $x \in X$ and $V \in \mathcal{J}$ be an open set containing $f(x)$. Since f is contra $R - \mathcal{I}$ -continuous, there exists a $R - \mathcal{I}$ -open set $U \in \mathcal{I}$ containing x such that $f(U) \subset Cl(V)$. Since f is $R - \mathcal{I}$ -open, $f(U)$ is $R - \mathcal{J}$ -open in Y . So $f(U) = R - \mathcal{J} - Int(f(U))$. Hence $f(U) \subset R - \mathcal{J} - Int(Cl(V))$ and f is almost- $R - \mathcal{I}$ -continuous.

Theorem 3.8. If function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is contra $R - \mathcal{I}$ -continuous then f is almost weakly- $R - \mathcal{I}$ -continuous.

Proof.

Let $x \in X$ and $V \in Y$ be an open set containing $f(x)$. Since f is contra $R - \mathcal{I}$ -continuous, $f^{-1}(Cl(V))$ is $R - \mathcal{I}$ -open in X . Take $U = f^{-1}(Cl(V))$. Then $f(U) \subset Cl(V)$. Thus f is almost weakly- $R - \mathcal{I}$ -continuous.

Theorem 3.9. *The set of all points $x \in X$ at which $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is not contra- $R - \mathcal{I}$ -continuous is identical with the union of $R - \mathcal{I}$ -frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof.

First assume that f is not contra $R - \mathcal{I}$ -continuous at $x \in X$. Then there exists a closed set $F \subset Y$ containing $f(x)$ such that $f(U) \cap (Y \setminus F) \neq \emptyset$ for every $R - \mathcal{I}$ -open set $U \subset X$ containing x . So $U \cap f^{-1}(Y \setminus F) \neq \emptyset$. Thus $x \in R - \mathcal{I} - Cl(f^{-1}(Y \setminus F)) = R - \mathcal{I} - Cl(X \setminus f^{-1}(F))$. Since $x \in f^{-1}(F)$, $x \in R - \mathcal{I} - Cl(f^{-1}(F))$. This implies $x \in R - \mathcal{I} - \partial(f^{-1}(F))$.

Now suppose that $x \in R - \mathcal{I} - \partial(f^{-1}(F))$ for some closed set $F \subset Y$ containing $f(x)$. Assume that f is contra $R - \mathcal{I}$ -continuous at x . Then there exists a $R - \mathcal{I}$ -open set $U \subset X$ containing x such that $f(U) \subset F$. This implies $x \in U \subset f^{-1}(F)$. So $x \notin R - \mathcal{I} - \partial(f^{-1}(F))$, a contradiction. Thus f is not contra $R - \mathcal{I}$ -continuous at x .

Theorem 3.10. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a contra $R - \mathcal{I}$ -continuous function and $g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta)$ be a continuous function. Then $g \circ f : (X, \tau, \mathcal{I}) \rightarrow (Z, \eta)$ is contra $R - \mathcal{I}$ -continuous.*

Proof.

Let $x \in X$ and $W \in Z$ be a closed set containing $(g \circ f)(x)$. Then $V = g^{-1}(W)$ is closed in Y by the continuity of g . Also, since f is contra $R - \mathcal{I}$ -continuous, there exists a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset V = g^{-1}(W)$. Thus $(g \circ f)(U) \subset W$. Hence $g \circ f : (X, \tau, \mathcal{I}) \rightarrow (Z, \eta)$ is contra $R - \mathcal{I}$ -continuous.

Theorem 3.11. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a $R^* - \mathcal{I}$ -irresolute function and $g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta)$ be a contra $R - \mathcal{I}$ -continuous function. Then $g \circ f : (X, \tau, \mathcal{I}) \rightarrow (Z, \eta)$ is contra $R - \mathcal{I}$ -continuous.*

Proof.

Let $x \in X$ and $W \in Z$ be a closed set containing $(g \circ f)(x)$. Since g is contra $R - \mathcal{I}$ -continuous, there exists a $R - \mathcal{J}$ -open set $V \in Y$ containing $f(x)$ such that $g(V) \subset W$. Also, since f is $R^* - \mathcal{I}$ -irresolute, $f^{-1}(V)$ is $R - \mathcal{I}$ -open in X containing x . Take $U = f^{-1}(V)$. Thus $(g \circ f)(U) \subset W$. Hence $g \circ f : (X, \tau, \mathcal{I}) \rightarrow (Z, \eta)$ is contra $R - \mathcal{I}$ -continuous.

Theorem 3.12. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a surjective $R^* - \mathcal{I}$ -irresolute and $R - \mathcal{I}$ -open function and $g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta)$ be any function. Then $g \circ f : (X, \tau, \mathcal{I}) \rightarrow (Z, \eta)$ is contra $R - \mathcal{I}$ -continuous if and only if g is contra $R - \mathcal{I}$ -continuous.*

Proof.

If g is contra $R - \mathcal{I}$ -continuous, then the proof is clear. Now assume $g \circ f$ is contra $R - \mathcal{I}$ -continuous.

Let W be closed in Z . Then $(g \circ f)^{-1}(W)$ is $R-\mathcal{I}$ -open in X . Since f is $R-\mathcal{I}$ -open, $f(f^{-1}(g^{-1}(W)))$ is $R-\mathcal{J}$ -open in Y . Hence $g^{-1}(W)$ is $R-\mathcal{J}$ -open in Y and so g is contra $R-\mathcal{I}$ -continuous.

Definition 3.8. [8] A topological space $(X, \tau, \mathcal{I})$ is said to be $R-\mathcal{I}-T_1$ if for each pair of distinct points x and y in X , there exists $R-\mathcal{I}$ -open sets U and V of X , such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Definition 3.9. [8] A topological space $(X, \tau, \mathcal{I})$ is said to be $R-\mathcal{I}-T_2$ if for each pair of distinct points x and y in X , there exists disjoint $R-\mathcal{I}$ -open sets U and V in X such that $x \in U$ and $y \in V$.

Definition 3.10. [8] An ideal topological space X is $R-\mathcal{I}$ -normal if for each pair A, B of disjoint $R-\mathcal{I}$ -closed sets of X , there exists disjoint $R-\mathcal{I}$ -open sets containing A and B respectively.

Definition 3.11. [9] A space (X, τ) is said to be ultra normal if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Definition 3.12. A topological space (X, τ) is ultra Hausdorff [9] if for each pair of distinct points x and y of X there exist closed sets U and V such that $x \in U, y \in V$ and $U \cap V = \phi$.

A topological space (X, τ) is said to be weakly Hausdorff [T. Soundararajan, *Weakly Hausdorff spaces and the cardinality of topological spaces in general topology and its relations to modern analysis and algebra, III Proc. Conf. Kanpur, (1968), Academia, Prague, (1971), 301-306.*] if each element of X is the intersection of regular closed sets of X .

Definition 3.13. [10] A topological space (X, τ) is called Urysohn if, for each $x, y \in X$ with $x \neq y$, there exist open sets P and Q of X containing x and y respectively such that $Cl(P) \cap Cl(Q) = \phi$.

Theorem 3.13. Let $(X, \tau, \mathcal{I})$ be a $R-\mathcal{I}$ -connected space and (Y, σ) a T_1 space. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is contra $R-\mathcal{I}$ -continuous, then f is constant.

Proof.

Let X be a $R-\mathcal{I}$ -connected space and Y be a T_1 space. Then $\mathcal{P} = \{f^{-1}(y : y \in Y)\}$ is a disjoint collection of $R-\mathcal{I}$ -open sets which partitions X . If $|\mathcal{P}| \geq 2$, then X is the union of two disjoint non-empty $R-\mathcal{I}$ -open sets. This implies $|\mathcal{P}| = 1$ and hence f is constant.

Theorem 3.14. Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a contra $R-\mathcal{I}$ -continuous and one-one function. If Y is a Urysohn space, then X is $R-\mathcal{I}-T_2$.

Proof.

Let $x \neq y \in X$. Then $f(x) \neq f(y)$. Since Y is a Urysohn space, there exist open sets $P, Q \in Y$ such that $f(x) \in P, f(y) \in Q$ and $Cl(P) \cap Cl(Q) = \phi$. Since f is contra $R-\mathcal{I}$ -continuous at there exist $R-\mathcal{I}$ -open sets $U, V \in X$ such that $x \in U, y \in V$ and $f(U) \subset Cl(P), f(V) \subset Cl(Q)$. Thus $U \cap V = \phi$, since $f(U) \cap f(V) = \phi$. Hence X is $R-\mathcal{I}-T_2$.

Theorem 3.15. Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a contra $R-\mathcal{I}$ -continuous, closed and one-one function where Y is ultra normal, then X is $R-\mathcal{I}$ -normal.

Proof.

Let $A, B \in X$ be disjoint closed subsetsets. Since f is cosed and one-one, $f(A)$ and $f(B)$ are disjoint closed subsets in Y . Since Y is ultra normal, $f(A)$ and $f(B)$ are separated by disjoint clopen sets, say, U and V . Since f is contra $R - \mathcal{I}$ -continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $R - \mathcal{I}$ -open and $A \subset f^{-1}(U)$, $B \subset f^{-1}(V)$. This gives $f^{-1}(U) \cap f^{-1}(V) = \phi$ and so X is $R - \mathcal{I}$ -normal.

Theorem 3.16. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a contra $R - \mathcal{I}$ -continuous and one-one function where Y is ultra Hausdorff, then X is $R - \mathcal{I} - T_2$.*

Proof.

Let $x \neq y \in X$. Then $f(x) \neq f(y)$. Since Y is ultra Hausdorff, there exists closed sets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \phi$. Since f is contra $R - \mathcal{I}$ -continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $R - \mathcal{I}$ -open in X containing x and y respectively with $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence X is $R - \mathcal{I} - T_2$.

4 Almost contra $R - \mathcal{I}$ -continuous Functions

Definition 4.1. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called almost contra $R - \mathcal{I}$ -continuous if $f^{-1}(V)$ is $R - \mathcal{I}$ -closed in $(X, \tau, \mathcal{I})$ for every regular open set V in (Y, σ) .

Definition 4.2. An ideal topological space is said to be countably $R - \mathcal{I}$ -compact if every countable $R - \mathcal{I}$ -open cover admits a finite subcover.

Definition 4.3. An ideal topological space is said to be $R - \mathcal{I}$ -closed compact ($R - \mathcal{I}$ -closed Lindeloff) if every $R - \mathcal{I}$ -closed cover admits a finite (countable) subcover.

An ideal topological space is said to be countably $R - \mathcal{I}$ -closed compact if every countable $R - \mathcal{I}$ -closed cover admits a finite subcover.

Definition 4.4. [5] An ideal topological space is said to be $R - \mathcal{I}$ -compact ($R - \mathcal{I}$ -Lindeloff) if every $R - \mathcal{I}$ -open cover admits a finite (countable) subcover.

Definition 4.5. A topological space X is called S-closed [11] (resp. countably S-closed [12], S-Lindeloff [3]) if every regular closed (resp. countably regular closed, regular closed) cover of X has a finite (resp. finite, countable) subcover.

A topological space X is said to be nearly compact (resp. nearly countably compact, nearly Lindeloff) if every regular open (resp. countable regular open, regular open) cover of X has a finite (resp. finite, a countably) subcover. [10]

Theorem 4.1. *The following statements are equivalent for a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$:*

1. f is almost contra $R - \mathcal{I}$ -continuous
2. $f^{-1}(F)$ is $R - \mathcal{I}$ -open in X for each regular closed set F in Y
3. for each $x \in X$ and each regular closed set F in Y containing $f(x)$ there exist a $R - \mathcal{I}$ -open set U in X containing x such that $f(U) \subset F$
4. for each $x \in X$ and each regular open set V in Y not containing $f(x)$ there exist a $R - \mathcal{I}$ -closed set C in X not containing x such that $f^{-1}(V) \subset C$

Proof.

1. $\Leftrightarrow$ 2.

Let F be a regular closed set in Y . Then $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is $R - \mathcal{I}$ -closed in X . Hence $f^{-1}(F)$ is $R - \mathcal{I}$ -open in X . In a similar manner, the converse part can be proved.

2. $\Leftrightarrow$ 3.

Let F be a regular closed set in Y containing $f(x)$. Then $f^{-1}(F)$ is $R - \mathcal{I}$ -open in X and contains x . Take $U = f^{-1}(F)$. Hence U is a $R - \mathcal{I}$ -open set in X containing x such that $f(U) \subset F$.

Let F be a regular closed set in Y and let $x \in f^{-1}(F)$. Then there exists a $R - \mathcal{I}$ -open set U_x containing x such that $f(U_x) \subset F$. Hence $f^{-1}(F) = \cup \{U_x : x \in f^{-1}(F)\}$. So $f^{-1}(F)$ is $R - \mathcal{I}$ -open in X .

3. $\Leftrightarrow$ 4.

Let $x \in X$ and V be a regular open set not containing $f(x)$. Then $Y \setminus V$ is a regular closed set containing $f(x)$. So there will exist a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset (Y \setminus V)$. Hence $U \subset f^{-1}(Y \setminus V) \subset X \setminus f^{-1}(V)$. So $f^{-1}(V) \subset X \setminus U$. Taking $C = X \setminus U$ gives a $R - \mathcal{I}$ -closed set C in X not containing x such that $f^{-1}(V) \subset C$. Similarly, the converse can be proved.

Theorem 4.2. *If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is almost contra $R - \mathcal{I}$ -continuous, then f is almost weakly- $R - \mathcal{I}$ -continuous.*

Proof.

Let $x \in X$ and V be an open set in Y containing $f(x)$. Obviously $Cl(V)$ is regular closed in Y containing $f(x)$. Then, since f is almost contra $R - \mathcal{I}$ -continuous, there exist a $R - \mathcal{I}$ -open set U in X containing x such that $f(U) \subset Cl(V)$. Hence f is almost weakly- $R - \mathcal{I}$ -continuous.

Lemma 4.3. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is almost $R - \mathcal{I}$ -continuous, if and only if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset V$.*

Proof.

The 'if part' is trivial. The 'only if' part can be proved by taking V as a open set in Y and hence $R - \mathcal{J} - Int(Cl(V))$ will be regular open .

Theorem 4.4. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a function where Y is extremally disconnected. Then f is almost contra $R - \mathcal{I}$ -continuous if and only if f is almost $R - \mathcal{I}$ -continuous.*

Note. A topological (X, τ) is extremally disconnected if the closure of every open set in X is open in X .

Proof.

Let $x \in X$ and $V \in Y$ be a regular open set containing $f(x)$. Now Y is extremally disconnected $\rightarrow Cl(V)$ is open $\rightarrow Int(Cl(V)) = V \rightarrow Cl(V) = V \rightarrow V$ is regular closed. Since f is almost contra $R - \mathcal{I}$ -continuous, there exist a $R - \mathcal{I}$ -open set U in X containing x such that $f(U) \subset V$. Then by the above lemma, f is almost $R - \mathcal{I}$ -continuous.

Conversely assume f is almost $R - \mathcal{I}$ -continuous. Let $x \in X$ and $V \in Y$ be a regular closed set containing $f(x)$. This implies $Cl(Int(V)) = V$. Since Y is extremally disconnected, we get V is open in Y and thus is regular open in Y . Since f is almost $R - \mathcal{I}$ -continuous, there exist a $R - \mathcal{I}$ -open

set $U \in X$ containing x such that $f(U) \subset V$ by lemma 3. Thus f is almost contra $R - \mathcal{I}$ -continuous.

Theorem 4.5. *If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is an almost contra $R - \mathcal{I}$ -continuous and one-one function where Y is weakly Hausdorff, then X is $R - \mathcal{I} - T_1$.*

Proof.

Let $x \neq y \in X$. Since f is one-one, $f(x) \neq f(y)$ in Y . Since Y is weakly Hausdorff there exist disjoint regular closed sets C and D such that $f(x) \in C$, $f(y) \notin C$ and $f(y) \in D$, $f(x) \notin D$. But f is almost contra $R - \mathcal{I}$ -continuous. So $f^{-1}(U)$ and $f^{-1}(V)$ are $R - \mathcal{I}$ -open sets in X containing x and y respectively. Also $x \notin f^{-1}(V)$ and $y \notin f^{-1}(U)$. Hence X is $R - \mathcal{I} - T_1$.

Remark 4.1. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is a contra $R - \mathcal{I}$ -continuous and one-one function where Y is weakly Hausdorff, then X is $R - \mathcal{I} - T_1$.

Theorem 4.6. *If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is an almost contra $R - \mathcal{I}$ -continuous and onto function where X is $R - \mathcal{I}$ -connected, then Y is connected.*

Proof.

On the contrary assume that Y is disconnected. Then $Y = A \cup B$, where A and B are disjoint non-empty clopen subsets of Y . A and B are regular open in Y since they are clopen. Since f is almost contra $R - \mathcal{I}$ -continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $R - \mathcal{I}$ -open in X . Since f is onto, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \phi$ which contradicts to the fact that X is $R - \mathcal{I}$ -connected. Hence Y is connected.

Theorem 4.7. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a surjective almost contra- $R - \mathcal{I}$ -continuous function. Then the following statements are true:*

1. X is $R - \mathcal{I}$ -compact implies Y is S -closed
2. X is $R - \mathcal{I}$ -Lindeloff implies Y is S -Lindeloff
3. X is countably $R - \mathcal{I}$ -compact implies Y is countably S -closed

Proof.

1. Let $\{V_j : j \in J\}$ be an regular closed cover of Y . Then $\{f^{-1}(V_j) : j \in J\}$ is a $R - \mathcal{I}$ -open cover of X since f is almost contra- $R - \mathcal{I}$ -continuous. But X is $R - \mathcal{I}$ -compact. Hence there exists a finite subset J_κ of J such that $X = \cup_{j \in J_\kappa} \{f^{-1}(V_j)\}$. So $Y = \cup\{V_j : j \in J_\kappa\}$ and thus Y is S -closed.
2. and 3. can be proved in a similar fashion.

Theorem 4.8. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a surjective almost contra- $R - \mathcal{I}$ -continuous function. Then the following statements are true:*

1. X is R - $\mathcal{I}$ -closed compact implies Y is nearly compact
2. X is $R - \mathcal{I}$ -closed Lindeloff implies Y is nearly Lindeloff
3. X is countably $R - \mathcal{I}$ -closed compact implies Y is nearly countably compact

Proof.

1. Let $\{V_j : j \in J\}$ be a regular open cover of Y . Then $\{f^{-1}(V_j) : j \in J\}$ is a $R - \mathcal{I}$ -closed cover of X since f is almost contra- $R - \mathcal{I}$ -continuous. But X is $R - \mathcal{I}$ -closed compact. Hence there exists a finite subset J_κ of J such that $X = \bigcup_{j \in J_\kappa} \{f^{-1}(V_j)\}$. So $Y = \bigcup \{V_j : j \in J_\kappa\}$ and thus Y is nearly compact.

2. and 3. can be proved in a similar fashion.

5 Graphs

Definition 5.1. For a function $f : X \rightarrow Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 5.2. The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $R - \mathcal{I}$ -closed (resp. contra $R - \mathcal{I}$ -closed) if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x and an open (resp. a closed) set $V \in Y$ containing y such that $(U \times V) \cap G(f) = \phi$.

Definition 5.3. The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be strongly contra $R - \mathcal{I}$ -closed if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x and a regular closed set $V \in Y$ containing y such that $(U \times V) \cap G(f) = \phi$.

Definition 5.4. [13] A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called $R - \mathcal{I}$ -dense if $R - \mathcal{I} - Cl(A) = X$.

Lemma 5.1. The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $R - \mathcal{I}$ -closed (resp. contra $R - \mathcal{I}$ -closed) if and only if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x and an open (resp. a closed) set $V \in Y$ containing y such that $f(U) \cap V = \phi$.

Proof.

Suppose that $(U \times V) \cap G(f) \neq \phi$. Then there exists at least one $(x, y) \in X \times Y$ such that $(x, y) \in U \times V$ and $(x, y) \in G(f)$. That means, $x \in U$, $y \in V$ and $V \ni y = f(x) \in f(U)$. Hence $f(U) \cap V \neq \phi$.

Lemma 5.2. The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be strongly contra $R - \mathcal{I}$ -closed if and only if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x and a regular closed set $V \in Y$ containing y such that $f(U) \cap V = \phi$.

Theorem 5.3. If a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is contra $R - \mathcal{I}$ -continuous and Y is Urysohn, then $G(f)$ is contra $R - \mathcal{I}$ -closed in $X \times Y$.

Proof.

Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Y is Urysohn implies that there exist open sets P and Q of Y containing $f(x)$ and y respectively such that $Cl(P) \cap Cl(Q) = \phi$. Since $Cl(P)$ is a closed set containing $f(x)$ and since f is contra $R - \mathcal{I}$ -continuous, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset Cl(P)$. Hence $f(U) \cap Cl(Q) = \phi$ and so $G(f)$ is contra $R - \mathcal{I}$ -closed in $X \times Y$ by lemma 4.

Theorem 5.4. *If a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is $R - \mathcal{I}$ -continuous and Y is T_1 , then $G(f)$ is contra $R - \mathcal{I}$ -closed in $X \times Y$.*

Proof.

Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Since Y is T_1 there exists an open set $V \in Y$ such that $f(x) \in V$, $y \notin V$. Since f is $R - \mathcal{I}$ -continuous, there exists an $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset V$. Then $f(U) \cap (Y \setminus V) = \phi$. But $(Y \setminus V)$ is a closed set in Y containing y . Thus $G(f)$ is contra $R - \mathcal{I}$ -closed in $X \times Y$ by lemma 4.

Theorem 5.5. *Let Y be a Urysohn space. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ and $g : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ are contra $R - \mathcal{I}$ -continuous functions, then $S = \{x \in X : f(x) = g(x)\}$ is $R - \mathcal{I}$ -closed in X .*

Proof.

Let $x \notin S$. That is $x \in (X \setminus S)$ and so $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets P and Q of Y containing $f(x)$ and $g(x)$ respectively such that $Cl(P) \cap Cl(Q) = \phi$. Since f is contra $R - \mathcal{I}$ -continuous, $f^{-1}(Cl(P))$ and $f^{-1}(Cl(Q))$ are $R - \mathcal{I}$ -open in X containing x . Name $f^{-1}(Cl(P)) = A$ and $f^{-1}(Cl(Q)) = B$ and let $R = A \cap B$. Then R is a $R - \mathcal{I}$ -open set in X containing x . Now, $f(R) \cap g(R) = f(A \cap B) \cap g(A \cap B) \subset f(A) \cap g(B) = Cl(P) \cap Cl(Q) = \phi$. Hence $S \cap R = \phi \rightarrow S \subset (X \setminus R) \rightarrow R - \mathcal{I} - Cl(S) \subset (X \setminus R) \rightarrow x \notin R - \mathcal{I} - Cl(S)$. Thus $S = \{x \in X : f(x) = g(x)\}$ is $R - \mathcal{I}$ -closed in X .

Theorem 5.6. *Let Y be a Urysohn space. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ almost weakly $R - \mathcal{I}$ -continuous, then $G(f)$ is strongly contra $R - \mathcal{I}$ -closed in $X \times Y$.*

Proof.

Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Y is Urysohn implies that there exist open sets P and Q of Y containing $f(x)$ and y respectively such that $Cl(P) \cap Cl(Q) = \phi$. Since f is almost weakly $R - \mathcal{I}$ -continuous, there exist a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset Cl(P)$. Hence $f(U) \cap Cl(Q) = \phi \rightarrow f(U) \cap Cl(Int(Q)) = \phi$. Since $Cl(Int(Q))$ is regular closed in Y , $G(f)$ is strongly contra $R - \mathcal{I}$ -closed in $X \times Y$ by lemma 5.

Theorem 5.7. *If a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is almost $R - \mathcal{I}$ -continuous and Y is T_2 , then $G(f)$ is strongly contra $R - \mathcal{I}$ -closed in $X \times Y$.*

Proof.

Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Since Y is T_2 there exists disjoint open sets A and B containing $f(x)$ and y respectively. Clearly $Cl(B)$ is regular closed and $Int(Cl(A))$ is regular open. A and B are disjoint implies $Cl(B) \cap Int(Cl(A)) = \phi$. Since f is almost $R - \mathcal{I}$ -continuous, by lemma 3., there exists a $R - \mathcal{I}$ -open set $U \in X$ containing x such that $f(U) \subset Int(Cl(A))$. Hence $f(U) \cap Cl(B) = \phi$ and thus $G(f)$ is strongly contra $R - \mathcal{I}$ -closed in $X \times Y$ by lemma 5.

Theorem 5.8. *Let Y be a Urysohn space and $D \subset X$ be a $R - \mathcal{I}$ -dense set. Let $f, g : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be any two contra $R - \mathcal{I}$ -continuous functions. If $f = g$ on D , then $f = g$ on X .*

Proof.

By theorem 5.5, $S = \{x \in X : f(x) = g(x)\}$ is $R - \mathcal{I}$ -closed in X . Clearly $D \subset S$. Since D is $R - \mathcal{I}$ -dense, $R - \mathcal{I} - Cl(D) = X$. Thus $X = R - \mathcal{I} - Cl(D) \subset R - \mathcal{I} - Cl(S) = S$. Hence $f = g$ on X .

Declaration of AI Use

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