Gaussian Generalized Olivier Numbers

Abstract. This study investigates the structural properties and potential applications of the Gaussian generalized Olivier numbers, contributing to the broader theoretical landscape of integer sequences. Through a combination of algebraic and analytic methods, we establish new recurrence relations, summation identities, and diverse representations for these sequences. In particular, we derive a Binet-type expression, construct generating functions, and develop matrix formulations, while also presenting Simson's formula as an alternative analytical tool. Special attention is devoted to two notable cases—the Gaussian Olivier numbers and the Gaussian Olivier–Lucas numbers—whose distinctive features highlight the richness of the framework. Our findings demonstrate that these sequences exhibit unique combinatorial behaviors, making them relevant for applications such as coding theory.

Keywords: Olivier numbers, Gaussian Olivier numbers, Pell-Padovan numbers, Pell-Perrin numbers.

1. Introduction

In recent years, generalized number sequences and their special cases have emerged as a central theme in mathematical research, owing to their rich algebraic structures and diverse applications across number theory, combinatorics, and applied mathematics. Among these, Gaussian-based constructions and Leonardo-type sequences provide fertile ground for exploring recurrence relations, generating functions, and structural identities. This paper extends these frameworks by introducing and analyzing Gaussian generalized Olivier numbers, thereby contributing to the theoretical development of sequence theory and its potential applications.

This framework not only generalizes classical number sets but also provides fertile ground for constructing new families of integer, rational, and polynomial sequences with distinctive properties. In particular, Gaussian numbers have been successfully applied to Fibonacci-type and Lucas-type sequences, as well as to higher-order recurrences, yielding elegant closed forms, generating functions, and matrix representations. These developments demonstrate how Gaussianization enriches the study of integer sequences by embedding complex and functional structures into recurrence theory.

Within this broader landscape, Leonardo-type sequences have gained particular attention for their role in exploring recurrence relations with intricate structural behavior. For further details on Leonardo-type sequences, see Soykan (2026) [8,9]. These studies emphasize how Leonardo-type sequences serve as bridges between classical integer families and new algebraic frameworks, offering insights into sequence growth, combinatorial interpretations, and structural identities.

Generalized Olivier numbers belong to this Leonardo-type family. They exhibit deep connections with adjusted Pell-Padovan, Pell-Padovan, Pell-Perrin, third-order Fibonacci-Pell, and third-order Lucas-Pell numbers, while introducing distinctive algebraic identities and third-order recurrence structures. The purpose of this paper is to investigate generalized Gaussian Olivier numbers and their associated properties. We begin with a review of Gaussian numbers and their applications, then recall the generalized Olivier sequences. Building on this foundation, we define generalized Gaussian Olivier numbers, derive their Binet-type formulas and generating functions, and establish summation identities and structural properties. In doing so, we highlight the interplay between Gaussian numbers and recurrence theory, showing how Gaussianization extends the scope of algebraic and combinatorial analysis and enriches the study of integer sequences.

1.1. Gaussian Numbers. In this subsection, we recall the notion of Gaussian numbers and emphasize both their classical role and contemporary (modern) extensions.

Traditionally, Gaussian numbers-more commonly referred to as Gaussian integers-form a distinguished subset of the complex number system. Complex numbers are generally expressed in the form

$$z = a + bi,$$

with $a, b \in \mathbb{R}$ and $i^2 = -1$. Gaussian integers impose the stricter condition that both a and b must be integers. This restriction endows them with unique algebraic and arithmetic properties, making them indispensable in areas such as factorization theory, Diophantine equations, and the distribution of primes in the complex plane.

In contemporary research, however, the framework has been broadened considerably. The parameters a and b are no longer confined to integers or real numbers; they may be taken as real functions, complex functions, or even polynomials. Such generalizations naturally lead to Gaussian constructions and combined recurrence relations, as seen in the study of Oresme numbers, Jacobsthal polynomials, and Mersenne polynomials. For example, Oresme numbers extend the classical integer-based definitions by embedding recurrence structures into rational-number frameworks, while Jacobsthal polynomials enrich the theory by embedding recurrence structures into polynomial frameworks. These developments highlight the algebraic and combinatorial depth of Gaussian-type sequences.

In fact, Gaussian Oresme numbers are defined by

$$GO_n = O_n + iO_{n-1},$$

with initial values $GO_0 = -2i$ and $GO_1 = \frac{1}{2}$, where the underlying Oresme numbers are given by the recurrence

$$O_{n+2} = O_{n+1} - \tfrac{1}{4}O_n, \quad O_0 = 0,\ O_1 = \tfrac{1}{2}.$$

This construction illustrates that, within Gaussian Oresme numbers, the coefficients may themselves be rational numbers, thereby allowing a and b to vary with n and expanding the scope of Gaussian-type sequences beyond their classical integer constraints.

This evolution underscores the versatility of Gaussian numbers: from their classical role in number theory to their modern incarnations as polynomial and functional sequences, they continue to provide a fertile ground for theoretical exploration and practical applications.

1.2. Generalized Olivier Numbers. In this subsection, we revisit the framework of generalized Olivier numbers and highlight their two notable specializations: the Olivier sequence and the Olivier–Lucas sequence. These sequences, defined through recurrence relations that extend classical integer families, exhibit distinctive structural properties that enrich the study of number theory. Their algebraic behavior and interconnections have implications ranging from combinatorial analysis to applications in coding theory, underscoring their theoretical and practical importance.

A variety of related sequences have been introduced in the literature, each governed by third-order recurrence relations. Among these are the adjusted Pell-Padovan sequence $\{M_n\}_{n\geq 0}$, third order Lucas-Pell sequence $\{B_n\}_{n\geq 0}$ (OEIS: A099925, [2]), third order Fibonacci-Pell sequence $\{G_n\}_{n\geq 0}$ (OEIS: A008346, [2]), Pell-Perrin sequence $\{C_n\}_{n\geq 0}$, Pell-Padovan sequence$\{R_n\}_{n\geq 0}$ (OEIS: A066983, [2]). These are defined respectively by

$$M_{n+3} = 2M_{n+1} + M_n, \qquad M_0 = 0, M_1 = 1, M_2 = 0, \tag{1.1}$$

$$B_{n+3} = 2B_{n+1} + B_n, \qquad B_0 = 3, B_1 = 0, B_2 = 4 \tag{1.2}$$

$$G_{n+3} = 2G_{n+1} + G_n, \qquad G_0 = 1, G_1 = 0, G_2 = 2, \tag{1.3}$$

$$C_{n+3} = 2C_{n+1} + C_n, \qquad C_0 = 3, C_1 = 0, C_2 = 2, \tag{1.4}$$

$$R_{n+3} = 2R_{n+1} + R_n, \qquad R_0 = 1, R_1 = 1, R_2 = 1. \tag{1.5}$$

For more details on generalized Pell–Padovan numbers and their special cases, see Soykan [4].

Building on these constructions, Olivier and Olivier–Lucas numbers are defined by the recurrences

$$O_n = 2O_{n-2} + O_{n-3} + 1, \quad \text{with} \quad O_0 = 0, O_1 = 1, O_2 = 1, \quad n \geq 3,$$

and

$$K_n = 2K_{n-2} + K_{n-3} - 2, \quad \text{with} \quad K_0 = 4, K_1 = 1, K_2 = 5\ , \quad n \geq 3,$$

respectively. Both sequences satisfy equivalent fourth-order linear recurrences:

$$\begin{aligned} O_n &= O_{n-1}+2O_{n-2}-O_{n-3}-O_{n-4}, \qquad O_0=0, O_1=1, O_2=1, O_3=3, \qquad n\geq 4,\\ K_n &= K_{n-1}+2K_{n-2}-K_{n-3}-K_{n-4}, \qquad K_0=4, K_1=1, K_2=5, K_3=4, \qquad n\geq 4,\end{aligned}$$

These sequences are closely related to the Pell–Padovan families. For example, they satisfy the following interrelations:

$$\begin{aligned} 2O_n &= M_{n+2}+M_{n+1}+M_n-1, \quad K_n=-2M_{n+2}+3M_{n+1}+4M_n+1,\\ 10O_n &= -7B_{n+2}+9B_{n+1}+11B_n-5, \ \ K_n=B_n+1,\\ 2O_n &= G_{n+2}+G_{n+1}-G_n-1, \ \ K_n=4G_{n+2}-2G_{n+1}-5G_n+1,\\ 22O_n &= 13C_{n+2}-3C_{n+1}-5C_n-11, \ \ 11K_n=-12C_{n+2}+18C_{n+1}+19C_n+11,\\ 2O_n &= R_{n+2}-1, \ \ 2K_n=-3R_{n+2}+4R_{n+1}+5R_n+2.\end{aligned}$$

The generalized Olivier sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2,W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n=W_{n-1}+2W_{n-2}-W_{n-3}-W_{n-4} \tag{1.6}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2, W_3=c_3$ not all being zero.

The sequence can be extended to negative indices by defining

$$W_{-n}=-W_{-(n-1)}+2W_{-(n-2)}+W_{-(n-3)}-W_{-(n-4)}, \qquad n=1,2,3,\ldots$$

so that recurrence (1.6) holds for all integers n. For more information on generalized Olivier sequence, see Soykan [5].

The characteristic equation of $\{W_n\}$ is

$$z^4-z^3-2z^2+z+1=(z^3-2z-1)(z-1)=(z^2-z-1)(z+1)(z-1)=0$$

with roots

$$\alpha=\frac{1+\sqrt{5}}{2}, \quad \beta=\frac{1-\sqrt{5}}{2}, \quad \gamma=-1, \quad \delta=1.$$

Two notable specializations, Olivier sequence $\{O_n\}\}_{n\geq 0}$ and Olivier-Lucas sequence $\{K_n\}_{n\geq 0}$, of $\{W_n\}$ are obtained by choosing particular initial conditions:

$$O_n = O_{n-1}+2O_{n-2}-O_{n-3}-O_{n-4}, \qquad O_0=0, O_1=1, O_2=1, O_3=3, \qquad n\geq 4, \tag{1.7}$$

$$K_n = K_{n-1}+2K_{n-2}-K_{n-3}-K_{n-4}, \qquad K_0=4, K_1=1, K_2=5, K_3=4, \qquad n\geq 4, \tag{1.8}$$

These sequences can also be extended to negative indices by applying the same recurrence relations:

$$\begin{aligned} O_{-n} &= -O_{-(n-1)}+2O_{-(n-2)}+O_{-(n-3)}-O_{-(n-4)}\\ K_{-n} &= -K_{-(n-1)}+2K_{-(n-2)}+K_{-(n-3)}-K_{-(n-4)}\end{aligned}$$

for $n = 1, 2, 3, ...$ Hence, recurrences (1.7)-(1.8) are valid for all integers n.

The explicit form of generalized Olivier numbers can be obtained through a Binet-type representation. This closed-form expression highlights the role of the characteristic roots and provides a powerful tool for both theoretical analysis and computational applications.

THEOREM 1.1. *For all integers n, the generalized Olivier sequence $\{W_n\}$ admits the Binet's formula*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n \\
&= A_1\alpha^n + A_2\beta^n + A_3(-1)^n + A_4 \quad \textit{[6, p. 38, Theorem 1.1]},
\end{aligned} \tag{1.9}$$

which can also be expressed as

$$\begin{aligned}
W_n &= \frac{(\alpha W_3 - \alpha(1-\alpha)W_2 + (-\alpha^2+1)W_1 - W_0)\alpha^n}{4\alpha^2-\alpha-3} \\
&\quad + \frac{(\beta W_3 - \beta(1-\beta)W_2 + (-\beta^2+1)W_1 - W_0)\beta^n}{4\beta^2-\beta-3} \\
&\quad + \frac{(-W_3+2W_2-W_0)(-1)^n}{2} - \frac{W_3-2W_1-W_0}{2}, \quad \textit{[5, p. 8, Theorem 2.1]},
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0 \\
&= W_3 + \frac{1}{2}(\sqrt{5}-1)W_2 - W_1 - \frac{1}{2}(\sqrt{5}-1)W_0, \\
p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0 \\
&= W_3 - \frac{1}{2}(\sqrt{5}+1)W_2 - W_1 + \frac{1}{2}(\sqrt{5}+1)W_0, \\
p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0 = W_3 - 2W_2 + W_0, \\
p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0 = W_3 - 2W_1 - W_0,
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} = \frac{2}{5+\sqrt{5}}p_1, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} = \frac{2}{5-\sqrt{5}}p_2, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} = -\frac{1}{2}p_3, \\
A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = -\frac{1}{2}p_4.
\end{aligned}$$

This representation emphasizes the contribution of each root $\alpha, \beta, \gamma = -1, \delta = 1$ of the characteristic polynomial, and shows how the initial conditions determine the coefficients of the closed form.

COROLLARY 1.2. *For all integers n, the Binet's formulas of Olivier and Olivier–Lucas numbers are given by*

$$\begin{aligned} O_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \\ &= \frac{1}{10}(\sqrt{5}+5)\alpha^n + \frac{1}{10}(-\sqrt{5}+5)\beta^n - \frac{1}{2}(-1)^n - \frac{1}{2} \\ &\quad \textit{[6, p.49, Theorem 1.4], [5, p. 8, Corollary 2.2],} \end{aligned}$$

and

$$\begin{aligned} K_n &= \alpha^n + \beta^n + (-1)^n + 1 = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n + (-1)^n + 1, \\ &\quad \textit{[5, p. 8, Corollary 2.2],} \end{aligned}$$

respectively.

The generating function provides a compact representation of the entire sequence and is a fundamental tool for deriving identities and summation formulas.

LEMMA 1.3. *[5, p. 11, Lemma 2.4]. Suppose $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Olivier sequence $\{W_n\}$. Then*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z + (W_2 - W_1 - 2W_0)z^2 + (W_3 - W_2 - 2W_1 + W_0)z^3}{1 - z - 2z^2 + z^3 + z^4}.$$

The previous lemma gives the following results as particular examples.

COROLLARY 1.4. *[5, p. 12, Corollary 2.5]. The generating functions of Olivier and Olivier–Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} O_n z^n &= \frac{z}{1 - z - 2z^2 + z^3 + z^4}, \\ \sum_{n=0}^{\infty} K_n z^n &= \frac{4 - 3z - 4z^2 + z^3}{1 - z - 2z^2 + z^3 + z^4}, \end{aligned}$$

respectively.

The exponential generating function offers an alternative perspective, particularly useful for analytic and combinatorial interpretations.

LEMMA 1.5. *Suppose $\sum_{n=0}^{\infty} W_n \frac{z^n}{n!}$ is the exponential generating function of the generalized Olivier sequence $\{W_n\}$. Then*

$$\sum_{n=0}^{\infty} W_n \frac{z^n}{n!} = A_1 e^{\alpha z} + A_2 e^{\beta z} + A_3 e^{-z} + A_4 e^{z}$$

where A_1, A_2, A_3, A_4 are as defined in (1.9).

Proof: Using the Binet's formula of generating Olivier numbers we get

$$\sum_{n=0}^{\infty} W_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3(-1)^n + A_4)\frac{z^n}{n!}$$

Separating terms yields

$$\sum_{n=0}^{\infty} W_n \frac{z^n}{n!} = A_1 \sum_{n=0}^{\infty} \frac{(\alpha z)^n}{n!} + A_2 \sum_{n=0}^{\infty} \frac{(\beta z)^n}{n!} + A_3 \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} + A_4 \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which simplifies to

$$\sum_{n=0}^{\infty} W_n \frac{z^n}{n!} = A_1 e^{\alpha z} + A_2 e^{\beta z} + A_3 e^{-z} + A_4 e^{z}. \quad \square$$

It is worth emphasizing that Lemma 1.5 does not appear in the earlier treatment of Olivier numbers given in [5]. To ensure that our exposition remains both complete and self-contained, we have taken the initiative to present the lemma here together with its full proof. In doing so, we clarify the underlying rationale of the exponential generating function and reinforce the theoretical framework within which our subsequent results are developed. This addition not only fills a gap in the existing literature but also provides a stronger foundation for the identities, summation formulas, and structural properties that will be established in the following sections.

The previous Lemma gives the following results as particular examples.

Corollary 1.6. *Exponential generating function of Olivier and Olivier-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} O_n \frac{z^n}{n!} = \frac{1}{10}(\sqrt{5}+5)e^{\alpha z} + \frac{1}{10}(-\sqrt{5}+5)e^{\beta z} - \frac{1}{2}e^{-z} - \frac{1}{2}e^{z}.$$

b):

$$\sum_{n=0}^{\infty} K_n \frac{z^n}{n!} = e^{\alpha z} + e^{\beta z} + e^{-z} + e^{z}.$$

Next, we turn our attention to summation identities associated with the generalized Olivier numbers. Such formulas are of particular importance because they provide compact expressions for partial sums of the sequence and reveal deeper structural relationships between consecutive terms. They also serve as useful tools for deriving combinatorial identities and for establishing connections with other recurrence families. In this way, summation identities enrich both the algebraic and analytic perspectives of recurrence theory.

Theorem 1.7. *[7, Theorem 3.4] For $n \geq 0$, the following sum formulas for generalized Olivier numbers hold:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{2}(-(n+3)W_{n+3} + W_{n+2} + (2n+7)W_{n+1} + (n+4)W_n + 3W_3 - W_2 - 7W_1 - 2W_0)$.

(b): $\sum_{k=0}^{n} W_{2k} = W_{2n+2} - (n+2)W_{2n+1} + (n+1)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_2 - 2W_1 + 2W_0$.

(c): $\sum_{k=0}^{n} W_{2k+1} = -(n+1)W_{2n+2} + (n+3)W_{2n+1} + (n+1)W_{2n} - W_{2n-1} - W_3 + 2W_2 - 2W_0$.

(d): $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-(n+1)W_{-n+3} - W_{-n+2} + (2n+1)W_{-n+1} + (n+2)W_{-n} + W_3 + W_2 - W_1 - 2W_0)$.

(e): $\sum\limits_{k=1}^{n} W_{-2k} = -W_{-2n+2} - (n+1)W_{-2n+1} + (n+3)W_{-2n} + (n+1)W_{-2n-1} + W_3 - W_1 - 2W_0.$

(f): $\sum\limits_{k=1}^{n} W_{-2k+1} = -(n+2)W_{-2n+2} + (n+1)W_{-2n+1} + (n+2)W_{-2n} + W_{-2n-1} + W_3 + W_2 - 3W_1 - W_0.$

In summary, generalized Olivier numbers serve as a unifying framework that brings together several classical and modern recurrence families, including Pell-type, Padovan-type, and higher-order Fibonacci–Lucas structures, while simultaneously introducing new algebraic identities. Their capacity to accommodate integer, rational, and even polynomial coefficients, together with the possibility of extending definitions to negative indices, demonstrates the remarkable flexibility of the model and its potential for uncovering deeper combinatorial relationships. By situating Olivier and Olivier–Lucas sequences within this broader landscape, we emphasize their role as a conceptual bridge between established recurrence systems and innovative generalizations, thereby opening new avenues for theoretical exploration and practical applications in number theory, combinatorics, and related fields. Furthermore, the explicit Binet's formulas, generating functions, and summation identities provide complementary analytic perspectives: the closed forms highlight the influence of characteristic roots, while the generating functions and sum formulas encapsulate the entire sequence in compact algebraic expressions. Taken together, these tools form a robust foundation for deriving identities, matrix representations, and structural properties, ensuring that generalized Olivier numbers occupy a central position in the ongoing development of recurrence theory and its applications.

2. Gaussian Generalized Olivier Numbers

Having established the structural framework, recurrence relations, and analytic tools for generalized Olivier numbers, we are now prepared to extend our investigation into their Gaussian counterparts. In particular, we turn to the definition and analysis of generalized Gaussian Olivier numbers and their two notable special cases, the Gaussian Olivier and Gaussian Olivier–Lucas sequences. This transition marks the point where classical recurrence theory meets Gaussianization, opening the way to new closed forms, generating functions, and identities that enrich the broader study of integer sequences. In what follows, we introduce Gaussian generalized Olivier numbers and examine their fundamental properties. Central to this discussion are Binet's formula, which provides an explicit closed-form representation of the sequence, and the generating function, which encapsulates the entire sequence within a single algebraic expression. Together, these tools clarify the recursive structure of the sequence, establish a foundation for further theoretical exploration, and offer computational techniques that extend the scope of recurrence theory into new algebraic and combinatorial domains.

Gaussian generalized Olivier numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2, GW_3)\}_{n\geq 0}$ are defined by

$$GW_n = GW_{n-1} + 2GW_{n-2} - GW_{n-3} - GW_{n-4} \tag{2.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + iW_{-1} = W_0 + i(2W_1 - W_0 + W_2 - W_3), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \\ GW_3 &= W_3 + iW_2, \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = -GW_{-(n-1)} + 2GW_{-(n-2)} + GW_{-(n-3)} - GW_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) holds for all integer n. Note that for all integers n, we get,

$$GW_n = W_n + iW_{n-1}, \tag{2.2}$$

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{2.3}$$

In the following table, we present the first few generalized Gaussian Olivier numbers, including those with both positive and negative subscripts, thereby providing a clear illustration of how the sequence behaves across its entire domain and highlighting the natural extension of the indexing scheme to encompass both the standard positive indices and their negative counterparts.

Table 1. The first few generalized Gaussian Olivier numbers.

n	GW_n	GW_{-n}
0	GW_0	GW_0
1	GW_1	$2GW_1 - GW_0 + GW_2 - GW_3$
2	GW_2	$3GW_0 - GW_1 - 2GW_2 + GW_3$
3	GW_3	$4GW_1 - 4GW_0 + 4GW_2 - 3GW_3$
4	$2GW_2 - GW_1 - GW_0 + GW_3$	$8GW_0 - 4GW_1 - 7GW_2 + 4GW_3$
5	$GW_2 - 2GW_1 - GW_0 + 3GW_3$	$9GW_1 - 12GW_0 + 12GW_2 - 8GW_3$
6	$4GW_2 - 4GW_1 - 3GW_0 + 4GW_3$	$21GW_0 - 12GW_1 - 20GW_2 + 12GW_3$
7	$4GW_2 - 7GW_1 - 4GW_0 + 8GW_3$	$22GW_1 - 33GW_0 + 33GW_2 - 21GW_3$
8	$9GW_2 - 12GW_1 - 8GW_0 + 12GW_3$	$55GW_0 - 33GW_1 - 54GW_2 + 33GW_3$
9	$12GW_2 - 20GW_1 - 12GW_0 + 21GW_3$	$56GW_1 - 88GW_0 + 88GW_2 - 55GW_3$
10	$22GW_2 - 33GW_1 - 21GW_0 + 33GW_3$	$144GW_0 - 88GW_1 - 143GW_2 + 88GW_3$
11	$33GW_2 - 54GW_1 - 33GW_0 + 55GW_3$	$145GW_1 - 232GW_0 + 232GW_2 - 144GW_3$
12	$56GW_2 - 88GW_1 - 55GW_0 + 88GW_3$	$377GW_0 - 232GW_1 - 376GW_2 + 232GW_3$
13	$88GW_2 - 143GW_1 - 88GW_0 + 144GW_3$	$378GW_1 - 609GW_0 + 609GW_2 - 377GW_3$

Gaussian Olivier numbers, $GW_n : GW_n(0, 1, 1+i, 3+i) = GO_n$, are defined by

$$GO_n = GO_{n-1} + 2GO_{n-2} - GO_{n-3} - GO_{n-4}, \tag{2.4}$$

with the initial conditions

$$GO_0 = 0, GO_1 = 1, GO_2 = 1+i, GO_3 = 3+i,$$

and Gaussian Olivier-Lucas numbers, $GW_n(4-i, 1+4i, 5+i, 4+5i) = GK_n$, are defined by

$$GK_n = GK_{n-1} + 2GK_{n-2} - GK_{n-3} - GK_{n-4}, \tag{2.5}$$

with the initial conditions

$$GK_0 = 4-i, GK_1 = 1+4i, GK_2 = 5+i, GK_3 = 4+5i.$$

.

These sequences can also be extended to negative indices by applying the same recurrence relations:

$$\begin{aligned} GO_{-n} &= -GO_{-(n-1)} + 2GO_{-(n-2)} + GO_{-(n-3)} - GO_{-(n-4)} \\ GK_{-n} &= -GK_{-(n-1)} + 2GK_{-(n-2)} + GK_{-(n-3)} - GK_{-(n-4)} \end{aligned}$$

for $n = 1, 2, 3, ...$ Hence, recurrences (2.4)-(2.5) are valid for all integers n.

Note that for all integers n, we have

$$\begin{aligned} GO_n &= O_n + iO_{n-1}, \\ GK_n &= K_n + iK_{n-1}. \end{aligned}$$

In Table 2 and 3, we present the first few Gaussian Olivier and Gaussian Olivier-Lucas numbers with positive and negative subscripts.

Table 2. Gaussian Olivier numbers

n	GO_n	GO_{-n}
0	0	0
1	1	0
2	$1+i$	$-i$
3	$3+i$	$-1+i$
4	$4+3i$	$1-3i$
5	$8+4i$	$-3+4i$
6	$12+8i$	$4-8i$
7	$21+12i$	$-8+12i$
8	$33+21i$	$12-21i$
9	$55+33i$	$-21+33i$
10	$88+55i$	$33-55i$
11	$144+88i$	$-55+88i$
12	$232+144i$	$88-144i$
13	$377+232i$	$-144+232i$

Table 3. Gaussian Olivier-Lucas numbers

n	GK_n	GK_{-n}
0	$4-i$	$4-i$
1	$1+4i$	$-1+5i$
2	$5+i$	$5-4i$
3	$4+5i$	$-4+9i$
4	$9+4i$	$9-11i$
5	$11+9i$	$-11+20i$
6	$20+11i$	$20-29i$
7	$29+20i$	$-29+49i$
8	$49+29i$	$49-76i$
9	$76+49i$	$-76+125i$
10	$125+76i$	$125-199i$
11	$199+125i$	$-199+324i$
12	$324+199i$	$324-521i$
13	$521+324i$	$-521+845i$

Next, we present the Binet's formula for the Gaussian generalized Olivier numbersa concise closed form expression derived from solving the characteristic equation associated with their recurrence relation, which not only encapsulates the inherent recursive structure of these numbers but also provides valuable insights into their algebraic properties, thereby serving as a powerful analytical tool for further theoretical exploration and computational applications.

Theorem 2.1. *The Binet's formula for the Gaussian generalized Olivier numbers is*

$$GW_n = (A_1\alpha^n + A_2\beta^n + A_3(-1)^n + A_4) + i(A_1\alpha^{n-1} + A_2\beta^{n-1} + A_3(-1)^{n-1} + A_4)$$

where A_1, A_2, A_3, A_4 are as defined in (1.9).

Proof. The proof follows from (2.2) and (1.9). □

In light of the preceding theorem which establishes a comprehensive framework for analyzing the intrinsic properties of these sequences the ensuing results emerge naturally as special cases, thereby illustrating not only the far reaching generality of the theorem but also its capacity to encapsulate specific, elegantly distilled instances as direct corollaries of the broader theoretical framework.

Corollary 2.2. *For all integers n, we have following identities,*

- **(a):** $GO_n = (\frac{1}{10}(\sqrt{5}+5)\alpha^n + \frac{1}{10}(-\sqrt{5}+5)\beta^n - \frac{1}{2}(-1)^n - \frac{1}{2}) + i(\frac{1}{10}(\sqrt{5}+5)\alpha^{n-1} + \frac{1}{10}(-\sqrt{5}+5)\beta^{n-1} - \frac{1}{2}(-1)^{n-1} - \frac{1}{2})$.
- **(b):** $GK_n = (\alpha^n + \beta^n + (-1)^n + 1) + i(\alpha^{n-1} + \beta^{n-1} + (-1)^{n-1} + 1)$.

Building on the preceding discussion and analytical developments, the following theorem introduces the generating function of the Gaussian generalized Olivier numbers a single, unified algebraic expression that encapsulates the entire infinite sequence, thereby revealing the intricate interplay between the sequence's recursive properties and its combinatorial structure while also furnishing a powerful tool for further theoretical exploration and practical computational analysis.

Theorem 2.3. *Let $f_{GW_n}(z) = \sum_{n=0}^{\infty} GW_n z^n$ denote the generating function of Gaussian generalized Olivier numbers. Then,*

$$\sum_{n=0}^{\infty} GW_n z^n = \frac{GW_0 + (GW_1 - GW_0)z + (GW_2 - GW_1 - 2GW_0)z^2 + (GW_3 - GW_2 - 2GW_1 - (-1)GW_0)z^3}{1 - z - 2z^2 + z^3 + z^4} \tag{2.6}$$

Proof. Using the definition of generalized Gaussian Olivier numbers, and substracting $z\sum_{n=0}^{\infty} GW_n z^n$, $2z^2\sum_{n=0}^{\infty} GW_n z^n$, $(-1)z^3\sum_{n=0}^{\infty} GW_n z^n$ and $(-1)z^4\sum_{n=0}^{\infty} GW_n z^n$ from $\sum_{n=0}^{\infty} GW_n z^n$ we obtain

$$\begin{aligned}
& (1-z-2z^2-(-1)z^3-(-1)z^4)\sum_{n=0}^{\infty} GW_n z^n \\
= & \sum_{n=0}^{\infty} GW_n z^n - z\sum_{n=0}^{\infty} GW_n z^n - 2z^2\sum_{n=0}^{\infty} GW_n z^n - (-1)z^3\sum_{n=0}^{\infty} GW_n z^n - (-1)z^4\sum_{n=0}^{\infty} GW_n z^n \\
= & \sum_{n=0}^{\infty} GW_n z^n - \sum_{n=0}^{\infty} GW_n z^{n+1} - 2\sum_{n=0}^{\infty} GW_n z^{n+2} - (-1)\sum_{n=0}^{\infty} GW_n z^{n+3} - (-1)\sum_{n=0}^{\infty} GW_n z^{n+4} \\
= & \sum_{n=0}^{\infty} GW_n z^n - \sum_{n=1}^{\infty} GW_{n-1} z^n - 2\sum_{n=2}^{\infty} GW_{n-2} z^n - (-1)\sum_{n=3}^{\infty} GW_{n-3} z^n - (-1)\sum_{n=4}^{\infty} GW_{n-4} z^n \\
= & (GW_0+GW_1z+GW_2z^2+GW_3z^3)-(GW_0z+GW_1z^2+GW_2z^3)-2(GW_0z^2+GW_1z^3)-(-1)GW_0z^3 \\
& +\sum_{n=4}^{\infty}(GW_n-GW_{n-1}-2GW_{n-2}-(-1)GW_{n-3}-(-1)GW_{n-4})z^n \\
= & GW_0+(GW_1-GW_0)z+(GW_2-GW_1-2GW_0)z^2+(GW_3-GW_2-2GW_1-(-1)GW_0)z^3.
\end{aligned}$$

Rearranging above equation, we obtain (2.6). □

Theorem 2.3 gives the following results as special cases.

COROLLARY 2.4. *The generating functions of Gausian Olivier and Gausian Olivier–Lucas numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} GO_n z^n &= \frac{z+iz^2}{1-z-2z^2+z^3+z^4}, \\
\sum_{n=0}^{\infty} GK_n z^n &= \frac{(4-i)+(-3+5i)z-(4+i)z^2+(1-5i)z^3}{1-z-2z^2+z^3+z^4},
\end{aligned}$$

respectively.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} GW_n \frac{z^n}{n!}$ of the sequence GW_n.

LEMMA 2.5. *Suppose that $f_{EGW_n}(z) = \sum\limits_{n=0}^{\infty} GW_n \frac{z^n}{n!}$ is the exponential generating function of the Gaussian generalized Olivier sequence $\{GW_n\}$. Then,*

$$\sum_{n=0}^{\infty} GW_n \frac{z^n}{n!} = (A_1e^{\alpha z}+A_2e^{\beta z}+A_3e^{-z}+A_4e^{z})+i(\frac{A_1}{\alpha}e^{\alpha z}+\frac{A_2}{\beta}e^{\beta z}-A_3e^{-z}+A_4e^{z}).$$

Proof: Using the Binet's formula of Gaussian generalized Olivier numbers or exponential generating function of the generalized Olivier sequence we get

$$\begin{aligned}
\sum_{n=0}^{\infty} GW_n \frac{z^n}{n!} &= \sum_{n=0}^{\infty}(W_n+iW_{n-1})\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} W_n\frac{z^n}{n!}+i\sum_{n=0}^{\infty} W_{n-1}\frac{z^n}{n!} \\
&= (A_1e^{\alpha z}+A_2e^{\beta z}+A_3e^{-z}+A_4e^{z})+i(\frac{A_1}{\alpha}e^{\alpha z}+\frac{A_2}{\beta}e^{\beta z}-A_3e^{-z}+A_4e^{z}). \ \square
\end{aligned}$$

The previous Lemma gives the following results as particular examples.

COROLLARY 2.6. *Exponential generating function of Gaussian Olivier and Gaussian Olivier-Lucas numbers are*

a):

$$\begin{aligned}\sum_{n=0}^{\infty} GO_n \frac{z^n}{n!} &= (\frac{1}{10}(\sqrt{5}+5)e^{\alpha z} + \frac{1}{10}(-\sqrt{5}+5)e^{\beta z} - \frac{1}{2}e^{-z} - \frac{1}{2}e^{z}) \\ &\quad +i(\frac{\sqrt{5}+5}{5(\sqrt{5}+1)}e^{\alpha z} + \frac{\sqrt{5}-5}{5(\sqrt{5}-1)}e^{\beta z} + \frac{1}{2}e^{-z} - \frac{1}{2}e^{z}).\end{aligned}$$

b):

$$\sum_{n=0}^{\infty} GK_n \frac{z^n}{n!} = (e^{\alpha z} + e^{\beta z} + e^{-z} + e^{z}) + i(\frac{1}{\alpha}e^{\alpha z} + \frac{1}{\beta}e^{\beta z} - e^{-z} + e^{z}).$$

3. Some Identities About Recurrence Relations of Gaussian Generalized Olivier Numbers

In this section, we present a systematic collection of identities involving Gaussian Olivier numbers and Gaussian Olivier–Lucas numbers. These relations, derived from their recurrence definitions and generating functions, reveal intricate algebraic and combinatorial structures. Such identities not only deepen our theoretical understanding of the sequences but also provide useful tools for further exploration and potential applications in areas such as coding theory and discrete mathematics.

THEOREM 3.1. *For all integers n, the following identities hold:*

$$10GO_n = 9GK_{n+3} - 7GK_{n+2} - 9GK_{n+1} + 2GK_n, \tag{3.1}$$

$$GK_n = 5GO_{n+3} - 6GO_{n+2} - 5GO_{n+1} + 4GO_n. \tag{3.2}$$

Proof. To establish identity (3.1), we assume a linear relation of the form

$$GO_n = aGK_{n+3} + bGK_{n+2} + cGK_{n+1} + dGK_n.$$

By substituting $n = 0, 1, 2, 3$ and using the initial values of GO_n and GK_n, we obtain the system

$$\begin{aligned} GO_0 &= aGK_3 + bGK_2 + cGK_1 + dGK_0 \\ GO_1 &= aGK_4 + bGK_3 + cGK_2 + dGK_1 \\ GO_2 &= aGK_5 + bGK_4 + cGK_3 + dGK_2 \\ GO_3 &= aGK_6 + bGK_5 + cGK_4 + dGK_3 \end{aligned}$$

Solving this system yields $a = \frac{9}{10}$, $b = -\frac{7}{10}$, $c = -\frac{9}{10}$, $d = \frac{1}{5}$, which confirms the stated identity.

Similarly, to prove identity (3.2), we assume

$$GK_n = aGO_{n+3} + bGO_{n+2} + cGO_{n+1} + dGO_n.$$

Substituting $n = 0, 1, 2, 3$ and applying the initial conditions gives

$$\begin{aligned}
GK_0 &= aGO_3 + bGO_2 + cGO_1 + dGO_0 \\
GK_1 &= aGO_4 + bGO_3 + cGO_2 + dGO_1 \\
GK_2 &= aGO_5 + bGO_4 + cGO_3 + dGO_2 \\
GK_3 &= aGO_6 + bGO_5 + cGO_4 + dGO_3
\end{aligned}$$

Solving this system yields $a = 5,\ b = -6,\ c = -5,\ d = 4$, thereby establishing the second identity. $\square$

These identities illustrate the deep interconnections between Gaussian Olivier and Gaussian Olivier–Lucas numbers. By expressing each sequence in terms of the other, they highlight the structural symmetry inherent in their recurrence relations. Such results not only enrich the algebraic theory of Gaussian generalized Olivier numbers but also provide a foundation for deriving further identities, summation formulas, and matrix representations in subsequent sections.

In the study of recurrence sequences, it is often useful to separate the contributions of even and odd indices, as this decomposition reveals additional structural symmetries and provides analytic simplifications. By examining subsequences defined by even and odd terms, one can derive generating functions that encapsulate their behavior in compact closed forms. This approach not only clarifies the recursive nature of the sequences but also offers powerful tools for deriving identities and exploring combinatorial interpretations.

Lemma 3.2. *([1]) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function of the sequence $\{a_n\}_{n\geq 0}$. Then the generating functions of the subsequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$ are given by*

$$\begin{aligned}
f_{a_{2n}}(z) &= \sum_{n=0}^{\infty} a_{2n} z^n = \frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}, \\
f_{a_{2n+1}}(z) &= \sum_{n=0}^{\infty} a_{2n+1} z^n = \frac{f(\sqrt{z}) - f(-\sqrt{z})}{2\sqrt{z}},
\end{aligned}$$

respectively.

The above lemma provides a general framework for decomposing generating functions into their even and odd components. Applying this principle to Gaussian generalized Olivier numbers yields explicit closed-form expressions for the subsequences, thereby illuminating the interplay between their recursive definitions and combinatorial structures.

THEOREM 3.3. *The generating functions of the subsequences $\{GW_{2n}\}$ and $\{GW_{2n+1}\}$ are given by*

$$\sum_{n=0}^{\infty} GW_{2n}z^n \tag{3.3}$$
$$= \frac{GW_0 + (-5GW_0 + GW_2)z + (7GW_0 - GW_1 - 3GW_2 + GW_3)z^2 + (-3GW_0 + GW_1 + 2GW_2 - GW_3)z^3}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$

$$\sum_{n=0}^{\infty} GW_{2n+1}z^n \tag{3.4}$$
$$= \frac{GW_1 + (-5GW_1 + GW_3)z + (-GW_0 + 6GW_1 + GW_2 - 2GW_3)z^2 + (GW_0 - 2GW_1 - GW_2 + GW_3)z^3}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$

respectively.

Proof. From Lemma 1.3, we obtain

$$f_{GW_n}(\sqrt{z}) = \frac{GW_0 + (GW_1 - GW_0)\sqrt{z} + (GW_2 - GW_1 - 2GW_0)z + (GW_3 - GW_2 - 2GW_1 + GW_0)\sqrt{z^3}}{1 - \sqrt{z} - 2z + \sqrt{z^3} + z^2},$$
$$f_{GW_n}(-\sqrt{z}) = \frac{GW_0 - (GW_1 - GW_0)\sqrt{z} + (GW_2 - GW_1 - 2GW_0)z - (GW_3 - GW_2 - 2GW_1 - (-1)GW_0)\sqrt{z}^3}{1 + \sqrt{z} - 2z - \sqrt{z}^3 + z^2}.$$

Applying Lemma 3.2, we obtain

$$\sum_{n=0}^{\infty} GW_{2n}z^n = \frac{f_{GW_n}(\sqrt{z}) + f_{GW_n}(-\sqrt{z})}{2},$$

and

$$\sum_{n=0}^{\infty} GW_{2n+1}z^n = \frac{f_{GW_n}(\sqrt{z}) - f_{GW_n}(-\sqrt{z})}{2},$$

which simplify to (3.3) and (3.4). □

From Theorem 3.3, we obtain the following Corollary.

COROLLARY 3.4. *The generating functions of the subsequences $GO_{2n}, GO_{2n+1}, GK_{2n}, GK_{2n+1}$ are given by:*

a):

$$f_{GO_{2n}}(z) = \frac{(1+i)z - (1+2i)z^2 + iz^3}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$
$$f_{GO_{2n+1}}(z) = \frac{1 + (-2+i)z + (1-i)z^2}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

b):

$$f_{GK_{2n}}(z) = \frac{(4-i) + (-15+6i)z + (16-9i)z^2 + (-5+4i)z^3}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$
$$f_{GK_{2n+1}}(z) = \frac{(1+4i) - (1+15i)z + (-1+16i)z^2 + (1-5i)z^3}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

These results demonstrate how the decomposition into even and odd subsequences provides additional analytic clarity. By isolating the contributions of $GO_{2n}, GO_{2n+1}, GK_{2n}, GK_{2n+1}$, we uncover structural symmetries that are otherwise obscured in the full sequence. Such generating functions not only enrich the algebraic theory of Gaussian generalized Olivier numbers but also serve as powerful tools for deriving further identities and exploring applications in combinatorics and discrete mathematics.

The decomposition of generating functions into even and odd subsequences, as established in Corollary 3.4, provides a fertile ground for deriving additional algebraic identities. By carefully manipulating these generating functions, one can uncover relations that connect Gaussian Olivier numbers and Gaussian Olivier–Lucas numbers in nontrivial ways. Such identities highlight the structural symmetry between the two families and demonstrate how generating functions serve not only as compact analytic representations but also as engines for producing deeper recurrence relations and cross-sequence connections.

Remark 3.5. *From Corollary 3.4 we obtain*

$$((1+i)z-(1+2i)z^2+iz^3)f_{GK_{2n}}(z)=((4-i)+(-15+6i)z+(16-9i)z^2+(-5+4i)z^3)f_{GO_{2n}}(z).$$

The left-hand side (LHS) expands as

$$\begin{aligned}
LHS &= ((1+i)z-(1+2i)z^2+iz^3)\sum_{n=0}^{\infty}GK_{2n}z^n\\
&= (1+i)z\sum_{n=0}^{\infty}GK_{2n}z^n-(1+2i)z^2\sum_{n=0}^{\infty}GK_{2n}z^n+iz^3\sum_{n=0}^{\infty}GK_{2n}z^n\\
&= (1+i)\sum_{n=0}^{\infty}GK_{2n}z^{n+1}-(1+2i)\sum_{n=0}^{\infty}GK_{2n}z^{n+2}+i\sum_{n=0}^{\infty}GK_{2n}z^{n+3}\\
&= (1+i)(GK_0z+GK_2z^2)-(1+2i)GK_0z^2+\sum_{n=3}^{\infty}((1+i)GK_{2n-2}-(1+2i)GK_{2n-4}+iGK_{2n-6})z^n.
\end{aligned}$$

The right-hand side (RHS) expands as

$$\begin{aligned}
RHS &= ((4-i)+(-15+6i)z+(16-9i)z^2+(-5+4i)z^3)\sum_{n=0}^{\infty}GO_{2n}z^n\\
&= ((4-i)\sum_{n=0}^{\infty}GO_{2n}z^n+(-15+6i)z\sum_{n=0}^{\infty}GO_{2n}z^n+(16-9i)z^2\sum_{n=0}^{\infty}GO_{2n}z^n+(-5+4i)z^3\sum_{n=0}^{\infty}GO_{2n}z^n\\
&= ((4-i)\sum_{n=0}^{\infty}GO_{2n}z^n+(-15+6i)\sum_{n=0}^{\infty}GO_{2n}z^{n+1}+(16-9i)\sum_{n=0}^{\infty}GO_{2n}z^{n+2}+(-5+4i)\sum_{n=0}^{\infty}GO_{2n}z^{n+3}\\
&= (4-i)(GO_2z+GO_4z^2)+(-15+6i)GO_2z^2\\
&\quad +\sum_{n=3}^{\infty}((4-i)GO_{2n}+(-15+6i)GO_{2n-2}+(16-9i)GO_{2n-4}+(-5+4i)GO_{2n-6})z^n
\end{aligned}$$

Since

$$(1+i)(GK_0z+GK_2z^2)-(1+2i)GK_0z^2=(4-i)(GO_0+GO_2z+GO_4z^2),$$

comparing coefficients yields the identity

$$(1+i)GK_{2n-2}-(1+2i)GK_{2n-4}+iGK_{2n-6}=(4-i)GO_{2n}+(-15+6i)GO_{2n-2}+(16-9i)GO_{2n-4}+(-5+4i)GO_{2n-6}. \tag{3.5}$$

Similarly, from Corollary 3.4, we obtain

$$((1+i)z-(1+2i)z^2+iz^3)f_{GK_{2n+1}}(z) = ((1+4i)-(1+15i)z+(-1+16i)z^2+(1-5i)z^3)f_{GO_{2n}}(z), \tag{3.6}$$

$$(1+(-2+i)z+(1-i)z^2)f_{GK_{2n}}(z) = ((4-i)+(-15+6i)z+(16-9i)z^2+(-5+4i)z^3)f_{GO_{2n+1}}(z), \tag{3.7}$$

$$(1+(-2+i)z+(1-i)z^2)f_{GK_{2n+1}}(z) = ((1+4i)-(1+15i)z+(-1+16i)z^2+(1-5i)z^3)f_{GO_{2n+1}}(z). \tag{3.8}$$

As above, one can derive identities analogous to (3.5) by applying (3.6)-(3.8).

This remark illustrates how generating function manipulations yield nontrivial cross-relations between Gaussian Olivier and Gaussian Olivier–Lucas subsequences. Such identities not only reinforce the algebraic symmetry between the two families but also provide a framework for deriving further recurrence relations and combinatorial interpretations in subsequent analysis.

One of the most fruitful approaches in the study of recurrence sequences is to establish convolution-type identities that connect different families of sequences. Such relations not only highlight the algebraic interplay between Gaussian Olivier numbers and their classical counterparts but also provide powerful tools for deriving summation formulas and cross-sequence transformations. In particular, identities of Honsberger type reveal how Gaussianized structures inherit and extend the combinatorial richness of the original Olivier framework.

Theorem 3.6. ***(Honsberger's Identity)*** *For all integers m,n, the following identity holds:*

$$GW_{n+m}=GW_nO_{m+1}+GW_{n-1}(2O_m-O_{m-1}-O_{m-2})+GW_{n-2}(-O_m-O_{m-1})-GW_{n-3}O_m,$$

i.e.,

$$GW_{m+n}=GW_{m+3}O_{n-2}+GW_{m+2}(2O_{n-3}-O_{n-4}-O_{n-5})+GW_{m+1}(-O_{n-3}-O_{n-4})-GW_mO_{n-3}.$$

Proof. The proof can be established either by direct application of the Binet's formulas for GW_n and O_n or by induction on n. Both approaches confirm the validity of the stated identity. □

The general identity above can be specialized to obtain relations involving Gaussian Olivier numbers and Gaussian Olivier–Lucas numbers. By substituting $GW_n = GO_n$ and $GW_n = GK_n$, respectively, we obtain the following corollary.

Corollary 3.7. *For all integers m,n, the following identities hold:*

$$GO_{n+m} = GO_nO_{m+1}+GO_{n-1}(2O_m-O_{m-1}-O_{m-2})+GO_{n-2}(-O_m-O_{m-1})-GO_{n-3}O_m,$$

$$GK_{n+m} = GK_nO_{m+1}+GK_{n-1}(2O_m-O_{m-1}-O_{m-2})+GK_{n-2}(-O_m-O_{m-1})-GK_{n-3}O_m,$$

i.e.,

$$\begin{aligned} GO_{m+n} &= GO_{m+3}O_{n-2} + GO_{m+2}(2O_{n-3} - O_{n-4} - O_{n-5}) + GO_{m+1}(-O_{n-3} - O_{n-4}) - GO_mO_{n-3}, \\ GK_{m+n} &= GK_{m+3}O_{n-2} + GK_{m+2}(2O_{n-3} - O_{n-4} - O_{n-5}) + GK_{m+1}(-O_{n-3} - O_{n-4}) - GK_mO_{n-3}. \end{aligned}$$

These identities demonstrate how Gaussian Olivier numbers and their Lucas-type counterparts inherit convolutional structures from the classical Olivier sequence. They provide a unified framework for expressing mixed-index terms, thereby enriching the algebraic theory and offering new pathways for combinatorial interpretations and applications.

4. Simson's Formula

We now turn our attention to Simson's formula in the context of Gaussian generalized Olivier numbers. This classical determinant identity, when adapted to Gaussianized structures, provides an elegant and compact relation that further enriches the analytical framework of these sequences. In particular, it demonstrates how Gaussian Olivier numbers preserve deep structural symmetries across indices, linking forward and backward terms through determinant evaluations. The result appears as a special case of [3, Theorem 4.1], and its adaptation here underscores the versatility of Simson-type identities in the study of generalized recurrence families.

Theorem 4.1 (Simpson's formula of generalized Gaussian Olivier numbers). *For all integers n, the following identity holds:*

$$\begin{vmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1} & GW_{n-2} & GW_{n-3} \end{vmatrix} = \begin{vmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{vmatrix}.$$

Proof. The result follows directly from Theorem 4.1 of [3], applied to the Gaussian generalized Olivier framework. □

The general determinant identity above can be specialized to obtain explicit evaluations for Gaussian Olivier and Gaussian Olivier–Lucas numbers. These special cases highlight how Simson's formula translates into concrete algebraic constants when applied to particular initial conditions.

Corollary 4.2. *For all integers n, the following identities hold:*

(a): $$\begin{vmatrix} GO_{n+3} & GO_{n+2} & GO_{n+1} & GO_n \\ GO_{n+2} & GO_{n+1} & GO_n & GO_{n-1} \\ GO_{n+1} & GO_n & GO_{n-1} & GO_{n-2} \\ GO_n & GO_{n-1} & GO_{n-2} & GO_{n-3} \end{vmatrix} = 4 - 2i.$$

(b): $$\begin{vmatrix} GK_{n+3} & GK_{n+2} & GK_{n+1} & GK_n \\ GK_{n+2} & GK_{n+1} & GK_n & GK_{n-1} \\ GK_{n+1} & GK_n & GK_{n-1} & GK_{n-2} \\ GK_n & GK_{n-1} & GK_{n-2} & GK_{n-3} \end{vmatrix} = 80 - 40i.$$

These results illustrate how Simson's formula, when applied to Gaussian Olivier and Gaussian Olivier–Lucas numbers, yields elegant determinant identities with explicit complex values. Such evaluations not only reinforce the algebraic coherence of Gaussianized sequences but also provide a bridge between classical determinant identities and modern generalizations in recurrence theory.

5. Summation Formulas for Gaussian Generalized Olivier Numbers

We now extend the summation identities obtained for generalized Olivier numbers to their Gaussian counterparts. These formulas are particularly valuable because they provide closed expressions for partial sums of Gaussian sequences, expressed in terms of only a few consecutive Gaussian numbers and initial conditions. Such identities highlight the structural efficiency of the recurrence relation and allow for deeper algebraic and combinatorial interpretations. Moreover, they demonstrate how Gaussianization preserves the summation structure of classical sequences while introducing new algebraic features through complex coefficients.

THEOREM 5.1. *For $n \geq 0$, the following sum formulas for Gaussian generalized Olivier numbers hold:*

(a): $\sum_{k=0}^{n} GW_k = \frac{1}{2}(-(n+3)GW_{n+3} + GW_{n+2} + (2n+7)GW_{n+1} + (n+4)GW_n + 3GW_3 - GW_2 - 7GW_1 - 2GW_0)$.

(b): $\sum_{k=0}^{n} GW_{2k} = GW_{2n+2} - (n+2)GW_{2n+1} + (n+1)GW_{2n} + (n+2)GW_{2n-1} + 2GW_3 - 3GW_2 - 2GW_1 + 2GW_0$.

(c): $\sum_{k=0}^{n} GW_{2k+1} = -(n+1)GW_{2n+2} + (n+3)GW_{2n+1} + (n+1)GW_{2n} - GW_{2n-1} - GW_3 + 2GW_2 - 2GW_0$.

(d): $\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-(n+1)GW_{-n+3} - GW_{-n+2} + (2n+1)GW_{-n+1} + (n+2)GW_{-n} + GW_3 + GW_2 - GW_1 - 2GW_0)$.

(e): $\sum_{k=1}^{n} GW_{-2k} = -GW_{-2n+2} - (n+1)GW_{-2n+1} + (n+3)GW_{-2n} + (n+1)GW_{-2n-1} + GW_3 - GW_1 - 2GW_0$.

(f): $\sum_{k=1}^{n} GW_{-2k+1} = -(n+2)GW_{-2n+2} + (n+1)GW_{-2n+1} + (n+2)GW_{-2n} + GW_{-2n-1} + GW_3 + GW_2 - 3GW_1 - GW_0$.

Proof. These identities follow directly from Theorem 1.7, which provides summation formulas for the scalar sequence $\{W_n\}$, together with the recurrence relation $W_n = W_{n-1} + 2W_{n-2} - W_{n-3} - W_{n-4}$. The Gaussian structure is preserved under summation since addition is defined componentwise. □

By substituting the initial conditions of the Gaussian Olivier and Gaussian Olivier–Lucas numbers into the general summation identities, we obtain explicit closed-form formulas for each case. These corollaries illustrate how Gaussianization modifies the constants while preserving the overall summation structure, with the first specialization tailored specifically to the Gaussian Olivier sequence.

Corollary 5.2. *For $n \geq 0$, the Gaussian Olivier numbers satisfy:*

(a): $\sum\limits_{k=0}^{n} GO_k = \frac{1}{2}(-(n+3)GO_{n+3} + GO_{n+2} + (2n+7)GO_{n+1} + (n+4)GO_n + 1 + 2i.$

(b): $\sum\limits_{k=0}^{n} GO_{2k} = GO_{2n+2} - (n+2)GO_{2n+1} + (n+1)GO_{2n} + (n+2)GO_{2n-1} + 1 - i.$

(c): $\sum\limits_{k=0}^{n} GO_{2k+1} = -(n+1)GO_{2n+2} + (n+3)GO_{2n+1} + (n+1)GO_{2n} - GO_{2n-1} - 1 + i.$

(d): $\sum\limits_{k=1}^{n} GO_{-k} = \frac{1}{2}(-(n+1)GO_{-n+3} - GO_{-n+2} + (2n+1)GO_{-n+1} + (n+2)GO_{-n} + 3 + 2i.$

(e): $\sum\limits_{k=1}^{n} GO_{-2k} = -GO_{-2n+2} - (n+1)GO_{-2n+1} + (n+3)GO_{-2n} + (n+1)GO_{-2n-1} + 2 + i.$

(f): $\sum\limits_{k=1}^{n} GO_{-2k+1} = -(n+2)GO_{-2n+2} + (n+1)GO_{-2n+1} + (n+2)GO_{-2n} + GO_{-2n-1} + 1 + 2i.$

Likewise, by applying the initial conditions of the Gaussian Olivier–Lucas numbers, the general summation identities specialize to the following explicit formulas.

Corollary 5.3. *For $n \geq 0$, the Gaussian Olivier-Lucas numbers satisfy:*

(a): $\sum\limits_{k=0}^{n} GK_k = \frac{1}{2}(-(n+3)GK_{n+3} + GK_{n+2} + (2n+7)GK_{n+1} + (n+4)GK_n - 8 - 12i).$

(b): $\sum\limits_{k=0}^{n} GK_{2k} = GK_{2n+2} - (n+2)GK_{2n+1} + (n+1)GK_{2n} + (n+2)GK_{2n-1} - 1 - 3i.$

(c): $\sum\limits_{k=0}^{n} GK_{2k+1} = -(n+1)GK_{2n+2} + (n+3)GK_{2n+1} + (n+1)GK_{2n} - GK_{2n-1} - 2 - i.$

(d): $\sum\limits_{k=1}^{n} GK_{-k} = \frac{1}{2}(-(n+1)GK_{-n+3} - GK_{-n+2} + (2n+1)GK_{-n+1} + (n+2)GK_{-n} + 4i).$

(e): $\sum\limits_{k=1}^{n} GK_{-2k} = -GK_{-2n+2} - (n+1)GK_{-2n+1} + (n+3)GK_{-2n} + (n+1)GK_{-2n-1} - 5 + 3i.$

(f): $\sum\limits_{k=1}^{n} GK_{-2k+1} = -(n+2)GK_{-2n+2} + (n+1)GK_{-2n+1} + (n+2)GK_{-2n} + GK_{-2n-1} + 2 - 5i.$

These summation identities demonstrate how Gaussian Olivier and Gaussian Olivier–Lucas numbers inherit the summation structure of their classical counterparts while introducing new constants that reflect the Gaussian framework. They provide compact closed forms for partial sums, reveal deeper algebraic relationships, and serve as a foundation for further exploration of combinatorial identities and applications in complex recurrence theory.

6. Matrix Formulation of GW_n

Building on the matrix formulation of Olivier numbers, we now extend the representation to their Gaussian counterparts. This approach encodes the recursive structure directly into matrix powers, allowing us to express Gaussian Olivier and Gaussian Olivier–Lucas numbers in compact algebraic form. The following lemma formalizes this connection.

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which satisfies $\det(A) = 1$. Note that, according to [5, Theorem 8.1. (a)],

$$A^n = \begin{pmatrix} O_{n+1} & 2O_n - O_{n-1} - O_{n-2} & -O_n - O_{n-1} & -O_n \\ O_n & 2O_{n-1} - O_{n-2} - O_{n-3} & -O_{n-1} - O_{n-2} & -O_{n-1} \\ O_{n-1} & 2O_{n-2} - O_{n-3} - O_{n-4} & -O_{n-2} - O_{n-3} & -O_{n-2} \\ O_{n-2} & 2O_{n-3} - O_{n-4} - O_{n-5} & -O_{n-3} - O_{n-4} & -O_{n-3} \end{pmatrix}.$$

We now present the following lemma, which establishes the matrix formulation of Gaussian generalized Olivier numbers.

Lemma 6.1. *For $n \geq 0$, the following identity holds:*

$$\begin{pmatrix} GW_{n+3} \\ GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

Proof. The proof proceeds by mathematical induction on n. For $n = 0$, we have

$$\begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix},$$

which is trivially true. Assume the identity holds for $n = k$, i.e.,

$$\begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k + 1$, we obtain

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix}$$
$$= \begin{pmatrix} GW_{k+4} \\ GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.$$

Thus, by induction, the identity holds for all $n \geq 0$. □

We define the matrices

$$N_{GW} = \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix}, \tag{6.1}$$

and

$$E_{GW} = \begin{pmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1} & GW_{n-2} & GW_{n-3} \end{pmatrix}. \tag{6.2}$$

With these definitions, we obtain the following theorem.

THEOREM 6.2. *Using N_{GW} and E_{GW}, we have*

$$A^n N_{GW} = E_{GW}.$$

Proof. We compute

$$A^n N_{GW} = \begin{pmatrix} O_{n+1} & 2O_n - O_{n-1} - O_{n-2} & -O_n - O_{n-1} & -O_n \\ O_n & 2O_{n-1} - O_{n-2} - O_{n-3} & -O_{n-1} - O_{n-2} & -O_{n-1} \\ O_{n-1} & 2O_{n-2} - O_{n-3} - O_{n-4} & -O_{n-2} - O_{n-3} & -O_{n-2} \\ O_{n-2} & 2O_{n-3} - O_{n-4} - O_{n-5} & -O_{n-3} - O_{n-4} & -O_{n-3} \end{pmatrix} \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= GW_3O_{n+1} - GW_0O_n - GW_1\left(O_n + O_{n-1}\right) - GW_2\left(O_{n-1} - 2O_n + O_{n-2}\right), \\ a_{21} &= GW_3O_n - GW_1\left(O_{n-1} + O_{n-2}\right) - GW_2\left(O_{n-2} - 2O_{n-1} + O_{n-3}\right) - GW_0O_{n-1}, \\ a_{31} &= GW_3O_{n-1} - GW_2\left(O_{n-3} - 2O_{n-2} + O_{n-4}\right) - GW_0O_{n-2} - GW_1\left(O_{n-2} + O_{n-3}\right), \\ a_{41} &= GW_3O_{n-2} - GW_2\left(O_{n-4} - 2O_{n-3} + O_{n-5}\right) - GW_0O_{n-3} - GW_1\left(O_{n-3} + O_{n-4}\right), \end{aligned}$$

$$\begin{aligned} a_{12} &= GW_2O_{n+1} - GW_0\left(O_n + O_{n-1}\right) - GO_nW_{-1} - GW_1\left(O_{n-1} - 2O_n + O_{n-2}\right), \\ a_{22} &= GW_2O_n - GW_{-1}O_{n-1} - GW_0\left(O_{n-1} + O_{n-2}\right) - GW_1\left(O_{n-2} - 2O_{n-1} + O_{n-3}\right), \\ a_{32} &= GW_2O_{n-1} - GW_{-1}O_{n-2} - GW_1\left(O_{n-3} - 2O_{n-2} + O_{n-4}\right) - GW_0\left(O_{n-2} + O_{n-3}\right), \\ a_{42} &= GW_2O_{n-2} - GW_{-1}O_{n-3} - GW_1\left(O_{n-4} - 2O_{n-3} + O_{n-5}\right) - GW_0\left(O_{n-3} + O_{n-4}\right), \end{aligned}$$

$$\begin{aligned} a_{13} &= GW_1O_{n+1} - GW_{-1}\left(O_n + O_{n-1}\right) - GO_nW_{-2} - GW_0\left(O_{n-1} - 2O_n + O_{n-2}\right), \\ a_{23} &= GW_1O_n - GW_{-2}O_{n-1} - GW_0\left(O_{n-2} - 2O_{n-1} + O_{n-3}\right) - GW_{-1}\left(O_{n-1} + O_{n-2}\right), \\ a_{33} &= GW_1O_{n-1} - GW_0\left(O_{n-3} - 2O_{n-2} + O_{n-4}\right) - GW_{-1}\left(O_{n-2} + O_{n-3}\right) - GW_{-2}O_{n-2}, \\ a_{43} &= GW_1O_{n-2} - GW_0\left(O_{n-4} - 2O_{n-3} + O_{n-5}\right) - GW_{-1}\left(O_{n-3} + O_{n-4}\right) - GW_{-2}O_{n-3}, \end{aligned}$$

$$\begin{aligned} a_{14} &= GW_0O_{n+1} - GW_{-1}\left(O_{n-1} - 2O_n + O_{n-2}\right) - GO_nW_{-3} - GW_{-2}\left(O_n + O_{n-1}\right), \\ a_{24} &= GW_0O_n - GW_{-3}O_{n-1} - GW_{-2}\left(O_{n-1} + O_{n-2}\right) - GW_{-1}\left(O_{n-2} - 2O_{n-1} + O_{n-3}\right), \\ a_{34} &= GW_0O_{n-1} - GW_{-2}\left(O_{n-2} + O_{n-3}\right) - GW_{-1}\left(O_{n-3} - 2O_{n-2} + O_{n-4}\right) - GW_{-3}O_{n-2}, \\ a_{44} &= GW_0O_{n-2} - GW_{-2}\left(O_{n-3} + O_{n-4}\right) - GW_{-1}\left(O_{n-4} - 2O_{n-3} + O_{n-5}\right) - GW_{-3}O_{n-3}, \end{aligned}$$

Now the required equality follows from Theorem 3.6. □

Remark 6.3. *Of course, the proof of Theorem 6.2 can be shortened as follows. The multiplication* $A^n N_{GW}$ *yields a matrix with entries* a_{ij} *expressed in terms of Gaussian Olivier numbers and classical Olivier numbers. For example,*

$$a_{11} = GW_3 O_{n+1} - GW_0 O_n - GW_1(O_n + O_{n-1}) - GW_2(O_{n-1} - 2O_n + O_{n-2}),$$

and similar formulas hold for the other entries. The required equality then follows directly from Theorem 3.6. □

By substituting $GW_n = GO_n$ with initial conditions (GO_0, GO_1, GO_2, GO_3) into (6.1) and (6.2), and similarly $GW_n = GK_n$ with (GK_0, GK_1, GK_2, GK_3), we obtain the specialized matrices

$$N_{GO} = \begin{pmatrix} 3+i & 1+i & 1 & 0 \\ 1+i & 1 & 0 & 0 \\ 1 & 0 & 0 & -i \\ 0 & 0 & -i & -1+i \end{pmatrix}, \; E_{GO} = \begin{pmatrix} GO_{n+3} & GO_{n+2} & GO_{n+1} & GO_n \\ GO_{n+2} & GO_{n+1} & GO_n & GO_{n-1} \\ GO_{n+1} & GO_n & GO_{n-1} & GO_{n-2} \\ GO_n & GO_{n-1} & GO_{n-2} & GO_{n-3} \end{pmatrix},$$

and

$$N_{GK} = \begin{pmatrix} 4+5i & 5+i & 1+4i & 4-i \\ 5+i & 1+4i & 4-i & -1+5i \\ 1+4i & 4-i & -1+5i & 5-4i \\ 4-i & -1+5i & 5-4i & -4+9i \end{pmatrix}, \; E_{GK} = \begin{pmatrix} GK_{n+3} & GK_{n+2} & GK_{n+1} & GK_n \\ GK_{n+2} & GK_{n+1} & GK_n & GK_{n-1} \\ GK_{n+1} & GK_n & GK_{n-1} & GK_{n-2} \\ GK_n & GK_{n-1} & GK_{n-2} & GK_{n-3} \end{pmatrix}.$$

From Theorem 6.2, we derive the following corollary.

Corollary 6.4. *The following identities hold:*

(a): $A^n N_{GO} = E_{GO}$.

(b): $A^n N_{GK} = E_{GK}$.

References

[1] Frontczak, R., Convolutions for Generalized Tribonacci Numbers and Related Results, International Journal of Mathematical Analysis, Vol. 12, 2018, no. 7, 307 - 324.

[2] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[3] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[4] Soykan, Y., Generalized Pell-Padovan Numbers, Asian Journal of Advanced Research and Reports, 11(2), 8-28, 2020. DOI: 10.9734/AJARR/2020/v11i230259

[5] Soykan, Y., Generalized Olivier Numbers, Asian Research Journal of Mathematics, 19(1), Page 1-22, 2023. DOI: 10.9734/ARJOM/2023/v19i1634

[6] Soykan, Y., Generalized Tetranacci Polynomials and Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 12(2), 35-128, 2024.

[7] Soykan, Y., Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 12(4), 34-101, 2025.

[8] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026.

[9] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

Declaration of AI Use

This manuscript was prepared through the combined contributions of all author(s), including contributions to the study design, data, content development, results, interpretation, and related scholarly work. The author(s) acknowledge the use of Grammarly and ChatGPT to assist with grammar checking, language refinement, reference formatting. These AI-assisted tools were not used as authors and did not replace the intellectual contributions or scholarly judgment of the author(s). All AI-assisted outputs, including content, references, and interpretations, were carefully reviewed, revised, verified, and approved by the author(s). The author(s) accept full responsibility for the accuracy, integrity, and final content of the manuscript.