A theorem on general matrix summability method

ABSTRACT. The paper presents a new general theorem on $\phi - |B;\delta|_k$ summability of an infinite series, where B is a positive normal matrix, (ϕ_n) is any sequence of positive real numbers, $k \geq 1$ and $\delta \geq 0$.

Keywords:summability factor, absolute matrix summability, infinite series, matrix transformation.
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1 Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and $B = (b_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then B defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $Bs = (B_n(s))$, where $B_n(s) = \sum_{v=0}^{n} b_{nv} s_v, \quad n = 0, 1, ...$ Let (ϕ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\phi - |B;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [9])

$$\sum_{n=1}^{\infty} \phi_n^{\delta k+k-1} |B_n(s) - B_{n-1}(s)|^k < \infty.$$

For $\delta = 0$, $\phi_n = \frac{P_n}{p_n}$ and $b_{nv} = \frac{p_v}{P_n}$, $|\bar{N}, p_n|_k$ summability (see [2]) is obtained.

2 Known Results

Concerning $|\bar{N}, p_n|_k$ summability, Bor [1] proved the following theorem.

Theorem 1 *Let the sequences* (λ_n) *and* (p_n) *satisfy the following conditions:*

$$\sum_{n=2}^{m} \frac{p_n|\lambda_n|}{P_n} = O(1), \tag{1}$$

$$P_m|\Delta\lambda_m| = O(p_m|\lambda_m|) \quad as \quad m \to \infty. \tag{2}$$

If

$$\sum_{v=1}^{n} p_v|s_v|^k = O(P_n\gamma_n) \quad as \quad n \to \infty \tag{3}$$

where (γ_n) *is a positive non-decreasing sequence such that*

$$P_{n+1}\gamma_n\Delta\left(\frac{1}{\gamma_n}\right) = O(p_{n+1}) \quad as \quad n \to \infty \tag{4}$$

then the series $\sum a_n\lambda_n(\gamma_n)^{-1}$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$

Theorem 2 *[3] Let the sequences* (λ_n) *and* (p_n) *satisfy the condition (2) and*

$$\sum_{n=2}^{m} \frac{p_n|\lambda_n|^k}{P_n} = O(1) \quad as \quad m \to \infty. \tag{5}$$

If the condition (3) satisfies, then the series $\sum a_n\lambda_n(\gamma_n)^{-1}$ *is summable* $|\bar{N}, p_n|_k$, *where* $k \geq 1$ *and* (γ_n) *is as in Theorem 1.*

Lemma 1 *[4] If the sequences* (λ_n) *and* (p_n) *satisfy the conditions (2) and (5), then*

$$|\lambda_n| = O(1), \tag{6}$$

$$P_n\Delta\left(|\lambda_n|^k\right) = O(p_n|\lambda_n|^k) \quad as \quad n \to \infty. \tag{7}$$

3 Main Result

The aim of this paper is to establish a new theorem on $\phi - |B;\delta|_k$ summability. Some further papers on generalized absolute summability methods, we can refer to [5–8, 10–16]. Let $B = (b_{nv})$ be a normal matrix, we associate two lower semimatrices $\bar{B} = (\bar{b}_{nv})$ and $\hat{B} = (\hat{b}_{nv})$ as follows:

$$\bar{b}_{nv} = \sum_{i=v}^{n} b_{ni}, \quad n, v = 0, 1, ... \tag{8}$$

$$\hat{b}_{00} = \bar{b}_{00} = b_{00}, \quad \hat{b}_{nv} = \bar{b}_{nv} - \bar{b}_{n-1,v}, \quad n = 1, 2, ... \tag{9}$$

and

$$B_n(s) = \sum_{v=0}^{n} b_{nv} s_v = \sum_{v=0}^{n} \bar{b}_{nv} a_v \tag{10}$$

$$\bar{\Delta} B_n(s) = \sum_{v=0}^{n} \hat{b}_{nv} a_v. \tag{11}$$

Now, we shall prove the following theorem.

Theorem 3 *Let $p_{v+1}/P_{v+1} = O\left(p_v/P_v\right)$ and $\phi_n p_n \asymp O(P_n)$. Let all conditions of Theorem 2 be satisfied with the condition (3) replaced by*

$$\sum_{v=1}^{n} \phi_v^{\delta k} p_v |s_v|^k = O(P_n \gamma_n) \quad as \quad n \to \infty. \tag{12}$$

If $B = (b_{nv})$ is a positive normal matrix such that

$$\bar{b}_{n0} = 1, \;\; n = 0, 1, ..., \tag{13}$$

$$b_{n-1,v} \geq b_{nv}, \;\; for \; n \geq v + 1, \tag{14}$$

$$b_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{15}$$

$$\sum_{n=2}^{m} \frac{p_n}{P_n} = O(1) \quad as \quad m \to \infty. \tag{16}$$

$$\sum_{n=v+1}^{m+1} \phi_n^{\delta k} |\hat{b}_{n,v+1}| = O\left(\phi_v^{\delta k}\right) \quad as \quad m \to \infty, \tag{17}$$

$$\sum_{n=v+1}^{m+1} \phi_n^{\delta k} |\Delta_v\left(\hat{b}_{nv}\right)| = O\left(\phi_v^{\delta k - 1}\right) \quad as \quad m \to \infty, \tag{18}$$

where $\Delta_v(\hat{b}_{nv}) = \hat{b}_{nv} - \hat{b}_{n,v+1}$, then the series $\sum a_n \lambda_n (\gamma_n)^{-1}$ is summable $\phi - |B; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

4 Proof of Theorem 3

Let (M_n) be the sequence of B-transform of the series $\sum a_n\lambda_n(\gamma_n)^{-1}$. Then, by (11), we have

$$\bar{\Delta}M_n = \sum_{v=1}^{n}\hat{b}_{nv}a_v\lambda_v(\gamma_v)^{-1}.$$

By using Abel's transformation for the above sum, we obtain

$$\begin{aligned}
\bar{\Delta}M_n &= \sum_{v=1}^{n-1}\Delta_v(\hat{b}_{nv}\lambda_v(\gamma_v)^{-1})\sum_{r=1}^{v}a_r + \hat{b}_{nn}\lambda_n(\gamma_n)^{-1}\sum_{v=1}^{n}a_v \\
&= \sum_{v=1}^{n-1}\Delta_v(\hat{b}_{nv}\lambda_v(\gamma_v)^{-1})s_v + \hat{b}_{nn}\lambda_n(\gamma_n)^{-1}s_n \\
&= b_{nn}\lambda_n(\gamma_n)^{-1}s_n + \sum_{v=1}^{n-1}\Delta_v(\hat{b}_{nv})\lambda_v(\gamma_v)^{-1}s_v + \sum_{v=1}^{n-1}\hat{b}_{n,v+1}\Delta\lambda_v(\gamma_{v+1})^{-1}s_v \\
&\quad + \sum_{v=1}^{n-1}\hat{b}_{n,v+1}\lambda_v\Delta\left((\gamma_v)^{-1}\right)s_v \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\end{aligned}$$

First, by (12) and (15), we achieve

$$\begin{aligned}
\sum_{n=1}^{m}\phi_n^{\delta k+k-1}|M_{n,1}|^k &= \sum_{n=1}^{m}\phi_n^{\delta k+k-1}b_{nn}^k|\lambda_n|^k\gamma_n^{-k}|s_n|^k \\
&= O(1)\sum_{n=1}^{m}\phi_n^{\delta k}\frac{|\lambda_n|^k p_n|s_n|^k}{P_n\gamma_n} \\
&= O(1)\sum_{n=1}^{m-1}\Delta\left(\frac{|\lambda_n|^k}{P_n\gamma_n}\right)\sum_{r=1}^{n}\phi_r^{\delta k}p_r\,|s_r|^k + O(1)\frac{|\lambda_m|^k}{P_m\gamma_m}\sum_{n=1}^{m}\phi_n^{\delta k}p_n\,|s_n|^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta\left(\frac{|\lambda_n|^k}{P_n\gamma_n}\right)P_n\gamma_n + O(1)\frac{|\lambda_m|^k}{P_m\gamma_m}P_m\gamma_m \\
&= O(1)\sum_{n=1}^{m-1}\Delta\left(|\lambda_n|^k\right) + O(1)\sum_{n=1}^{m-1}|\lambda_{n+1}|^k\,\gamma_n\Delta\left(\frac{1}{\gamma_n}\right) + O(1)\sum_{n=1}^{m-1}\frac{p_{n+1}|\lambda_{n+1}|^k}{P_{n+1}} \\
&+ \; O(1)|\lambda_m|^k.
\end{aligned}$$

Then by (4), (5), (6), (7), we get

$$\begin{aligned}
\sum_{n=1}^{m}\phi_n^{\delta k+k-1}|M_{n,1}|^k &= O(1)\sum_{n=1}^{m-1}\frac{p_n|\lambda_n|^k}{P_n} + O(1)\sum_{n=1}^{m-1}\frac{p_{n+1}|\lambda_{n+1}|^k}{P_{n+1}} + O(1)\sum_{n=1}^{m-1}\frac{p_{n+1}|\lambda_{n+1}|^k}{P_{n+1}} + O(1) \\
&= O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Secondly, by Hölder's inequality, (15), (18), we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,2}|^k &\leq \sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{b}_{nv})\right|\frac{|\lambda_v|\,|s_v|}{\gamma_v}\right)^k\\ &\leq \sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{b}_{nv})\right|\frac{|\lambda_v|^k\,|s_v|^k}{\gamma_v^k}\right)\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{b}_{nv})|\right)^{k-1}\\ &\leq \sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}b_{nn}^{k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{b}_{nv})\right|\frac{|\lambda_v|^k\,|s_v|^k}{\gamma_v^k}\right)\\ &= O(1)\sum_{v=1}^{m}\frac{|\lambda_v|^k\,|s_v|^k}{\gamma_v^k}\sum_{n=v+1}^{m+1}\phi_n^{\delta k}|\Delta_v(\hat{b}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\phi_v^{\delta k}\frac{|\lambda_v|^k\,p_v\,|s_v|^k}{P_v\gamma_v} = O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $M_{n,1}$. Now, by (2), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,3}|^k &\leq \sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|\frac{|\Delta\lambda_v||s_v|}{\gamma_v}\right)^k\\ &\leq \sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|^k|s_v|^k\,|\lambda_v|^k\,\frac{1}{\gamma_v^k}\frac{p_v}{P_v}\right)\left(\sum_{v=1}^{n-1}\frac{p_v}{P_v}\right)^{k-1}.\end{aligned}$$

Here, the condition (16) gives

$$\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|^{k-1}|\hat{b}_{n,v+1}|\frac{|\lambda_v|^k\,p_v\,|s_v|^k}{P_v\gamma_v}\right).$$

Thence, by (15) and (17), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}b_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|\frac{|\lambda_v|^k\,p_v\,|s_v|^k}{P_v\gamma_v}\right)\\ &= O(1)\sum_{v=1}^{m}\frac{|\lambda_v|^k\,p_v\,|s_v|^k}{P_v\gamma_v}\sum_{n=v+1}^{m+1}\phi_n^{\delta k}|\hat{b}_{n,v+1}|\\ &= O(1)\sum_{v=1}^{m}\phi_v^{\delta k}\frac{|\lambda_v|^k\,p_v\,|s_v|^k}{P_v\gamma_v} = O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $M_{n,1}$.

Finally, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,4}|^k &\leq \sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}||\lambda_v||s_v|\Delta\left(\frac{1}{\gamma_v}\right)\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}||\lambda_v||s_v|\frac{p_{v+1}}{P_{v+1}\gamma_v}\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|^k\,|\lambda_v|^k\,|s_v|^k\,\frac{1}{\gamma_v^k}\frac{p_{v+1}}{P_{v+1}}\right)\left(\sum_{v=1}^{n-1}\frac{p_{v+1}}{P_{v+1}}\right)^{k-1}.\end{aligned}$$

Now, using (16), we get

$$\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,4}|^k = O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}b_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|\frac{|\lambda_v|^k\,p_{v+1}\,|s_v|^k}{P_{v+1}\gamma_v}\right).$$

Then, by (17), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}|M_{n,4}|^k &= O(1)\sum_{v=1}^{m}\frac{|\lambda_v|^k\,p_{v+1}\,|s_v|^k}{P_{v+1}\gamma_v}\sum_{n=v+1}^{m+1}\phi_n^{\delta k}|\hat{b}_{n,v+1}|\\ &= O(1)\sum_{v=1}^{m}\phi_v^{\delta k}\frac{|\lambda_v|^k\,p_{v+1}\,|s_v|^k}{P_{v+1}\gamma_v} = O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $M_{n,1}$. Therefore, we prove that $\sum_{n=1}^{\infty}\phi_n^{\delta k+k-1}|M_{n,r}|^k < \infty$ for $r = 1, 2, 3, 4$ so the proof of Theorem 3 is completed.

5 Conclusion

In this paper, a new general theorem about absolute matrix summability of the series $\sum a_n\lambda_n(\gamma_n)^{-1}$ is proved for $k \geq 1$ and $0 \leq \delta < 1/k$.

Declaration of AI Use

This manuscript was prepared through the combined contributions of all author(s), including contributions to the study design, data, content development, results, interpretation, and related scholarly work. The author(s) acknowledge the use of Grammarly and ChatGPT to assist with grammar checking, language refinement, reference formatting. These AI-assisted tools were not used as authors and did not replace the intellectual contributions or scholarly judgment of the author(s). All AI-assisted outputs, including content, references, and interpretations, were carefully reviewed, revised, verified, and approved by the author(s). The author(s) accept full responsibility for the accuracy, integrity, and final content of the manuscript.

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