Applications of power increasing sequences

ABSTRACT. In the present paper, a general theorem dealing with absolute matrix summability of an infinite series has been established by using a quasi power increasing sequence instead of an almost increasing sequence. Also, some results have been obtained from this theorem. The theorem broadens the applicability of existing summability methods by replacing almost increasing sequences with the more general class of quasi f-power increasing sequences.

Keywords: absolute matrix summability, almost increasing sequences, Hölder's inequality, infinite series, quasi power increasing sequences
MSC (2020): 26D15; 40D15; 40F05; 40G99.

1 Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad (n \to \infty), \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,.$$

The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (z_n) of the Riesz mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [10]).

Let $B = (b_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then B defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $Bs = (B_n(s))$, where

$$B_n(s) = \sum_{v=0}^{n} b_{nv} s_v, \quad n = 0, 1, ...$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |B, p_n|_k$, $k \geq 1$, if (see [15])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |B_n(s) - B_{n-1}(s)|^k < \infty.$$

For $\varphi_n = \frac{P_n}{p_n}$, the $\varphi - |B, p_n|_k$ summability reduces to $|B, p_n|_k$ summability (see [26]). Furthermore, for $\varphi_n = \frac{P_n}{p_n}$ and $b_{nv} = \frac{p_v}{P_n}$, the $|\bar{N}, p_n|_k$ summability method is obtained (see [2]).

Given a normal matrix $B = (b_{nv})$, two lower semimatrices $\bar{B} = (\bar{b}_{nv})$ and $\hat{B} = (\hat{b}_{nv})$ are defined as follows:

$$\bar{b}_{nv} = \sum_{i=v}^{n} b_{ni}, \quad n, v = 0, 1, ... \tag{1}$$

$$\hat{b}_{00} = \bar{b}_{00} = b_{00}, \quad \hat{b}_{nv} = \bar{b}_{nv} - \bar{b}_{n-1,v}, \quad n = 1, 2, ... \tag{2}$$

and

$$B_n(s) = \sum_{v=0}^{n} b_{nv} s_v = \sum_{v=0}^{n} \bar{b}_{nv} a_v$$

$$\bar{\Delta} B_n(s) = B_n(s) - B_{n-1}(s) = \sum_{v=0}^{n} \hat{b}_{nv} a_v. \tag{3}$$

A positive sequence (g_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constant M and N such that $Mc_n \leq g_n \leq Nc_n$ (see [1]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say, $b_n = ne^{(-1)^n}$. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $K f_n X_n \geq f_m X_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^{\delta} (\log n)^{\sigma}, \ \sigma \geq 0, \ 0 < \delta < 1\}$(see [27]). For σ=0, then a quasi δ-power increasing sequence is obtained (see [13]). It should be noted that, every almost increasing sequence is a quasi δ-power increasing sequence for any nonnegative δ, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\delta}$ for $\delta > 0$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

Summability factors of infinite series associated with Riesz means have been investigated by several authors using quasi power increasing sequences (see [4,5,7–9]). On the other hand, numerous results on absolute matrix summability factors and summability theorems involving almost increasing sequences may be found in the literature (see [11, 12, 17, 18, 21, 22, 24, 25]). Further extensions involving quasi power increasing sequences and their generalizations may be found in [19,20,23]. Since the class of quasi-f-power increasing sequences contains the class of almost increasing sequences, results established under the latter assumptions may be extended to a broader setting. Motivated by this observation, the present work generalizes Bor's theorem [3] by replacing the almost increasing sequence condition with the quasi-f-power increasing sequence condition. Therefore, the scope of applicability of the corresponding summability theorem is enlarged. The applicability of the main theorem is demonstrated through several consequences and special cases, which recover previously known summability results under suitable choices of the parameters.

2 Known Result

In [3], Bor proved a theorem on $|\bar{N}, p_n|_k$ summability factors of infinite series by using an almost increasing sequence.

Theorem 1 *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{4}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty, \tag{6}$$

$$|\lambda_n| X_n = O(1). \tag{7}$$

If

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad as \quad m \to \infty, \tag{8}$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad as \quad m \to \infty,$$

and (p_n) *is a sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty,$$

where (t_n) *is the n-th* $(C,1)$ *mean of the sequence* (na_n)*, then the series* $\sum a_n \lambda_n$ *is summable* $\left|\bar{N}, p_n\right|_k, k \geq 1$.

3 Main Result

The main result of this paper is a generalization of Theorem 1 obtained by employing quasi power increasing sequences.

Theorem 2 *Let* $B = (b_{nv})$ *be a positive normal matrix such that*

$$\bar{b}_{n0} = 1, \; n = 0, 1, ..., \tag{9}$$

$$b_{n-1,v} \geq b_{nv}, \; for \; n \geq v+1, \tag{10}$$

$$b_{nn} = O\left(\frac{p_n}{P_n}\right). \tag{11}$$

Let (X_n) *be a quasi-f-power increasing sequence for some* $0 < \delta < 1$, $\sigma \geq 0$ *and* $(\frac{\varphi_n p_n}{P_n})$ *be a non-increasing sequence. If* $(\lambda_n) \in \mathcal{BV}$*, the conditions (4)-(8) of Theorem 1 and*

$$\sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^{k-1} \frac{1}{n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

$$\sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^{k} |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{13}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |B, p_n|_k$, $k \geq 1$.

Conditions (9)–(13) are standard assumptions on the matrix B and are used in the study of absolute matrix summability. Conditions (12) and (13) provide the growth restrictions required for the application of Hölder's inequality and the convergence arguments used in the proof.

The following lemma is required for the proof of the theorem.

Lemma 1 *([6]) Under the conditions of Theorem 2, the following conditions hold;*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{15}$$

4 Proof of Theorem 2

Let (I_n) denotes B-transform of the series $\sum a_n \lambda_n$. Then, by (3), it follows that

$$\bar{\Delta} I_n \quad = \quad \sum_{v=0}^{n} \hat{b}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{b}_{nv}\lambda_v}{v} v a_v.$$

Applying Abel's transformation to this sum, it follows that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{b}_{nv}\lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{b}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{b}_{nv} \right) \lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{b}_{n,v+1} \Delta\lambda_v t_v \\
&+ \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} + \frac{n+1}{n} b_{nn} \lambda_n t_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 2, it is enough to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \mid I_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{16}$$

The case for $k = 1$ follows directly by applying linear estimates and changing the order of summation without using Hölder's inequality. Therefore, the details of the proof are presented for the case $k > 1$.

First, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$,

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |I_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v(\hat{b}_{nv}) \right| |\lambda_v| \, |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v(\hat{b}_{nv}) \right|^{\frac{1}{k}} \left| \Delta_v(\hat{b}_{nv}) \right|^{\frac{k-1}{k}} |\lambda_v| \, |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\left(\sum_{v=1}^{n-1} \left(\left| \Delta_v(\hat{b}_{nv}) \right|^{\frac{1}{k}} |\lambda_v| \, |t_v| \right)^k \right)^{\frac{1}{k}} \right)^k \\
&\times \left(\left(\sum_{v=1}^{n-1} \left(|\Delta_v(\hat{b}_{nv})|^{\frac{k-1}{k}} \right)^{k'} \right)^{\frac{1}{k'}} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v(\hat{b}_{nv}) \right| |\lambda_v|^k \, |t_v|^k \right) \\
&\times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{b}_{nv})| \right)^{k-1}.
\end{aligned}$$

By (1) and (2),

$$\begin{aligned}
\Delta_v(\hat{b}_{nv}) = \hat{b}_{nv} - \hat{b}_{n,v+1} &= \bar{b}_{nv} - \bar{b}_{n-1,v} - \bar{b}_{n,v+1} + \bar{b}_{n-1,v+1} \\
&= b_{nv} - b_{n-1,v}. \qquad (17)
\end{aligned}$$

Thus using (1), (9) and (10)

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{b}_{nv})| = \sum_{v=1}^{n-1} (b_{n-1,v} - b_{nv}) \leq b_{nn}.$$

Thus, by (11) and using the fact that $(\frac{\varphi_n p_n}{P_n})$ is a non-increasing sequence,

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} b_{nn}^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v(\hat{b}_{nv}) \right| |\lambda_v|^k \, |t_v|^k \right) \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v(\hat{b}_{nv}) \right| |\lambda_v|^k \, |t_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\Delta_v(\hat{b}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v(\hat{b}_{nv})|.
\end{aligned}$$

By (10) and (17), it follows that

$$\sum_{n=v+1}^{m+1} |\Delta_v(\hat{b}_{nv})| = \sum_{n=v+1}^{m+1} (b_{n-1,v} - b_{nv}) \leq b_{vv},$$

and so

$$
\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1} \mid I_{n,1} \mid^k &= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}|\lambda_v|^{k-1}|\lambda_v||t_v|^k b_{vv}\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k}|\lambda_v|\,|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{n=1}^{v}\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^{k}|t_n|^k+O(1)|\lambda_m|\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{n=1}^{m-1}\beta_v X_v+O(1)|\lambda_m|X_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}
$$

by (4), (7), (13) and (15).

By Hölder's inequality,

$$
\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1} \mid I_{n,2} \mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}||\Delta\lambda_v||t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}||\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}b_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|\beta_v|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{b}_{n,v+1}|\beta_v|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{b}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}|\hat{b}_{n,v+1}|.
\end{aligned}
$$

By (1), (2), (9) and (10),

$$|\hat{b}_{n,v+1}| = \sum_{i=0}^{v} (b_{n-1,i} - b_{ni}).$$

Thus, using (1) and (9),

$$\sum_{n=v+1}^{m+1} |\hat{b}_{n,v+1}| = \sum_{n=v+1}^{m+1} \sum_{i=0}^{v} (b_{n-1,i} - b_{ni}) \leq 1,$$

then

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid I_{n,2} \mid^k &= O(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v}\right)^{k-1} v\beta_v \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \varphi_r^{k-1} \left(\frac{p_r}{P_r}\right)^{k-1} \frac{1}{r} |t_r|^k \\
&+ O(1) m\beta_m \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v}\right)^{k-1} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (6), (12), (14) and (15).

Furthermore, again using Hölder's inequality,

$$
\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |I_{n,3}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{b}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{b}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \times \left(\sum_{v=1}^{n-1} |\hat{b}_{n,v+1}| \frac{|\lambda_{v+1}|}{v} \right)^{k-1} \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} b_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{b}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \times \left(\sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{b}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{b}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\hat{b}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} |\hat{b}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| \sum_{r=1}^{v} \varphi_r^{k-1} \left(\frac{p_r}{P_r} \right)^{k-1} \frac{1}{r} |t_r|^k \\
&+ O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^{k-1} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by using (4), (7), (8), (11), (12) and (15).

Finally,

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid I_{n,4} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} b_{nn}^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\lambda_n|^{k-1} |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \varphi_v^{k-1} \left(\frac{p_v}{P_v}\right)^k |t_v|^k + O(1)|\lambda_m| \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v}\right)^k |t_v|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (4), (7), (11), (13) and (15). This completes the proof of Theorem 2.

5 Conclusions

The main contribution of this paper is the extension of a known summability theorem from almost increasing sequences to the more general class of quasi f-power increasing sequences. This shows that the summability results remain valid under different assumptions, thereby enlarging the scope of applicability of the theorem. Consequently, the obtained result provides a broader framework for the study of absolute matrix summability factors of infinite series. Further extensions may be considered by using some of the conditions of Theorem 2 or by investigating other classes of summability methods.

If $\sigma = 0$ in Theorem 2, a theorem dealing with quasi δ-power increasing sequences is obtained (see [16]). Theorem 1 follows by taking $\varphi_n = \frac{P_n}{p_n}$, $b_{nv} = \frac{p_v}{P_n}$, and (X_n) as an almost increasing sequence. Similarly, for $\varphi_n = \frac{P_n}{p_n}$ and (X_n) an almost increasing sequence, a theorem dealing with $|B, p_n|_k$ summability is obtained (see [14]). In the last two cases, the condition $(\lambda_n) \in \mathcal{BV}$ is not required.

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