Counting methods of area integrals on the unit disk

Abstract

Suppose that a simple closed curve C is defined parametrically in the complex w-plane

$$u = u(\theta) \quad , v = v(\theta) \quad , 0 \leq \theta \leq 2\pi$$

Then the area A enclosed by C is given by,

$$A = \frac{1}{2}\int_{0}^{2\pi} \left(u\frac{dv}{d\theta} - v\frac{du}{d\theta} \right) d\theta$$

If there is a conformal mapping Ψ_r,from the exterior of D_r ,$r > 1$ to the exterior of C_r of the form

$$\Psi_r(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

then by Gronwall's area formula, the area A_r of the region B_r enclosed by C_r is given by,

$$A_r = \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right)$$

We use Gronwall's area formula to find the area of some differents regions as circles with radius r centred ot the origin lying in the complex w-plane , ellipses and lemniscates A lemniscate shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2, 3, 4.....$$

We use Laurent and Taylor series expansions of conformal mapping from the exterior of the unit disk to either of these regions to compute the area of them.

Keywords: Unit disk, Conformal mapping ,Area integral,Gronwall's area formula, Univalents functions,regular functions ,Circles, Ellipses ,Lemniscates, Differentiation.

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1 Introduction

In [18],intituled :Sur les formules de Cauchy et de Green et quelques problémes qui s'y rattachent, Bulletin de l'académie des sciences de l'U.R.S.S., 1932, 337-372. He considered examples of applications of the two formulas dues to Cauchy and Green ,Smirnov consider a simply connected domain B and closed rectifiable Jordan curve C ,the pole a is interior of the contour C and $f(z)$ is regular function on interior of B. Smirnov use the two Cauchy's and Green's formulas.

$$f(a) = \frac{1}{2i\pi} \int_C \frac{f(z)}{z-a} dz$$

and

$$f(a) = \frac{1}{2\pi} \int_C f(z) \frac{\partial G(z,a)}{\partial \eta} ds$$

$G(z,a)$,is the Green's function defined on the simply connected domain B at the pole$a$ equivalently that $G(z,a)$ is harmonic function real on interior of B equal zero on the contour C and with a singularity at the point a such that

$$G(z,a) - \log |z-a|$$

have not singulariry at the point a .Let us denote by $f(z)$ the limit values of $f(a)$ on the contour C in direction of curves no tangents and η is the direction of exterior normal on the contour C .

One of other counting methods of area integrals on the unit disk is so called Green's and Stokes formula,[7],[19],[16],[3],[4],[5]

$$\iint\limits_B f(z) \overline{g'(z)} dxdy = \frac{1}{2i} \int\limits_C f(z) \overline{g(z)} dz$$

where$f$ and g are continuous functions with $z = x + iy$ and the domain B is enclosed by closed curve C in the complex z-plane .Suppose the potentiel function:

$$V(z) = -\int \int_B \log |w-z| d\sigma_w$$

exist, where $w = u + iv = re^{i\phi}$, $d\sigma_w = dudv = rdrd\phi$.The first derivative of this potentiel function is continuous on the complex plane $\mathbb{C}$.This potentiel function is solution of partial differential equation :

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -2\pi$$

:

$$\frac{\partial V}{\partial z} = \frac{1}{2}\left(\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}\right) = \frac{1}{2} \int \int_B \frac{d\sigma_w}{w-z}$$

In help of Green's and Stokes formula:

$$\int \int_B f(z) dxdy = \frac{1}{2i} \int_C f(z) \overline{z} dz$$

Suppose the existence of Schwarz's function $S(z)$,

$$S(z) = \frac{1}{\pi} \sum_{i=1}^{n} \sum_{j=0}^{r_{i-1}} \frac{a_{i,j} j!}{(z-z_j)^{j+1}}$$

then

$$S(z) = \overline{z}$$

Applying the Cauchy's formula yields to

$$\int\int_D f(z)dxdy = \sum_{l=1}^{n}\sum_{k=1}^{r_{k-1}} a^{k,l} f^{(l)}(z_k)$$

The paper will be structured as follows : in section 1 we present some useful terminology as well as some necessary definitions regarding the counting methods of area integrals on the unit disc with respect to the two-dimensional Lebesgue planar measure $dm\,(x, y) = dxdy$,where $z = x + iy$. In section 2,first we present the Gronwall's area formula in the case of general simple closed curve C is defined parametrically in the complex w-plane

$$u = u\,(\theta) \quad ,v = v\,(\theta) \quad ,0 \leq \theta \leq 2\pi$$

Then the area A enclosed by C is given by,

$$A = \frac{1}{2}\int_{0}^{2\pi} \left(u\frac{dv}{d\theta} - v\frac{du}{d\theta} \right) d\theta$$

we prove If there is a conformal mapping Ψ_r,from the exterior of D_r ,$r > 1$ to the exterior of C_r of the form

$$\Psi_r\,(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

then by Gronwall's area formula, the area A_r of the region B_r enclosed by C_r is given by,

$$A_r \; = \pi \left(r^2 - \sum_{n=1}^{\infty} n\,|b_n|^2\, r^{-2n} \right)$$

Second,we use Gronwall's area formula to find the area of some differents regions as circles with radius r centred ot the origin lying in the complex w-plane,ellipses and lemniscates.

A lemniscate shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}\,, m = 2, 3, 4.....$$

We use Laurent and Taylor series expansions of conformal mapping from the exterior of the unit disk to either of these regions to compute the area of theses bounded regions.

One of other counting methods of area integrals on the unit disc and ellipses is so called Green's and Stokes formula,[7],[19],[16],[3],[4],[5]

$$\iint_D f\,(z)\,\overline{g'\,(z)}dxdy = \frac{1}{2i}\int_C f\,(z)\,\overline{g\,(z)}dz$$

where $z = x + iy$ and the domain D is enclosed by closed curve C in the complex z-plane .We apply this formula to get some results concerning counting methods of area integrals on the unit disc and ellipses.As application of Green's and Stokes formula on the ellipse $D : b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.

We consider a new system of orthogonal Tchebychev polynomials of second kind $\{U_n(z)\}_{n=0,1,2...}$,with respect to the Lebesgue planar measure $dm(x,y) = dxdy$,where $z = x + iy$ concentrated on the ellipse D: $b^2x^2 + a^2y^2 < a^2b^2$ where $a > b$,a system of orthogonal polynomials, given by :

$$U_n(z) = \frac{T'_{n+1}(z)}{n+1} = \frac{\sin\left((n+1)\cos^{-1} z\right)}{\sqrt{1-z^2}} \quad , \quad n = 0, 1, 2....$$

where $T_n(z) = \cos\left(n \cos^{-1} z\right), n = 0, 1, 2, 3....$is a polynomial of degree n . $T_n(z)$ is called the Tchebychev polynomial of degree n of first kind .

They satisfies

$$\iint\limits_D U_n(z)\overline{U_m(z)}dxdy = \frac{4(n+1)}{\pi(\rho^{n+1} - \rho^{-n-1})}\delta_{n,m} \qquad ,n, m = 0, 1, 2....$$

where $\delta_{n,m}$, is the symbol of Kronecker , $z = x + iy$ and $(a + b)^2 = \rho$.

Let $f(z)$ be regular in a domain D.If

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$$

which converges uniformly and absolutely in any closed subdomain of ellipse D.By applying the Green's and Stokes formula,we show that

$$a_{n+1} = \frac{i\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8(n+1)^2} \int\limits_C f(z)\overline{U_n(z)}d\overline{z} \ \ , \ \rho = (a+b)^2 \ , \ n = 0, 1, 2, ...$$

where C: is the ellipse :$b^2x^2 + a^2y^2 = a^2b^2$.

Suppose $G \subset \mathbb{C}$ is an arbitrary domain ,f is analytic in G and set,[7],[19],[16],[1],[10]

$$I(f) = \iint\limits_G |f(z)|^2 \, dm \qquad ,z = x + iy \tag{1}$$

where $dm(x,y) = dxdy$ is the two-dimensional area Lebesgue measure.$I(f)$ can be represented explicititly in terms of the coefficients of f .

Let us compute $I(f)$ for a special case .Let $G = \{z : r < |z| < R\}, 0 \leq r < R < \infty$,and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad ,z \in G$$

it follows that,

$$I(f) = \int\limits_{\rho=r}^{R}\int\limits_{\phi=0}^{2\pi} \left(\sum_{n=-\infty}^{\infty} a_n \rho^n e^{in\phi}\right)\left(\sum_{m=-\infty}^{\infty} \overline{a_m}\rho^m e^{-im\phi}\right)\rho d\rho d\phi$$

$$= 2\pi \int\limits_{\rho=r}^{R} \sum_{n=-\infty}^{\infty} |a_n|^2 \rho^{2n+1} d\rho$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 \int_{\rho=r}^{R} \rho^{2n+1} d\rho$$

i-e

$$I(f) = \pi \sum_{n=-\infty}^{\infty} \frac{R^{2n+2} - r^{2n+2}}{n+1} |a_n|^2 \tag{2}$$

Since

$$a_0 = f(0), a_1 = f'(0) .... a_k = \frac{f^{(k)}(0)}{k!}$$

We can state

$$|f(0)|^2 \leq \frac{1}{R^2 - r^2} I(f) \tag{3}$$

and

$$\left|f'(0)\right|^2 \leq \frac{2}{R^4 - r^4} I(f) \tag{4}$$

Hence

$$\left|f^{(k)}(0)\right| \leq \frac{(k+1)!}{R^{2k+2} - r^{2k+2}} I(f) \tag{5}$$

Definition 1

$$L^2(G) = \{f : f \ \ analytic \ in \ G \ \ and \ \ I(f) < \infty\} \tag{6}$$

in this case,if $d_z = dist(z, \partial G)$

$$|f(z)|^2 \leq \frac{I(f)}{\pi d_z^2} \tag{7}$$

Theorem 2 ***Caratheodory*** *theorem [17],[16],[13],[1],[7],[2].*

The conformal mapping φ *of* G *onto the disk* $D_R = \{w, |w| < R\}$ *such that*

$$\varphi(\xi) = 0, \qquad and \quad \varphi'(\xi) = 1$$

can be conformally continued by ***Caratheodory*** *theorem,[17],[16],[13],[1],[7],[2] If we suppose that*

$$\psi : D_R = \{w, |w| < R\} \rightarrow G$$

is its inverse map, then we have,

$$\varphi(t) = u \Longleftrightarrow t = \psi(u) \qquad\qquad (t \in E)$$

where

$$\varphi(t) = \lim_{\substack{z \longrightarrow t \\ z \in G}} \varphi(z) \qquad\qquad (\ t \in E)$$

and

$$\psi\left(e^{i\theta}\right) = \lim_{r \longrightarrow 1^-} \psi\left(re^{i\theta}\right) \qquad\qquad (0 \leq \theta \leq 2\pi)$$

These limit boundary values of φ *and* ψ *are respectively bijections and inversely from the contour* E *onto unit circle* $\{u : |u| = 1\}$.

If $w = u + iv = f(z)$, $z = x + iy \in D$, whose boundary is the closed curve C ,the Jacobien of the transformation , $u, v \longrightarrow x, y$ is $\left|f'(z)\right|^2$.

Hence the area of $f(D)$ can be expressed as integral **:**

$$A = \iint\limits_{f(D)} dudv = \iint\limits_{D} \left|f'(z)\right|^2 dxdy \tag{8}$$

By Green's formula,[7],[19],[16],[3],[4],[5]

$$\iint\limits_{D} f(z)\,\overline{g'(z)}dxdy = \frac{1}{2i}\int\limits_{C} f(z)\,\overline{g(z)}dz \tag{9}$$

Notice that

$$A = \iint\limits_{D} \left|f'(z)\right|^2 dxdy = \iint\limits_{D} f(z)\,\overline{f'(z)}dxdy \tag{10}$$

ie

$$A = \iint\limits_{D} \left|f'(z)\right|^2 dxdy = \frac{1}{2i}\int\limits_{C} f'(z)\,\overline{f(z)}dz \tag{11}$$

If f is regular and univalent in a domain D and satisfies $|f(z)| \leq 1$ then ,

$$\iint\limits_{D} \left|f'(z)\right|^2 dxdy \leq \pi \tag{12}$$

Suppose that we are given the function analytique on the unit disk $D = \{|z| \leq 1\}$ function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ .Regarding the magnitude of the coefficients can be obtanined by means of Parseval's identity,[7],[19],[16],[6]

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{13}$$

which has also many other useful applications in the theory of univalents and holomorphics functions.To prove (13),we observe that

$$\left|f\left(re^{i\theta}\right)\right|^2 = f\left(re^{i\theta}\right)\overline{f\left(re^{i\theta}\right)} = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-in\theta}$$

i-e

$$\left|f\left(re^{i\theta}\right)\right|^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta}$$

The rearrangement of the terms are permissible.Hence,

$$\int\limits_{0}^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} \int\limits_{0}^{2\pi} e^{i(n-m)\theta} d\theta$$

reduces to

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{14}$$

If $|f(z)| \leq M(r)$ for $|z| = r$, then obviously,[7],[19],[16],[10]

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq M^2(r)$$

We known that If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad , |z| \leq r$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \qquad , |z| \leq \rho$$

then,[7],[19],[16],[11],[12],[13],[15]

$$\frac{1}{2\pi i} \int_{|w|=r} f(w)\, g\left(\frac{z}{w}\right) \frac{dw}{w} = \sum_{n=0}^{\infty} a_n b_n z^n \qquad , |z| < \rho r \tag{15}$$

In polar coordinates r, θ of $|z| = r$,expressed as R, ϕ in the w-plane ,when f maps $|z| = r$, $r < 1$ in the positive direction , to the unit circle,$f\left(re^{i\theta}\right) = R(r,\theta) + i\phi(r,\theta)$

$$A = -\frac{1}{2} \int_0^{2\pi} R^2(r,\theta) \frac{\partial \phi}{\partial \theta}(r,\theta)\, d\theta \tag{16}$$

Since by the Cauchy Riemann equations in polar coordinates r, θ

$$\frac{\partial \phi}{\partial \theta}(r,\theta) = \frac{r}{R} \frac{\partial R}{\partial r}(r,\theta) \tag{17}$$

it follows that,[7],[19],[16],[22]

$$A = -\frac{r}{4} \frac{\partial}{\partial r} \left(\int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta \right) \tag{18}$$

Example 3 *Let M is a region whose boundary is the cardioid has a parametric polar equation,for* $0 \leq \theta \leq 2\pi$

$$r = \frac{1}{2}(1 + \cos\theta)$$

The area $A_{cardioid}$ can be found by integration .We have

$$A_{cardioid} = \int_0^{\pi} r^2 d\theta = \frac{1}{4} \int_0^{\pi} (1 + \cos\theta)^2 d\theta$$

$$= \frac{1}{4} \int_0^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \right) d\theta = \frac{3\pi}{8}$$

Proposition 4 *The class of univalents functions in $|z|<1$ which have a Laurent expansion,[7],[19],[16],,*

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}b_n z^n \tag{19}$$

The coefficients $b_0, b_1, b_2.....b_n....$ are subject to the inequality

$$\sum_{n=0}^{\infty}n\left|b_n\right|^2\leq 1 \tag{20}$$

Proof.

$$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta=\int_0^{2\pi}f\left(re^{i\theta}\right)\overline{f\left(re^{i\theta}\right)}d\theta$$

$$=\int_0^{2\pi}\left[\frac{1}{re^{i\theta}}+\sum_{n=0}^{\infty}b_n r^n e^{in\theta}\right]\left[\frac{1}{re^{-i\theta}}+\sum_{n=0}^{\infty}b_n r^n e^{-in\theta}\right]d\theta$$

$$=2\pi\left[\frac{1}{r^2}+\sum_{n=0}^{\infty}\left|b_n\right|^2 r^{2n}\right]$$

Hence,

$$A=-\frac{r}{4}\frac{\partial}{\partial r}\left(\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta\right)$$

then

$$A=-\frac{\pi r}{2}\frac{\partial}{\partial r}\left[\frac{1}{r^2}+\sum_{n=0}^{\infty}\left|b_n\right|^2 r^{2n}\right]$$

and thus

$$\frac{1}{\pi}A=\frac{1}{r^2}-\sum_{n=1}^{\infty}n\left|b_n\right|^2 r^{2n}$$

Since the area can not negative it follows that

$$\sum_{n=1}^{\infty}n\left|b_n\right|^2 r^{2n}\leq\frac{1}{r^2}$$

holds for all values of r between 0 and 1,see,[7],[19],[16],[21],[8].It therefore must also be true for $r\longrightarrow 1$.This proves ,(20). ■

Our first result is to prove and apply Gronwall's area formula in the next theorem,we show that If there is a conformal mapping Ψ_r ,from the exterior of D_r ,$r>1$ to the exterior of C_r of the form

$$\Psi_r(z)=z+\sum_{n=0}^{\infty}\frac{b_n}{z^n}$$

then the area A_r of the region B_r enclosed by C_r is given by,

$$A_r=\pi\left(r^2-\sum_{n=1}^{\infty}n\left|b_n\right|^2 r^{-2n}\right)$$

2 Gronwall's area formula

Lemma 5 *Suppose that a simple closed curve C is defined parametrically in the complex w-plane*

$$u = u(\theta) \quad ,v = v(\theta) \quad ,0 \leq \theta \leq 2\pi \tag{21}$$

Then the area A enclosed by C is given by,[7],[19],[6],[20],[21].

$$A = \frac{1}{2}\int_{0}^{2\pi} \left(u\frac{dv}{d\theta} - v\frac{du}{d\theta} \right) d\theta \tag{22}$$

Proof. By Green's theorem ,if $p = p(u, v)$ and $q = q(u, v)$ are functions with continuous partial derivatives on a region B enclosed by C ,[7],[19],[16]then

$$A = \iint_{B} \left(\frac{\partial p}{\partial u} + \frac{\partial q}{\partial v} \right) dudv = \int_{C} (-qdu + pdv)$$

Choosing , $p = u$,$q = v$ gives

$$A = \iint_{B} dudv = \frac{1}{2}\int_{C} (-vdu + udv) = \frac{1}{2}\int_{0}^{2\pi} \left(u\frac{dv}{d\theta} - v\frac{du}{d\theta} \right) d\theta$$

We shall use this result to prove Gronwall's area theorem. ■

Theorem 6 *If there is a conformal mapping Ψ_r ,from the exterior of D_r ,$r > 1$ to the exterior of C_r of the form*

$$\Psi_r(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + ...\frac{b_n}{z^n} + ... \tag{23}$$

then the area A_r of the region B_r enclosed by C_r is given by,[7],[19],[16],[21],[15],

$$A_r = \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right) \tag{24}$$

Proof. Firstly,we write the mapping Ψ_r as

$$w = \Psi_r(z) = u(r, \theta) + iv(r, \theta) \qquad , \quad z = re^{i\theta}$$

where u and v are real-valued functions.By (22) and since r is constant, the area enclosed by C_r is given by

$$A_r = \frac{1}{2}\int_{0}^{2\pi} \left(u\frac{dv}{d\theta} - v\frac{du}{d\theta} \right) d\theta \tag{25}$$

We now want to prove that

$$A_r = \operatorname{Im}\left\{ \frac{1}{2}\int_{0}^{2\pi} \overline{\Psi_r(z)}\Psi_r'(z)\, dz \right\} \tag{26}$$

where Im denote the imaginary part .Since

$$\Psi_r'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{e^{-i\theta}}{r}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right) \tag{27}$$

on using the Cauchy Riemann equations in polar coordinates form,.In fact $dz = ire^{i\theta}d\theta$,we see that

$$\operatorname{Im}\left\{\frac{1}{2}\int_0^{2\pi}\overline{\Psi_r(z)}\Psi_r'(z)\,dz\right\} = \operatorname{Im}\left\{\frac{1}{2}\int_0^{2\pi}(u-iv)\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)d\theta\right\}$$

$$= \frac{1}{2}\int_0^{2\pi}\left(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}\right)d\theta = A_r$$

We can now rewrite equation (23) as

$$\Psi_r(z) = \sum_{n=-1}^{\infty}\frac{b_n}{z^n} \qquad , b_{-1} = 1 \tag{28}$$

Using (28) and (26) we have

$$A_r = \operatorname{Im}\left\{\frac{1}{2}\int_0^{2\pi}\left(\sum_{n=-1}^{\infty}\frac{\overline{b}_n}{\overline{z}^n}\sum_{m=-1}^{\infty}\frac{-mb_m}{z^{m+1}}\right)dz\right\}$$

$$= \operatorname{Im}\left\{-\frac{1}{2}\sum_{m=-1}^{\infty}\sum_{n=-1}^{\infty}mb_m\overline{b}_n\int_{|z=r|}\overline{z}^{-n}z^{m-1}dz\right\}$$

By writing $z = re^{i\theta}$,we find

$$\int_{|z=r|}\overline{z}^{-n}z^{m-1}dz = ir^{-m-n}\int_0^{2\pi}e^{i(n-m)}d\theta = 0 \qquad , n \neq m$$

and

$$\int_{|z=r|}\overline{z}^{-n}z^{n-1}dz = 2\pi i r^{-2n}$$

therefore

$$A_r = \operatorname{Im}\left\{-i\pi\sum_{n=-1}^{\infty}n|b_n|^2 r^{-2n}\right\} = -\pi\sum_{n=-1}^{\infty}n|b_n|^2 r^{-2n}$$

$$= \pi\left(r^2 - \sum_{n=1}^{\infty}n|b_n|^2 r^{-2n}\right)$$

The last relation proves the theorem. ■

The second result of paper is applying Gronwall's area formula to find the area of some differents regions as circles with radius r centred ot the origin lying in the complex

w-plane , ellipses and lemniscates A lemniscate shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2, 3, 4.....$$

We use Laurent and Taylor series expansions of conformal mapping from the exterior of the unit disk to either of these regions to compute the area of them.We obtain the following corollary of Gronwall's area formula property.

3 Circles, ellipses and lemniscates

In this case C_r is the circle with radius r centred ot the origin lying in the complex w-plane .The conformal mapping from the exterior of D_r onto the exterior of C_r is given by :

$$w = \Psi_r(z) = z \tag{29}$$

Thus we see that all the b_n's from equation(23) are zero.By Gronwall's area theorem,the area A_r of the circle C_r is given by

$$A_r = \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right) = \pi r^2 \tag{30}$$

The conformal mapping from the exterior of the unit circle centred at the origin of the complex $z-$plane D_r onto an ellipse C_r in the complex w-plane with semi axis $a = r + \frac{1}{r}$,$b = r - \frac{1}{r}$ is

given by

$$w = \Psi_r(z) = z + \frac{1}{z} \tag{31}$$

Thus we see that all the b_n's from equation(23) are zero except for $b_1 = 1$.By Gronwall's area theorem,the area A_r of the circle C_r is given by

$$A_r = \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right) = \pi \left(r + \frac{1}{r} \right) \left(r - \frac{1}{r} \right)$$

i-e

$$A_r = \pi ab \tag{32}$$

which agrees with the classical formula for the area of an ellipse.

A lemniscate is shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2, 3, 4..... \tag{33}$$

The conformal mapping from the exterior of the uniit circle centred at the origin of the complex $z-$plane D_r onto the exterior of the m-leafed symmetric lemniscate in the complex $w-$plane is given by

$$w = \Psi(z) = z\left(1 + \frac{1}{z^m}\right)^{\frac{1}{m}} \tag{34}$$

The image of this mapping to include the boundary of the lemniscate ,we look at the image on the unit circle in the the complex $z-$plane, that is, take $r = 1$.In this case

$$w^m = z^m + 1$$

so that

$$|w^m - 1| = |z|^m = 1$$

Theorem 7 *If z describe the unit circle.Then the area A of an m-leafed symetric lemniscate is given by*

$$A = \pi\left(\sum_{n=0}^{\infty}\left(C_n^{\frac{1}{m}}\right)^2 - m\sum_{n=1}^{\infty} n\left(C_n^{\frac{1}{m}}\right)^2\right) \tag{35}$$

Hence

$$A = 2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\sqrt{\pi} \tag{36}$$

Proof. If z describe the unit circle.Then by (34) for large $|z|$

$$\Psi(z) = z\sum_{n=0}^{\infty} C_n^{\frac{1}{m}}\frac{1}{z^{mn-1}} \tag{37}$$

According to Gronwall's area formula,(24) the area A of an m-leafed symetric lemniscate is given by

$$A = \pi\left[1 - \sum_{n=1}^{\infty} n\,|b_n|^2\right]$$

$$= \pi\left[1 - \sum_{n=1}^{\infty}(mn-1)\,|b_{mn-1}|^2\right]$$

by (37)

$$A = \pi m\sum_{n=1}^{\infty}\left(\frac{1}{m} - n\right)\left(C_n^{\frac{1}{m}}\right)^2$$

$$= \pi\left(\sum_{n=0}^{\infty}\left(C_n^{\frac{1}{m}}\right)^2 - m\sum_{n=1}^{\infty} n\left(C_n^{\frac{1}{m}}\right)^2\right)$$

and (35) is proved.To prove (36), we known ([6] page 5) ,we known that

$$\sum_{k=0}^{\infty}\left(C_k^{\alpha}\right)^2 = \frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2} \tag{38}$$

and

$$\sum_{k=0}^{\infty} k\left(C_k^{\alpha}\right)^2 = \frac{\Gamma(2\alpha)}{\left(\Gamma(\alpha)\right)^2} \tag{39}$$

Using these assumptions and proprietes of the Gamma function we find

$$A=\pi\left(\frac{\Gamma\left(\frac{2}{m}+1\right)}{\left(\Gamma\left(\frac{1}{m}+1\right)\right)^2}-m\frac{\Gamma\left(\frac{2}{m}\right)}{\left(\Gamma\left(\frac{1}{m}\right)\right)^2}\right)$$

By the Duplicate formula ([6] page 946),we get

$$A=2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\sqrt{\pi}$$

We will now confirm this result by using polar coordinates. Writing $w=\rho e^{i\phi}$,we have

$$|w^m-1|=1$$

or

$$\left(w^m-1\right)\left(\overline{w}^m-1\right)$$

i-e

$$\rho^{2m}-2\rho^m\cos m\phi=0$$

so that

$$\rho=2^{\frac{1}{m}}\cos^{\frac{1}{m}}\left(m\phi\right)$$

The area of one leaf of the m-leafed lemniscate is given by,[6]

$$\frac{1}{m}A=\frac{1}{2}\int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}}\rho^2d\phi$$

and thus the area of the m-leafed lemniscate is given by,[6]

$$A=m\int_0^{\frac{\pi}{2m}}\rho^2d\phi$$

$$=2^{\frac{2}{m}}m\int_0^{\frac{\pi}{2m}}\cos^{\frac{2}{m}}\left(m\phi\right)d\phi$$

If $u=\cos m\phi$,then

$$A=2^{\frac{2}{m}}\int_0^1\frac{u^{\frac{2}{m}}}{\sqrt{1-u^2}}du$$

If $v=u^2$,then

$$A=2^{\frac{2}{m}-1}\int_0^1 v^{\frac{1}{m}-\frac{1}{2}}\left(1-v\right)^{-\frac{1}{2}}dv$$

and therefore,by the Beta function,

$$A=2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}+\frac{1}{2}\right)}$$

$$= 2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\sqrt{\pi}$$

and the theorem is proved. ■

In some cases to be considered ,we will take the limit $r \overset{>}{\longrightarrow} 1$ from above ,and assume that the image of the unit circle is the boundary of the region for which we are trying to find the area.

In the next propositon we show that the area integral can be expressed as expansion of double integral on the unit circle.

Proposition 8 *Consider the area integral*

$$\Im(f) = \int_0^{2\pi}\int_0^{2\pi}\frac{\left|f\left(e^{i\varphi}\right)-f\left(e^{i\theta}\right)\right|^2}{\left|e^{i\varphi}-e^{i\theta}\right|^2}d\theta d\varphi \tag{40}$$

If

$$\Im_r(f) = \int_{C_r}\int_{C_1}\frac{\left|f(y)-f(x)\right|^2}{\left|y-x\right|^2}dxdy \tag{41}$$

where $,C_r = \{|z| = r, 0 \leq r \leq 1\}$ *,oriented in the positive direction then*

$$Lim_{r\overset{<}{\longrightarrow}1}\Im_r(f) = \Im(f) \tag{42}$$

if $r < 1$*.Then,*

$$\int_{C_r}\int_{C_1}\frac{\left|f(y)\right|^2}{(y-x)^2}dxdy = 0 \qquad ,r < 1 \tag{43}$$

then

$$\int_{C_r}\int_{C_1}\frac{\left|f(x)\right|^2}{(y-x)^2}dxdy = 0 \tag{44}$$

$$\int_{C_r}\int_{C_r}\frac{\overline{f(x)}f(y)}{(y-x)^2}dx\ dy\ = 0 \qquad (|z| = 1, |w| < 1) \tag{45}$$

If

$$K_r(f) = \int_{C_r}\int_{C_1}\frac{\overline{f(y)}f(x)}{(y-x)^2}dxdy \tag{46}$$

If $f\left(re^{i\theta}\right) = \sum_{q=0}^{\infty}a_q r^q e^{iq\theta}$*,then*

$$\Im(f) = 4\pi^2\sum_{p=0}^{\infty}p\left|a_p\right|^2 \tag{47}$$

and

$$A = \frac{1}{4\pi}\Im(f) \tag{48}$$

Proof. The function f is holomorphe on the disk $|z| \leq 1$,then the function

$$F(\theta, \varphi) = \frac{\left|f\left(e^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i\varphi} - e^{i\theta}\right|^2}$$

is continued on $[0 \quad 2\pi] \times [0 \quad 2\pi]$,then $\Im(f)$ exist.Setting,

$$x = e^{i\theta}, y = e^{i\varphi}$$

we have

$$\left|e^{i\varphi} - e^{i\theta}\right|^2 = \left(e^{i\varphi} - e^{i\theta}\right)\left(e^{-i\varphi} - e^{-i\theta}\right) = -e^{-i(\theta+\varphi)}(y-x)^2$$

$$dx = ie^{i\theta}d\theta, dy = ie^{i\varphi}d\varphi$$

then

$$\int_0^{2\pi}\int_0^{2\pi} \frac{\left|f\left(e^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i\varphi} - e^{i\theta}\right|^2} d\theta d\varphi = \int_{C_1}\int_{C_1} \frac{\left|f(y) - f(x)\right|^2}{|y-x|^2} dxdy$$

If

$$\Im_r(f) = -\int_0^{2\pi} d\varphi \int_0^{2\pi} \frac{\left|f\left(re^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|re^{i\varphi} - e^{i\theta}\right|^2} re^{i(\theta+\varphi)} d\theta$$

then

$$Lim_{r \overset{<}{\longrightarrow} 1} \int_{C_r} dy \int_{C_1} \frac{\left|f(y) - f(x)\right|^2}{|y-x|^2} dx = \int_{C_1} dy \int_{C_1} \frac{\left|f(y) - f(x)\right|^2}{|y-x|^2} dx$$

i-e

$$Lim_{r \overset{<}{\longrightarrow} 1} \Im_r(f) = \Im(f)$$

we have

$$\int_{C_1} \frac{dx}{(y-x)^2} = 0 \qquad , |y| < 1$$

and

$$\int_{C_r} \frac{dy}{(y-x)^2} = 0 \qquad , |x| = 1$$

also we have

$$\int_{C_r} f(y) \frac{dy}{(y-x)^2} = 0 \qquad , |x| = 1$$

thus proprietes (43),(44),(45) are proved.We have,

$$|f(y) - f(x)|^2 = |f(y)|^2 + |f(x)|^2 - \overline{f(y)} f(x) - \overline{f(x)} f(y)$$

then,in view (43),(44),(45) we get

$$\Im_r(f) = -\int_{C_r} dy \int_{C_1} \overline{f(y)} f(x) \frac{dx}{(y-x)^2} = -K_r(f)$$

By residu's theorem at the pole $x = y$, we have

$$\int_{C_1} f(x) \frac{dx}{(y-x)^2} = 2i\pi f'(y)$$

then if $|y| = r < 1$,we get

$$\Im_r(f) = -K_r(f) = -2i\pi \int_{C_r} \overline{f(y)} f'(y)\, dy \tag{49}$$

Because

$$f\left(re^{i\theta}\right) = \sum_{q=0}^{\infty} a_q r^q e^{iq\theta}$$

then

$$\Im_r(f) = -K_r(f) = -2i\pi \int_{C_r} \overline{\sum_{q=0}^{\infty} a_q r^q e^{iq\theta}} \sum_{p=0}^{\infty} p a_p r^p e^{ip\theta} d\theta$$

$$= 2\pi \int_0^{2\pi} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} p a_p \overline{a_q} r^{p+q} e^{i(p-q)\varphi} d\varphi$$

$$= 4\pi^2 \sum_{p=0}^{\infty} p |a_p|^2 r^{2p}$$

Thus

$$Lim_{r \overset{<}{\longrightarrow} 1} \Im_r(f) = 4\pi^2 \sum_{p=0}^{\infty} p |a_p|^2 \tag{50}$$

By (49),and (50),we get

$$Lim_{r \overset{<}{\longrightarrow} 1} \Im_r(f) = \frac{4\pi}{2i} \int_{C_1} f'(y)\, \overline{f(y)} dy = 4\pi A = \Im(f)$$

i-e

$$A = \frac{1}{4\pi} \Im(f)$$

The proposition is proved. ■

Proposition 9 *Let $z_1, z_2, z_3....z_p$ be n distincts points which are situated in the interior of $D = \{|z| < 1\}$,therefore*

$$\iint_D \left(\sum_{p=1}^{m} \left| \frac{1}{z - z_p} \right|^2 \right) dxdy = \pi \log \left| \prod_{p=1}^{m} \frac{1 - |z_p|^2}{|z_p|^2} \right| \tag{51}$$

and

$$\iint_D \left(\sum_{p=1}^{m} \left| \frac{z + z_p}{z - z_p} \right|^2 \right) dxdy = m\pi + 2\pi \log \left| \prod_{p=1}^{m} \frac{1 - |z_p|^2}{|z_p|^2} \right|^{2|z_p|^2} \tag{52}$$

Proof. If,

$$I_p = \frac{1}{\pi} \iint\limits_D \left| \frac{1}{z - z_p} \right|^2 dxdy \quad , \quad z = x + iy$$

suppose $z = re^{i\theta} = x + iy$

$$dxdy = rdrd\theta, \quad 0 \leq r < 1, \; 0 \leq \theta \leq 2\pi$$

then

$$I_p = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \frac{rdrd\theta}{(re^{i\theta} - z_p)\,(re^{-i\theta} - \overline{z_p})}$$

$$= \frac{1}{\pi} \int_0^1 rdr \int_0^{2\pi} \frac{d\theta}{(re^{i\theta} - z_p)\,(re^{-i\theta} - \overline{z_p})}$$

Setting

$$w = e^{i\theta}, dw = iwd\theta, 0 \leq \theta \leq 2\pi$$

In fact

$$|w| = 1$$

and

$$I_p = \frac{1}{\pi} \int_0^1 rdr \int_{C_1} \frac{\frac{dw}{iw}}{(rw - z_p)\left(\frac{r}{w} - \overline{z_p}\right)}$$

i-e

$$I_p = \frac{1}{\pi i} \int_0^1 dr \int_{C_1} \frac{dw}{\left(w - \frac{z_p}{r}\right)(r - w\overline{z_p})}$$

now we state

$$\frac{1}{\pi i} \int_{C_1} \frac{dw}{\left(w - \frac{z_p}{r}\right)(r - w\overline{z_p})} = \frac{2r}{r^2 - |z_p|^2}$$

hence

$$I_p = \int_0^1 \frac{2r}{r^2 - |z_p|^2} dr = \log \left| \frac{1 - |z_p|^2}{|z_p|^2} \right|$$

then

$$\iint\limits_D \left(\sum_{p=1}^m \left| \frac{1}{z - z_p} \right|^2 \right) dxdy = \pi \log \left| \prod_{p=1}^m \frac{1 - |z_p|^2}{|z_p|^2} \right|$$

To prove (52),setting

$$J_p = \frac{1}{2\pi} \iint\limits_D \left| \frac{z + z_p}{z - z_p} \right|^2 dxdy \quad , \quad z = x + iy$$

suppose $z = re^{i\theta} = x + iy$

$$dxdy = rdrd\theta, \quad 0 \leq r < 1, \ 0 \leq \theta \leq 2\pi$$

then

$$J_p = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{\left(re^{i\theta} + z_p\right)\left(re^{-i\theta} + \overline{z_p}\right)}{\left(re^{i\theta} - z_p\right)\left(re^{-i\theta} - \overline{z_p}\right)} rdrd\theta$$

$$= \frac{1}{2\pi} \int_0^1 rdr \int_0^{2\pi} \frac{\left(re^{i\theta} + z_p\right)\left(re^{-i\theta} + \overline{z_p}\right)}{\left(re^{i\theta} - z_p\right)\left(re^{-i\theta} - \overline{z_p}\right)} d\theta$$

Setting

$$w = e^{i\theta}, dw = iwd\theta, \qquad 0 \leq \theta \leq 2\pi$$

in fact

$$|w| = 1 \, |w| = 1$$

and

$$J_p = \frac{1}{2\pi} \int_0^1 rdr \int_{C_1} \frac{(rw + z_p)(r + w\overline{z_p})}{(rw - z_p)(r - w\overline{z_p})} \frac{dw}{iw}$$

i-e

$$J_p = \frac{1}{2\pi i} \int_0^1 dr \int_{C_1} \frac{(rw + z_p)(r + w\overline{z_p})}{\left(w - \frac{z_p}{r}\right)(r - w\overline{z_p})} \frac{dw}{w} = \frac{2r\left(r^2 + |z_p|^2\right) - r\left(r^2 - |z_p|^2\right)}{r^2 - |z_p|^2}$$

now we state

$$J_p = \int_0^1 r \frac{r^2 + 3|z_p|^2}{r^2 - |z_p|^2} dr = \int_0^1 \left(r + \frac{4r|z_p|^2}{r^2 - |z_p|^2}\right) dr$$

hence

$$J_p = \frac{1}{2} + 2|z_p|^2 \log\left|\frac{1 - |z_p|^2}{|z_p|^2}\right|$$

then

$$\iint_D \left(\sum_{p=1}^m \left|\frac{z + z_p}{z - z_p}\right|^2\right) dxdy = m\pi + 2\pi \log\left|\prod_{p=1}^m \frac{1 - |z_p|^2}{|z_p|^2}\right|^{2|z_p|^2}$$

and the proposition is proved. ■

The third result of paper is applying Green's and Stokes formula,[7],[19],[16],[3],[4],[5],[20]

$$\iint_D f(z)\,\overline{g'(z)}dxdy = \frac{1}{2i} \int_C f(z)\,\overline{g(z)}dz$$

where $z = x + iy$ and the domain D is he ellipse : $b^2x^2 + a^2y^2 < a^2b^2$enclosed by closed curve C in the complex z-plane on orthogonal Tchebychev polynomials of the second kind ,$\{U_n(z)\}_{n=0,1,2,3,...}$ with respect to the Lebesgue planar measure $dm(x,y) = dxdy$ concentrated on the ellipse $D : b^2x^2 + a^2y^2 < a^2b^2$ where $a > b$,$z = x + iy$ given by :

$$U_n(z) = \frac{T'_{n+1}(z)}{n+1} = \frac{\sin\left((n+1)\cos^{-1} z\right)}{\sqrt{1 - z^2}} \quad , \quad n = 0, 1, 2....$$

where $T_n(z) = \cos\left(n\cos^{-1} z\right), n = 0, 1, 2, 3....$is a polynomial of degree n . $T_n(z)$ is called the Tchebychev polynomial of degree n of first kind .

We obtain the following two corollaries of Green's and Stokes formula property.

Corollary 10 *If the ellipse :D : $b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.Let $f(z)$ be regular in a domain D.If*

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z) \tag{53}$$

which converges uniformly and absolutely in any closed subdomain of ellipse D.Then

$$a_{n+1} = \frac{i\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8(n+1)^2} \int\limits_C f(z)\,\overline{U_n(z)}d\overline{z} \quad , \rho = (a+b)^2, \; n = 0, 1, 2, ... \tag{54}$$

where C:$b^2x^2 + a^2y^2 = a^2b^2$.

Proof. The orthogonality of Tchebychev polynomials of the second kind ,$\{U_n(z)\}_{n=0,1,2,3,...}$ with respect to the Lebesgue planar measure $dxdy$ concentrated on the ellipse : D : $b^2x^2 + a^2y^2 < a^2b^2$ where $a > b$,we permits to show that

$$\iint\limits_D U_n(z)\,\overline{U_m(z)}dxdy = \frac{4(n+1)}{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}\delta_{n,m} \qquad ,n, m = 0, 1, 2, 3, ...$$

where $z = x + iy$.The coefficients a_n are the Fourier coefficients of the function $f'(z)$ corresponding to the orthogonal Tchebychev polynomials of second kind on the ellipse D.

$$(n+1)\,a_{n+1} = \frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{4(n+1)} \iint\limits_D f'(z)\,\overline{U_n(z)}dxdy \quad , \; n = 0, 1, 2, 3, ...$$

According to Eq.(9)this may be replaced,

$$a_{n+1} = -\frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8i(n+1)^2} \overline{\int\limits_C \overline{f(z)}U_n(z)\,dz} \quad , z = x + iy \; , n = 0, 1, 2, 3, ...$$

i-e

$$a_{n+1} = -\frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8i(n+1)^2} \int\limits_C f(z)\,\overline{U_n(z)}d\overline{z} \quad , \; n = 0, 1, 2, 3....$$

and the corollary is proved. ■

Corollary 11 *If the ellipse D : $b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.Let $f(z)$ be regular in a domain D.If*

$$\int\limits_a^z f(w)\,dw = \sum_{n=1}^{\infty} b_n T_n(z) \tag{55}$$

which converges uniformly and absolutely in any closed subdomain of ellipse D.Then

$$b_1 = a_0 - \frac{a_2}{2}$$

and

$$b_2 = \frac{a_1 - a_3}{4}$$

with

$$b_n = \frac{a_{n-1-}a_{n+1}}{2n} \qquad ,n = 3, 4, 5, ...$$

where

$$a_{n+1} = \frac{i\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8\left(n+1\right)^2}\int\limits_C f\left(z\right)\overline{U_n\left(z\right)}d\overline{z} \ \ , \ \rho = \left(a+b\right)^2 \qquad , \ n = 0, 1, 2, ... \tag{56}$$

Hence

$$b_n = -\frac{i\pi}{16n}\overline{\int\limits_C \overline{f\left(z\right)}\left(\frac{\rho^n - \rho^{-n}}{n^2}U_{n-1}\left(z\right) - \frac{\rho^{n+2} - \rho^{-n-2}}{\left(n+2\right)^2}U_{n+1}\left(z\right)\right)dz} \qquad , \ n = 0, 1, 2, ... \tag{57}$$

Proof. We have

$$\int T_n\left(z\right)dz = \frac{1}{2\left(n+1\right)}T_{n+1}\left(z\right) - \frac{1}{2\left(n-1\right)}T_{n-1}\left(z\right) + C \tag{58}$$

for $n \in \{0, 1\}$, we treat these cases individually

$$\int T_0\left(z\right)dz = T_1\left(z\right) + C, \quad \text{and} \int T_1\left(z\right)dz = \frac{1}{4}T_2\left(z\right) + C$$

If the ellipse :$D : b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.Let $f\left(z\right)$ be regular in a domain D.If

$$f\left(z\right) = \sum_{n=0}^{\infty} a_n T_n\left(z\right)$$

Becomes

$$\int\limits_a^z f\left(w\right)dw = a_0T_1\left(z\right) + \frac{a_1}{4}T_2\left(z\right) + \sum_{n=2}^{\infty} a_n\left(\frac{1}{2\left(n+1\right)}T_{n+1}\left(z\right) - \frac{1}{2\left(n-1\right)}T_{n-1}\left(z\right)\right)$$

Because the inifinite sum equal

$$\sum_{n=3}^{\infty}\frac{a_{n-1-}a_{n+1}}{2n}T_n\left(z\right) - \frac{a_2}{2}T_1\left(z\right) - \frac{a_3}{4}T_2\left(z\right)$$

yields

$$\int\limits_a^z f\left(w\right)dw = \left(a_0 - \frac{a_2}{2}\right)T_1\left(z\right) + \frac{a_1 - a_3}{4}T_2\left(z\right) + \sum_{n=3}^{\infty}\frac{a_{n-1-}a_{n+1}}{2n}T_n\left(z\right)$$

and the corollary is proved. ■

4 Conclusion

In this article,we demonstrate certain identities involving both the Gronwall's area theorem on the unit disc ,and Green's and Stokes formula on the ellipse.We use Gronwall's area formula to find the area of some differents regions as circles with radius r centred ot the origin lying in the complex w-plane , ellipses and lemniscates A lemniscate shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2, 3, 4.....$$

We use Laurent and Taylor series expansions of conformal mapping from the exterior of the unit disk to either of these regions to compute the area of them.We study also Tchebychev polynomials of second kind wich are of cmplex variable $z = x + iy$ and orthogonal with respect to the Lebesgue planar measure $dm(x, y) = dxdy$ concentrated on the ellipse D : $b^2x^2 + a^2y^2 < a^2b^2$ where $a > b$ .This study brings to light some significant results and defines the relationship between these polynomials and Tchebychev polynomials of first kind.General expressions are found for the primitives of inifinite linear combination of polynomials of Tchebychev polynomials of first kind on the ellipse .

It is worth mentioning here that the above-achieved results and analysis are fruitful. Some of their presumed uses are given below :

• The Gronwall's area theorem on the unit disk and its application are fruitful in in counting methods of area integrals on the unit disk.

• The Green's and Stokes formula on the unit disk and its application are fruitful in in counting methods of area integrals on the unit disk and interpolations.

•The orthgonal Tchebychev polynomials of second kind on the ellipse and their kernel polynomials are fruitful in counting methods of area integrals on the unit disk.

•These orthogonal Tchebychev polynomials of second kind on the ellipse are fruitful in applied to find the minimum value and the minimizing function for various definite area integrals

and solving some areas extremals problems.

•These results strengthen the knowledge of the kernel polynomials associated to the orthogonal Tchebychev polynomials of second kind on the ellipse.The kernel Tchebychev polynomials of second kind on the ellipse $\{U_n(z)\}_{n=0,1,2...}$given by

$$K(z, w) = \sum_{n=0}^{\infty} \frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{4(n+1)} U_n(z)U_n(w)$$

•They are also beneficial in studying problems connected to describe the approximation of continuous functions by kernel orthognal Tchebychev polynomials of second kind on the ellipse .

•They help to study finite linear combinations and defnites summations sequences and calculating general summations and Fourier development.

•These orthogonal polynomials are fruitful in applied to find the interpolation problem,we illustrates that one can use Gaussian quadratures for various definite integrals.

•They help to study and solving area extremal problems and Fourier series .

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References

[1] **G. Szego** , Orthogonal Polynomials, 4th ed., American Mathematical Society Colloquium Publications, Vol.23, Amer. Math. Soc. Providence,RI, 1978.

[2] **LV.Ahlfors,** Complex Analysis ,3 rd ed ,McGraw-Hill ,New York ,1979.

[3] **JH.Curtiss,** Faber polynomials and the Faber series ,Amer .Math .Monthly 78 (1971),pp 577-596.

[4] **WR.Derrick,**Comlpex analysis and Applications ,2nd ed.,Books /Cole Pub Co .,Pacine Grove, CA 1972.

[5] **M. Abramowitz and I.A. Stegun** (Eds.), Handbook of Mathematical Functions, 10th Edition, Dover, New York, 1972.

[6] **I. S. Gradshteyn, I. M. Ryzhik**, Table of Integrals, Series and Products, Seventh edition. Translated from Russian by Scripta Technica, Inc.Academic Press.2007.

[7] **D. Gaier,** Konstruktive Methoden der konformen Abbildung, SpringerVerlag, Berlin, 1964.

[8] ———, Polynomial approximation of conformal maps, Constr. Approx. 14 (1998), 27-40..

[9] **M. V. Keldysh and M. A. Lavrentiev**, On the theory of conformal mappings, Dokl. Akad. Nauk SSSR 1 (1935), 85-87. (Russian)

[10] **M. V. Keldysh and M. A. Lavrentiev**, Sur la representation conforme des domaines limités par des courbes rectifiables, Ann. Sci. Ecole ´ Norm.Sup. 54 (1937), 1-38..

[11] **YA.L.Gueronimus,**Sur des polynômes extrémaux dans l'espace $L^2(\sigma)$ .En russe .Mat .Sbornik.31 (73) 1952 , No 4 , 3-26.

[12] **G. Julia**, Lecons sur la représentation conforme des aires simplement connexes, Paris, 1931.

[13] **P.Koosis,**Introduction to H^p spaces.London Math.Soc.Lecture Notes,Series, Vol. 40, Cambridge University Press, Cambridge, 1980.

[14] **E.I.Pritsker,**On the local asymptotics of Faber polynomials .Proc.Ame.Math.Soc ,127 (10) (1999), 2953-2960.

[15] ————,Convergence of Julia polynomials, J. d'Analyse Math. 94 (2004), 343-361.

[16] **V.J.Smirnov,**Sur la théorie Théorie des polynômes orthogonaux à une variable complexe.Journ.Soc.Phys.Math.de Leningrad,2 (1928) 155-179.M.

[17] **V. J. Smirnov and N. A. Lebedev**, Functions of a Complex Variable:Constructive Theory, MIT Press, Cambridge, 1968.

[18] **V.J. SMIRNOV**, Sur les formules de Cauchy et de Green et quelques problémes qui s'y rattachent, Bulletin de l'académie des sciences de l'U.R.S.S., 1932, 337-372.

[19] **N.Zeev,** Conformal mapping ,Dover publications , INC. New York .1975.

[20] **Abdelhamid Rehouma**, Asymptotic of polar orthogonal polynomials on the unit circle and their reversed polynomials. Algerian Journal of Engineering Architecture-and Urbanism Vol. 5 Nr. 2, 2021.

[21] **Abdelhamid Rehouma and Manuel Malaver de la Fuente** ,Tchebychev Polynomials of SecondKind on the Ellipse andApproximations.Asian Journal of Mathematics and Computer Research, Volume 33, Issue 3, Page 50-65, 2026; Article no.AJOMCOR.14971.

[22] **K. Dilcher, K. B. Stolarsky**, Resultants and Discriminants of Chebyshev and related polynomials, Transactions of the Amer. Math. Soc. 357 (2004), 965-981.