Fixed point theorem utilizing rational-type integral contraction for multivalued mappings in cone $b-metric-like$ space

Abstract: We prove common fixed point theorem for multivalued mappings in a cone $b-metric-like$ space by using a rational-type integral contraction condition. Some sufficient conditions are obtained for the existence of common fixed points of multivalued mappings defined on this space . The obtained results generalize and extend some known results for fixed point theorems in cone $b-metric-like$ space. Some corollaries are obtained as special cases of the main theorem. An example is also given to illustrate the applicability of the obtained results.

Key words: Fixed point, Cone $b-metric-like$ space, Integral contraction, Multivalued Mapping.

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1. INTRODUCTION:

Fixed point theorems in generalized cone metric spaces are equal to those in ordinary metric spaces, according to Du [1]. Specifically, the author demonstrated that the well-known fixed-point theorems of Banach, Kannan, and Chatterjea in topological vector spaces and cone metric may be readily obtained from the standard metric space setup through a straightforward modification, specifically the use of a scalarization function.

As a generalization of $b-metric-like$ spaces, Fernandez et al. [2] introduced cone $b-metric-like$ spaces over Banach Algebra in 2017. They also studied the fixed point of generalized contractions and expansive mapping in the context of cone $b-metric-like$ spaces over Banach algebras with a nonnormal cone.

Reich [3] developed a novel kind of contraction in 1971 that we refer to as Reich contraction. The Banach contraction and the Kannan contraction are the two prominent contractions that it generalizes. The "Banach Contraction Principle" was expanded by Nadler [4] from single-valued to multivalued contraction maps, taking into account the metric defined on bounded and closed subsets of a non-empty set Z. Furthermore, by introducing the concepts of a $b-metric$ space [5], a $b-metric-like$ space [6], a partial metric space [7], a quasi $b-metric$ space [8 and 9], a cone rectangular metric space [10], a new extended $b-metric$ space [11], cone metric spaces [12 and 13], cone $b-metric$ spaces [14], and extended $b-metric$ spaces [15], the fixed point theory is widely accepted by altering the broad structure of underlying space.

Recent results of multivalued maps in extended $b-metric$ space and applications in Volterra-type integral equations have been provided by Dahhouch [16]. For single and multivalued mappings on partially extended $b-metric$ spaces, Ayadi [17] offered some novel fixed point solutions. Rational contractions are introduced in perturbed metric spaces by Alsahli [18]. Mebarki [19] establishes many fixed-point theorems for multivalued mappings and presents a Pompeiu–Hausdorff bipolar $b-metric$ spaces for multivalued contractions.

The purpose of this work is to extend fixed point theory to the more general setting of cone $b-metric-like$ spaces. It develops new common fixed point results for multivalued mappings using rational-type contraction conditions. The results generalize and unify several existing theorems in the literature. Ultimately, the work provides a broader framework for solving nonlinear problems and example arising in analysis.

2. PRELIMINARIES:

In this section, we recall some basic concepts and results which will be used throughout the paper. These notions form the foundation of fixed point theory in generalized metric structures, particularly in cone $b-metric-like$ spaces.

Let Y be a real Banach space and $V \subset Y$ be a subset.

Definition: 2.1. [16 20 and 17 21] A subset V of a real Banach space Y is called a *cone* if it satisfies the following conditions:

(i) V is closed, nonempty and $V \neq \{\vartheta\}$;

(ii) for all $c, d \geq 0$ and $z, x \in V$, we have $cz + dx \in V$;

(iii) $V \cap (-V) = \{\vartheta\}$.

The cone V induces a partial ordering $\preceq$ on Y defined by

$$z \preceq x \iff x - z \in V.$$

We write $z \prec x$ if $z \preceq x$ and $z \neq x$, and $z \ll x$ if $x - z$ belongs to the interior of V.

Definition: 2.2. [20] A cone V is said to be *normal* if there exists a constant $L > 0$ such that

$$\vartheta \preceq z \preceq x \implies \|z\| \leq L\|x\|,$$

for all $z, x \in Y$. The least such L is called the normal constant of V.

The concept of cone metric spaces was later generalized to cone $b-metric$ spaces, and further to cone $b-metric-like$ spaces, which allow more flexibility in handling nonlinear problems.

Definition: 2.3. [22] Let Z be a nonempty set and $S \geq 1$ be a real number. A mapping $d : Z \times Z \to Y$ is called a *cone b-metric* if for all $z, x, w \in Z$, the following conditions hold:

(i) $\vartheta \preceq d(z, x)$ and $d(z, x) = \vartheta \iff z = x$;

(i) $d(z, x) = d(x, z)$;

(i) $d(z, x) \preceq S\big(d(z, w) + d(w, x)\big)$.

Definition: 2.4. [19 23] A mapping $d : Z \times Z \to Y$ is called a *cone b-metric-like* if it satisfies all the conditions of a cone b-metric except that $d(z, z)$ need not be equal to ϑ, that is,

$$d(z,z) \neq \vartheta \quad \text{in general.}$$

The pair (Z,d) is called a *cone* $b - metric - like$ *space*.

This generalization plays an important role in extending fixed point results, since it allows self-distance to be non-zero, which appears naturally in various applications.

Definition: 2.5. [18 22] Let (Z,d) be a cone $b - metric - like$ space. A sequence $\{z_p\}$ for all p $\in$ n in Z is said to be:

(i) *Cauchy* if for every $c \in Y$ with $\vartheta \ll c$, there exists N such that for all $p, m \geq N$,

$$d(z_p, z_m) \ll c;$$

(i) *convergent* to $z \in Z$ if for every $c \in Y$ with $\vartheta \ll c$, there exists N such that for all $n \geq N$,

$$d(z_p, z) \ll c.$$

Definition: 2.6. [20 24] A cone $b - metric - like$ space (Z,d) is said to be *complete* if every Cauchy sequence in Z converges to a point in Z.

In cone b-metric-like spaces, the concept of a fixed point differs slightly from the classical case.

Definition: 2.7. [21 25] Let $G : Z \to Z$ be a mapping. A point $z \in Z$ is called a *metric-like fixed point* of G if

$$d(z, Gz) = d(z,z).$$

For multivalued mappings, we use the following notion.

Definition: 2.8. [22 26] Let $G : Z \to \mathcal{S}_M(Z)$ be a multivalued mapping. A point

$z \in Z$ is said to be a *fixed point* of G if

$$z \in Gz.$$

Throughout this paper, we denote by $\mathcal{S}_M(Z)$ the family of all nonempty subsets of Z. We shall consider mappings whose values are nonempty, closed and bounded subsets of Z.

We also make use of the following class of functions.

Definition: 2.9. [23 27] A function $\varphi : V \to [0, \infty)$ is said to belong to the class Φ if it satisfies:

(i) φ is continuous and non-decreasing;

(ii) $\varphi(w) = 0$ if and only if $w = \vartheta$.

These concepts will be used to establish our main results concerning common fixed points of multivalued mappings under integral-type rational contractions.

3. MAIN RESULT:

Theorem: 3.1. Let (Z, d) be a complete cone $b-metric-like$ space with coefficient $S \geq 1$ over a real Banach space Y with cone V. Assume that:

(i). $d(z, x) = \vartheta \Rightarrow z = x$ for all $z, x \in Z$

(ii). $G, H : Z \to S_M(Z)$ are two multivalued mappings.

(iii). There exists a function $\phi : S_M(Z) \to [0, \infty)$ which is continuous, non-decreasing and satisfies $\phi(w) = 0$ if and only if $w = \vartheta$, and a constant $k \in (0, \frac{1}{s})$ such that

$$\int_0^{d(Gz,Hx)} \phi(w)\,dw \preceq k \int_0^{d(z,x)} \phi(w)\,dw \quad \text{for all } z, x \in Z \tag{1}$$

(iv). For each $z \in Z$, Gz and Hz are nonempty, closed and bounded subsets of Z.

Then G and H have a common fixed point, that is, there exists $q \in Z$ such that

$$q \in Gq \cap Hq.$$

Proof: Let $z_0 \in Z$ be arbitrary. Choose $z_1 \in Gz_0$, $z_2 \in Hz_1$, $z_3 \in Gz_2$, and so on. Thus, we construct a sequence $\{z_p\}$ in Z such that

$$z_{2p+1} \in Gz_{2p}, \quad z_{2p+2} \in Hz_{2p+1}, \quad \text{for all } p \in \mathbb{N}.$$

By the stated contractive condition, we have

$$\int_0^{d(z_{p+1},z_{p+2})} \phi(w)\,dw \preceq k \int_0^{d(z_p,z_{p+1})} \phi(w)\,dw.$$

By the provided integral type contractive condition, for all $z, x \in Z$ we have

$$\int_0^{d(Gz,Hx)} \phi(w)\,dw \preceq k \int_0^{d(z,x)} \phi(w)\,dw. \tag{2}$$

Now, by using the construction of the sequence $\{z_p\}$ so that

$$z_{p+1} \in Gz_p, \quad z_{p+2} \in Hz_{p+1},$$

we obtain

$$d(z_{p+1}, z_{p+2}) \preceq d(Gz_p, Hz_{p+1}).$$

Given that ϕ is non-decreasing, it is evident that

$$\int_0^{d(z_{p+1},z_{p+2})} \phi(w)\,dw \preceq \int_0^{d(Gz_p,Hz_{p+1})} \phi(w)\,dw.$$

By using the contractive condition, we obtain

$$\int_0^{d(z_{p+1},z_{p+2})} \phi(w)\,dw \preceq k \int_0^{d(z_p,z_{p+1})} \phi(w)\,dw. \tag{3}$$

We now go over this inequality again. We have for $p = 0, p = n$

$$\int_0^{d(z_1,z_2)} \phi(w)\,dt \preceq k \int_0^{d(z_0,z_1)} \phi(w)\,dw.$$

For $p = 1$, we obtain

$$\int_0^{d(z_2,z_3)} \phi(w)\,dw \preceq k \int_0^{d(z_1,z_2)} \phi(w)\,dw \preceq k^2 \int_0^{d(z_0,z_1)} \phi(w)\,dw.$$

Proceeding inductively, assume that

$$\int_0^{d(z_p,z_{p+1})} \phi(w)\,dw \preceq k^p \int_0^{d(z_0,z_1)} \phi(w)\,dw.$$

Next, applying the contractive condition once more,

$$\int_0^{d(z_{p+1},z_{p+2})} \phi(w)\,dw \preceq k \int_0^{d(z_p,z_{p+1})} \phi(w)\,dw \preceq k^{p+1} \int_0^{d(z_0,z_1)} \phi(w)\,dw.$$

Thus, by mathematical induction, for all $p \in \mathbb{N}$,

$$\int_0^{d(z_p,z_{p+1})} \phi(w)\,dw \preceq k^p \int_0^{d(z_0,z_1)} \phi(w)\,dw. \tag{4}$$

Since $0 < k < \frac{1}{r} < 1$, we have $k^p \to 0$ as $p \to \infty$. Hence,

$$\int_0^{d(z_p,z_{p+1})} \phi(w)\,dw \to \vartheta.$$

Using the property of ϕ, it follows that

$$d(z_p, z_{p+1}) \to \vartheta.$$

Now, using the cone $b - metric - like$ inequality, for $m > n$ we have

$$d(z_p, z_m) \preceq r\big[d(z_p, z_{p+1}) + d(z_{p+1}, z_{p+2}) + \cdots + d(z_{m-1}, z_m)\big].$$

Using the above estimates and the fact that $k < 1$, we conclude that

$$d(z_p, z_m) \to \vartheta \quad \text{as } p, m \to \infty.$$

Thus, $\{z_p\}$ is a Cauchy sequence in Z.

Since (Z, d) is complete, there exists $u \in X$ such that

$$z_p \to q.$$

Next, we show that $q \in Hq$. From the contractive condition, we have

$$\int_0^{d(z_{2p+1}, Hq)} \phi(w)\, dw \preceq k \int_0^{d(z_{2p}, q)} \phi(w)\, dw.$$

Taking the limit as $p \to \infty$, the right-hand side tends to ϑ, hence

$$d(z_{2p+1}, Hq) \to \vartheta.$$

Since $z_{2p+1} \to q$ and Hq is closed, it follows that $q \in Hq$.

Similarly, we show that $q \in Gq$. Indeed,

$$\int_0^{d(z_{2p+2}, Gq)} \phi(w)\, dw \preceq k \int_0^{d(z_{2p+1}, q)} \phi(w)\, dw.$$

Taking the limit, we get

$$d(z_{2p+2}, Gq) \to \vartheta,$$

which implies that $q \in Gq$.

Therefore,

$$q \in Gq \cap SHq,$$

and hence q is a common fixed point of G and H.

Corollary: 3.2. Let (Z, d) be a complete cone $b-metric-like$ space with coefficient $S \geq 1$ over a real Banach space Y with cone V. Assume that:

(i). $d(z, x) = \vartheta \Rightarrow z = x$ for all $z, x \in Z$;

(ii). $G : Z \to S_M(Z)$ is a multivalued mapping;

(iii). There exists a function $\phi : P \to [0, \infty)$ which is continuous, non-decreasing and satisfies $\phi(w) = 0$ if and only if $w = \vartheta$, and a constant $k \in (0, \frac{1}{s})$ such that

$$\int_0^{d(Gz,Gx)} \phi(w)\, dw \preceq k \int_0^{d(z,x)} \phi(w)\, dw \quad \text{for all } z, x \in Z \tag{5}$$

(iv). For each $z \in Z$, Gz is nonempty, closed and bounded.

Then G has a fixed point, that is, there exists $q \in Z$ such that

$$q \in Gq.$$

Proof: The result follows directly from the main theorem by taking $H = G$. Then all the conditions of the theorem are satisfied, and hence G has a fixed point in Z.

Corollary: 3.3. Let (Z, d) be a complete cone $b-metric-like$ space with coefficient $S \geq 1$. Let $G : Z \to S_M(Z)$ be a multivalued mapping. Suppose there exists

$k \in (0, \frac{1}{s})$ such that

$$d(Gz, Gx) \preceq k\, d(z, x) \quad \text{for all } z, x \in Z.$$

Then G has a fixed point in Z.

Corollary: 3.4. Let (Z, d) be a complete cone $b{-}metric{-}like$ space with coefficient $S \geq 1$. Let $G, H : Z \to S_M(Z)$ be two multivalued mappings such that for some $k \in (0, \frac{1}{s})$,

$$\int_0^{d(Gz,Hx)} w\, dw \preceq k \int_0^{d(z,x)} w\, dw \quad \text{for all } z, x \in Z.$$

Then G and H have a common fixed point in Z.

Proof: Taking $\phi(w) = w$ in the main theorem, all conditions are satisfied. Hence, the result follows immediately.

Corollary: 3.5. Let (Z, d) be a complete cone $b{-}metric{-}like$ space with coefficient $S \geq 1$. Let $G, H : Z \to S_M(Z)$ be two multivalued mappings such that there exists $k \in (0, \frac{1}{s})$ satisfying

$$d(Gz, Hx) \preceq k\, d(z, x) \quad \text{for all } z, x \in Z.$$

Then G and H have a common fixed point in Z.

Proof: The result follows by choosing $\phi(w) = 1$ (or an appropriate constant function) in the integral type contraction and applying the main theorem.

Alternatively, the result can be obtained by considering a suitable constant function ϕ and reducing the integral inequality to the standard contraction form.

Example: 3.6. Let $Z = [0, 1]$ and let $\psi = C^1_{\mathbb{R}}[0, 1]$ be the set of all real-valued continuously differentiable functions on $[0, 1]$. Define the norm on ψ by

$$\|g\| = \|g\|_\infty + \|g'\|_\infty.$$

Let $V = \{g \in \psi : g(w) \geq 0 \text{ for all } w \in [0,1]\}$ be a cone in ψ.

Define $d : Z \times Z \to \psi$ by

$$d(z,x)(w) = |z-x|^m e^w + 1, \quad m > 1.$$

Then (Z,d) is a cone $b-metric-like$ space with coefficient $s = 2^m$, since for all $z, x, y \in Z$,

$$d(z,x)(t) \preceq r\big(d(z,x)(t) + d(x,y)(w)\big).$$

Note that $d(z,z)(w) = 1 \neq \vartheta$, hence (Z,d) is not a cone $b-metric$ space, but it is a cone $b-metric-like$ space.

Now define two multivalued mappings $G, H : Z \to PQ(Z)$ by

$$Gz = \left[0, \frac{z}{4}\right], \quad Hz = \left[0, \frac{z}{6}\right].$$

Clearly, for each $z \in Z$, Gz and Hz are non-empty, closed and bounded subsets of Z.

Now, for $z, x \in Z$ with $z \geq x$, we estimate

$$d(Gz, Hx) \preceq \left|\frac{z}{4} - \frac{x}{6}\right|^p e^w + 1.$$

Since

$$\left|\frac{z}{4} - \frac{x}{6}\right| \leq \frac{1}{4}|z-x|,$$

we obtain

$$d(Gz, Hx) \preceq \frac{1}{4^m}|z-x|^m e^w + 1.$$

Now, choose $\phi(w) = w$ (linear function). Then

$$\int_0^{d(Gz,Hx)} \phi(w)\, dw \preceq \frac{1}{4^m} \int_0^{d(z,x)} \phi(w)\, dw.$$

Let $k = \frac{1}{4^m}$. Since $m > 1$, we have $k < \frac{1}{r}$.

Thus, all the conditions of the main theorem are satisfied. Therefore, G and H have a common fixed point in Z.

In fact, $q = 0$ is a common fixed point since

$$0 \in G0 \cap H0.$$

Conclusion: The mappings G and H have a common fixed point are thoroughly supported by the rigorous step-by-step mathematical proofs provided in Theorem (3.1) and subsequent corollaries. The example (3.6) shows the application of the theorem, supporting the theoretical conclusion and the corollaries show that in some situations the main result can be simplified. These results are generalizations and extensions of several well known theorems of $b-metric$. The theory is more useful in situations where solutions may not be unique since it includes multivalued mappings.

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