Existence of solutions for a class of p-biharmonic equations with Navier boundary conditions

Abstract

This paper studies the existence of solutions for a class of p-biharmonic equations with Navier boundary conditions. Using Morse theory, critical point theory and the G-link theorem, we establish the existence and multiplicity of solutions under the nonquadratic type conditions and the asymptotic noncrossing condition.

Key words: p-biharmonic; G-link Theorem; Navier boundary value conditions; noncrossing condition

1 Introduction and main results

The p-biharmonic equation is a typical class of higher-order nonlinear elliptic partial differential equations. As a nonlinear generalization of the classical biharmonic equation, it has found wide applications in both practical modeling and pure mathematical research. In particular, the p-biharmonic equation with Navier boundary conditions is an important special case, widely employed to model diverse physical phenomena. For instance, Bernis [4] pointed out that such equations can accurately model mechanical behaviors in beam vibration theory. Lazer and McKenna [18] further verified that they provide a mathematical framework for the study of traveling wave propagation in suspension bridges. Therefore, a systematic investigation of the existence of solutions to p-biharmonic equations with Navier boundary conditions is not only of great academic value for improving the theoretical system of nonlinear elliptic PDEs, but also provides key support for applications in related engineering and physical fields.

In the past few decades, the study of the nonlinear fourth order differential equations has been an interesting topic which appears in various branches of mathematical physics (see [2,5,7,9,16,17,21,33,34] and references therein). Many authors have widely developed various methods and techniques, such as critical point theory, Morse theory to look for multiple solutions of elliptic equations involving p-biharmonic type operators (see [6, 8, 11, 12, 15, 19, 20, 25, 26, 28, 29, 32] and references therein). In particular, fruitful advances have been achieved during the recent decade from diverse research perspectives. Bueno, Paes-Leme, and Rodrigues [11] investigated p-biharmonic equations with Navier boundary conditions under critical growth. When the parameter range is sufficiently small, they established the existence of multiple positive solutions via variational methods. Bonanno, Chinnì and O'Regan [8] adopted variational critical point theory and proved the existence of at least two nontrivial weak solutions for constant exponent p-biharmonic equations. Going further to nonlocal framework, Lebrimchi, Talbi and Massar [19] investigated a $p(x)$-biharmonic Kirchhoff problem with Navier boundary conditions and determined the admissible parameter range ensuring the existence of nontrivial solutions. Safari and Razani [28] extended the research from Euclidean domains to the Heisenberg group and obtained radial positive solutions for weighted p-biharmonic equations.

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Motivated by these considerations, in this paper, we consider the following problem

$$\begin{cases}\Delta_p^2 u = f(x,u) & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega,\end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2p+1)$ with smooth boundary $\partial\Omega$, $p > 1$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ denotes the p-biharmonic operator, $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the following assumption:

(I_1) There exist $C > 0$ and $q \in (p, p_2^*)$ such that

$$|f(x,t)| \leq C\big(1 + |t|^{q-1}\big)$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where p_2^* is the critical Sobolev exponent defined as

$$p_2^* = \frac{Np}{N-2p}.$$

Let $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ be the Sobolev space with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

According to the Sobolev embedding theorem (see [1]), for all $1 \leq \gamma \leq p_2^*$, the embedding $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ is continuous, if $1 \leq \gamma < p_2^*$, the embedding $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ is compact. So there exists a positive constant C_0 such that

$$\|u\|_q \leq C_0\|u\|, \tag{2}$$

where $\|u\|_\gamma = \left(\int_\Omega |u|^\gamma \mathrm{d}x\right)^{\frac{1}{\gamma}}$ is the usual norm in $L^\gamma(\Omega)$. In what follows, we first recall some concepts related to the eigenvalues of $\Delta_p^2 u$ (see [26]). Consider the following manifold:

$$\Sigma = \left\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\colon \int_\Omega |u|^p \mathrm{d}x = 1\right\},$$

then Σ is a symmetric nonempty C^1- manifold in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Let

$$\Sigma_k = \big\{\Lambda \subset \Sigma\colon \text{there exists a continuous, odd and surjective } h\colon S^{k-1} \to \Lambda\big\},$$

where S^{k-1} denotes the unit sphere in $\mathbb{R}^k$. We say λ is an eigenvalue of the nonlinear eigenvalue problem if

$$\begin{cases}\Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{p-2}u & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega,\end{cases} \tag{3}$$

has nontrivial solutions. By the Ljusternik-Schnirelman theory on C^1- manifold (see [31]), we know that $\{\lambda_k\}_{k\in\mathbb{N}}$ is the eigenvalues sequence of (3), with the variational characterization:

$$\lambda_k = \inf_{\Lambda\in\Sigma_k}\sup_{u\in\Lambda}\int_\Omega |\Delta u|^p \mathrm{d}x.$$

In [15], Drábek and Ôtani proved the first eigenvalue λ_1 of (3) is simple, positive and isolated. Then, Liu and Squassina [26] divided the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into two complementary subspaces, i.e., $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = V \oplus W$, where $V = \operatorname{span}\{\phi_1\}$ is the eigenspace associated with λ_1 and W be a subspace of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ complementing V. By using Morse Theory and critical point theory, they studied the existence and multiplicity of nontrivial solutions for problem (1) and proved the following results.

Theorem A. . *Assume that* (I_1) *holds,* $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$ *satisfies the following conditions:*

(I_2) There exist $\delta > 0$ *and* $\bar{\lambda} \in \left]\lambda_1, \hat{\lambda}\right[$ *such that*

$$\lambda_1|t|^p \leq pF(x,t) \leq \bar{\lambda}|t|^p$$

for $x \in \Omega$ *and* $|t| \leq \delta$*, where* $\hat{\lambda} > \lambda_1$ *with* $\|u\|^p \geq \hat{\lambda}\|u\|_p^p$ *for any* $u \in W$.

(I_3) $\lim\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} < \lambda_1$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits two nontrivial solutions in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Theorem B. . *Assume that* (I_1) *and* (I_2) *hold,* $F(x,t)$ *satisfies the following conditions:*

(I_4) $\lim\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} = \lambda_1$ *uniformly for* $x \in \Omega$.

(I_5) $\lim\limits_{|t|\to\infty} \big(f(x,t)t - pF(x,t)\big) = +\infty$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits two nontrivial solutions in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Despite these fruitful achievements, most existing results impose relatively restrictive hypotheses on nonlinearities and weight functions, and few works discuss the asymptotic noncrossing eigenvalue conditions adopted in the present paper. Motivated by [26] and these unresolved problems, we devoted ourselves to studying a p-biharmonic equation with Navier boundary conditions. In this paper, we choose Morse theory (see [13, 27]) and G-link theorem (see [30]) as key tools. Compared with the classical techniques used in Theorem A and Theorem B from existing references, these tools allow us to remove the strict limitations related to λ_1 and establish the existence of weak solutions under the case that $F(x,t)$ lies between two consecutive higher eigenvalues $\lambda_k < \lambda_{k+1} (k \geq 1)$ of (3). Our main results are the following theorems.

Theorem 1.1. *Assume that* (I_1), (I_2) *and* (I_4) *hold,* $F(x,t)$ *satisfies the following condition:*

(I_5') $\lim\limits_{|t|\to\infty} \big(F(x,t) - \frac{1}{p}\lambda_1|t|^p\big) = -\infty$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits two nontrivial solutions in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Remark 1.2. *In fact, conditions* (I_4) *and* (I_5) *imply condition* (I_5') *by direct calculation. Hence, Theorem 1.1 can be regarded as a generalization of Theorem B.*

Theorem 1.3. *Assume that* $\lambda_k < \lambda_{k+1}$ $(k \geq 1)$ *are two consecutive eigenvalues of* (3)*,* (I_1) *and* (I_5) *hold,* $F(x,t)$ *satisfies the following condition:*

(I_6) $\lambda_k < \liminf\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} \leq \limsup\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} \leq \lambda_{k+1}$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits at least one weak solution in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Theorem 1.4. *Assume that* $\lambda_k < \lambda_{k+1}$ $(k \geq 1)$ *are two consecutive eigenvalues of* (3) *and* (I_1) *holds,* $F(x,t)$ *satisfies the following conditions:*

(I_7) $\lim\limits_{|t|\to\infty} \big(f(x,t)t - pF(x,t)\big) = -\infty$ *uniformly for* $x \in \Omega$.

(I_8) $\lambda_k \leq \liminf\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} \leq \limsup\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} < \lambda_{k+1}$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits at least one weak solution in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Remark 1.5. *We note that the nonquadratic type conditions such as* (I_5) *(or* (I_7)*) and the asymptotic noncrossing conditions such as* (I_6) *(or* (I_8)*) were originally introduced by Costa and* Magalhães *[14] for studying a class of semilinear elliptic equations. To the best of our knowledge, it is the first time that these assumptions are shown in p-biharmonic equations with Navier boundary value conditions and such relevant research content cannot be found in the related results established by Liu and Squassina. Finally, this is a theoretical study in pure mathematics without involving numerical or experimental data. The results presented in this paper can be further extended to more general fourth-order elliptic equations in future investigations.*

2 Proof of Main Results

Firstly, we define the energy functional Φ associated with problem (1) as

$$\Phi(u) = \frac{1}{p} \int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x$$

for every $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. It is well known that Φ is continuously differentiable and

$$\langle \Phi'(u), v \rangle = \int_\Omega |\Delta u|^{p-2} \Delta u \Delta v \mathrm{d}x - \int_\Omega f(x,u) v \mathrm{d}x$$

for every $u, v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Therefore, the weak solutions of problem (1) correspond to the critical points of Φ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [26, 32]).

Next, we recall some useful concepts about Morse Theory (see [13, 27]) and some abstract critical point theorems (see [22, 30]). Consider $\Phi\colon X \to \mathbb{R}$, a C^1-functional defined in a Banach space $(X, \|\cdot\|)$. Let $\mathcal{K} = \{u \in X\colon \Phi'(u) = 0\}$ be the critical set of Φ, and $\Phi^c = \{u \in X\colon \Phi(u) \le c\}$ denotes the sublevel sets of Φ. The k-th critical group of Φ at an isolated critical point $u \in \mathcal{K}$ is described by

$$C_k(\Phi, u) = H_k(\Phi^c \cap \mathcal{U}, \Phi^c \cap \mathcal{U} \setminus \{u\}), \quad k \in \mathbb{Z},$$

where $\mathcal{U}$ is a neighborhood of u containing no other critical points, and $H_k(A, B)$ denotes the k-th singular relative homology group of the topological pair (A, B) with the coefficients in a field $\mathbb{F}$. For any $a < \Phi(\mathcal{K})$, the k-th critical group of Φ at infinity is defined by

$$C_k(\Phi, \infty) = H_k(X, \Phi^a), \quad k \in \mathbb{Z}.$$

Definition 2.1. *[23]. Let* X *be a real Banach space such that* $X = V \oplus W$, *where* V *and* W *are closed subspaces of* X. *Let* $\Phi : X \to \mathbb{R}$ *be a* C^1*-functional. We say that* Φ *has a local linking near the origin 0 (with respect to the decomposition* $X = V \oplus W$*), if there exists* $\rho > 0$ *such that*

$$\begin{aligned} &u \in V : \|u\| \le \rho \Longrightarrow \Phi(u) \le 0, \\ &u \in W : 0 < \|u\| \le \rho \Longrightarrow \Phi(u) > 0. \end{aligned}$$

Theorem 2.2. *[24, Theorem 2.1]. Let* X *be a real Banach space and let* $\Phi \in C^1(X, \mathbb{R})$ *be bounded from below and satisfying the (PS) condition. Assume that* Φ *has a critical point* u *which is homologically nontrivial, that is,* $C_k(\Phi, u) \ncong \{0\}$ *for some* k, *and it is not a minimizer for* Φ. *Then* Φ *admits at least three critical points.*

Definition 2.3. *[30, Definition 2.1]. Let Q be a submanifold of a Banach space X with relative boundary ∂Q, S be a closed subset of a Banach space Y and G be a subset of $C^0(\partial Q, Y \setminus S)$. We say S and ∂Q are G-link if for any map $h \in C^0(Q, Y \setminus S)$ such that $h|_{\partial Q} \in G$ there holds $h(Q) \cap S \neq \emptyset$.*

Theorem 2.4. *[30, G-link Theorem]. Suppose that X, Y are Banach spaces. Consider a closed subset $S \subset Y$ and a submanifold $Q \subset X$ with relative boundary ∂Q, G is a subset of $C^0(\partial Q, Y \setminus S)$. Set $\Gamma = \{h \in C^0(Q, Y) : h|_{\partial Q} \in G\}$. Suppose S and ∂Q are G-link and $\Phi \in C^1(Y, \mathbb{R})$ satisfies*

(a) There exists $h_0 \in \Gamma$ and a positive constant r such that

$$\sup_{x\in Q} \Phi(h_0(x)) < r;$$

(b) There exist $\beta > \alpha$ such that

$$\inf_{y\in S} \Phi(y) \geq \beta \quad \text{and} \quad \sup_{x\in \partial Q} \Phi(h(x)) \leq \alpha, \quad \forall h \in \Gamma;$$

(c) Φ satisfies the (PS) condition. Then, the number

$$c := \inf_{h\in\Gamma} \sup_{x\in Q} \Phi(h(x))$$

defines a critical values $c \geq \beta$ of Φ.

Remark 2.5. *If the (PS) condition is replaced by the weaker (C) condition (see [3]), the deformation lemma (see [27]) can be proved. Hence, it is still sufficient to ensure that the above abstract critical point theorems hold.*

Lemma 2.6. *Assume that (I_1) and (I_5') hold, then Φ is coercive on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, that is, $\Phi(u) \to +\infty$ as $\|u\| \to \infty$.*

Proof. By contradiction, let $\{u_n\} \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ be such that

$$\begin{aligned} &\|u_n\| \to \infty \quad \text{as } n \to \infty, \quad \text{and} \\ &\Phi(u_n) \leq M_1, \end{aligned} \tag{4}$$

for some $M_1 \in \mathbb{R}$. Let $v_n := \frac{u_n}{\|u_n\|}$, then we can assume that there exists $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, up to a subsequence still denoted by $\{v_n\}$, such that

$$\begin{aligned} &v_n \rightharpoonup v \text{ weakly in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ &v_n \to v \text{ strongly in } L^\gamma(\Omega), \quad \gamma \in \big[1, p_2^*\big). \end{aligned} \tag{5}$$

From (4), it follows that there is $M_2 > 0$ such that

$$\begin{aligned} \frac{M_1}{\|u_n\|^p} &\geq \frac{\Phi(u_n)}{\|u_n\|^p} \\ &= \frac{1}{p}\int_\Omega |\Delta v_n|^p \mathrm{d}x - \frac{1}{\|u_n\|^p}\int_\Omega F(x, u_n)\mathrm{d}x \\ &= \frac{1}{p}\big(\|v_n\|^p - \lambda_1\|v_n\|_p^p\big) - \frac{1}{\|u_n\|^p}\int_\Omega \Big(F(x, u_n) - \frac{\lambda_1}{p}|u_n|^p\Big)\mathrm{d}x \\ &\geq \frac{1}{p}\big(\|v_n\|^p - \lambda_1\|v_n\|_p^p\big) - \frac{M_2}{\|u_n\|^p}. \end{aligned} \tag{6}$$

Therefore, the combination of (4), (5) and (6) implies that

$$\limsup_{n\to\infty} \|v_n\|^p \le \lambda_1 \|v\|_p^p. \tag{7}$$

From the lower semi-continuity of the norm and the Poincaré inequality, it follows that

$$\begin{aligned} \lambda_1\|v\|_p^p \le \|v\|^p &\le \liminf_{n\to\infty} \|v_n\|^p \\ &\le \limsup_{n\to\infty} \|v_n\|^p. \end{aligned} \tag{8}$$

Combining (7) and (8), we have $\|v\|^p = \lambda_1\|v\|_p^p$ and $v_n \to v$ in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ with $\|v\| = 1$. Thus, we have $v = \pm\phi_1$, and we take $v = \phi_1$. It follows from the definition of v_n and (I_5') that

$$u_n(x) \to +\infty \quad \text{uniformly for } x \in \Omega,$$

which implies that

$$M_1 \ge -\int_\Omega \Big(F(x,u_n) - \frac{\lambda_1}{p}|u_n|^p\Big)\mathrm{d}x \to +\infty \quad \text{as } n\to\infty,$$

a contradiction with (4). So Φ is coercive on $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. The proof of Lemma 2.6 is completed. □

Lemma 2.7. *Suppose that (I_1), (I_5) and (I_6) (or (I_7) and (I_8)) hold, then the functional Φ satisfies the (C) condition in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.*

Proof. Here, we only prove this lemma under conditions (I_5) and (I_6), because the proof under conditions (I_7) and (I_8) is similar.

Firstly, we prove that Φ satisfies the (C) condition, that is, for any $c \in \mathbb{R}$ and every sequence $\{u_n\} \subset W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that

$$\Phi(u_n) \to c \text{ and } \|\Phi'(u_n)\|(1+\|u_n\|) \to 0 \quad \text{as } n\to\infty \tag{9}$$

has a convergent subsequence. If $\{u_n\}$ is unbounded, without loss of generality, we can suppose that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from (9) that there exists a constant M_3 such that

$$\begin{aligned} M_3 &\ge \liminf_{n\to\infty} \big(p\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle\big) \\ &= \liminf_{n\to\infty} \int_\Omega \big(f(x,u_n)u_n - pF(x,u_n)\big)\mathrm{d}x. \end{aligned} \tag{10}$$

For any $\varepsilon_1 > 0$, from (I_6), there is a positive constant $M_4 = M_4(\varepsilon_1)$ such that

$$\big|F(x,t)\big| \le \frac{\lambda_{k+1}+\varepsilon_1}{p}|t|^p + M_4 \tag{11}$$

for all $(x,t) \in \Omega\times\mathbb{R}$. Combining (9) and (11), for n large enough, we have

$$\begin{aligned} c+1 &\ge \Phi(u_n) \\ &= \frac{1}{p}\int_\Omega |\Delta u_n|^p \mathrm{d}x - \int_\Omega F(x,u_n)\mathrm{d}x \\ &\ge \frac{1}{p}\int_\Omega |\Delta u_n|^p \mathrm{d}x - \Big(\frac{\lambda_{k+1}+\varepsilon_1}{p}\Big)\int_\Omega |u_n|^p\mathrm{d}x - M_4|\Omega| \\ &\ge \frac{1}{p}\|u_n\|^p - \Big(\frac{\lambda_{k+1}+\varepsilon_1}{p}\Big)\|u_n\|_p^p - M_4|\Omega|, \end{aligned} \tag{12}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Now, define $z_n := \frac{u_n}{\|u_n\|}$, then we can also assume, up to a subsequence, that there exists $z \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $z_n \rightharpoonup z$ weakly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $z_n \to z$ strongly in $L^\gamma(\Omega)$, $\gamma \in \big[1, p_2^*\big)$. Particularly, dividing by $\|u_n\|^p$ and letting $n \to \infty$ in (12), we obtain

$$1 \leq \big(\lambda_{k+1} + \varepsilon_1\big)\|z_n\|_p^p,$$

which implies that there exists $\widetilde{\Omega} \subset \Omega$ with $meas(\widetilde{\Omega}) > 0$ satisfying $z(x) \neq 0$ for $x \in \widetilde{\Omega}$. Therefore, for $x \in \widetilde{\Omega}$, we have

$$|u(x)| \to \infty \quad \text{as} \quad n \to \infty.$$

In addition, by Fatou's lemma and (I_5), it follows that there is a constant M_5 such that

$$\begin{aligned}
\liminf_{n\to\infty} \int_\Omega \big(f(x,u_n)u_n - pF(x,u_n)\big)\mathrm{d}x = & \liminf_{n\to\infty} \int_{\widetilde{\Omega}} \big(f(x,u_n)u_n - pF(x,u_n)\big)\mathrm{d}x \\
& + \liminf_{n\to\infty} \int_{\Omega\setminus\widetilde{\Omega}} \big(f(x,u_n)u_n - pF(x,u_n)\big)\mathrm{d}x \\
\geq & \int_{\widetilde{\Omega}} \liminf_{n\to\infty} \big(f(x,u_n)u_n - pF(x,u_n)\big)\mathrm{d}x \\
& + M_5|\Omega \setminus \widetilde{\Omega}| \\
\to & +\infty,
\end{aligned}$$

we obtain a contradiction with (11). Hence, we conclude that $\{u_n\}$ is bounded in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, i.e., there exists a constant $M_6 > 0$ such that

$$\|u_n\| \leq M_6 \tag{13}$$

for all $n \in \mathbb{N}$. Therefore, there is a $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
& u_n \rightharpoonup u \text{ weakly in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\
& u_n \to u \text{ strongly in } L^\gamma(\Omega), \quad \gamma \in \big[1, p_2^*\big).
\end{aligned} \tag{14}$$

It follows from (I_1), Hölder's inequality and (2) that

$$\begin{aligned}
\left|\int_\Omega f(x,u_n)(u_n - u)\mathrm{d}x\right| \leq & \int_\Omega \big|f(x,u_n)\big|\big|u_n - u\big|\mathrm{d}x \\
\leq & C\int_\Omega \big(1 + |u_n|^{q-1}\big)\big|u_n - u\big|\mathrm{d}x \\
\leq & C\bigg(\int_\Omega |u_n|^{(q-1)\frac{q}{q-1}}\mathrm{d}x\bigg)^{\frac{q-1}{q}}\bigg(\int_\Omega |u_n - u|^q\mathrm{d}x\bigg)^{\frac{1}{q}} \\
& + C\|u_n - u\|_1 \\
= & C\|u_n\|_q^{q-1}\|u_n - u\|_q + C\|u_n - u\|_1 \\
\leq & CC_0^{q-1}\|u_n\|^{q-1}\|u_n - u\|_q + C\|u_n - u\|_1.
\end{aligned}$$

By (13) and (14), we have

$$\lim_{n\to\infty}\left|\int_\Omega f(x,u_n)(u_n - u)\mathrm{d}x\right| = 0. \tag{15}$$

In addition, from (9), it holds that

$$\begin{aligned}\left|\left\langle \Phi'(u_n), u_n - u\right\rangle\right| &\le \|\Phi'(u_n)\|\|u_n - u\| \\ &\le \|\Phi'(u_n)\|\big(M_6 + \|u\|\big) \\ &\to 0\end{aligned} \tag{16}$$

as $n \to \infty$. By (15) and (16), one has

$$\begin{aligned}\left\langle \Phi'(u_n), u_n - u\right\rangle - \int_\Omega f(x,u_n)(u_n - u)\mathrm{d}x &= \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta\big(u_n - u\big)\mathrm{d}x \\ &\to 0\end{aligned} \tag{17}$$

as $n \to \infty$. From (15), it follows that

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\big(u_n - u\big)\mathrm{d}x \to 0 \tag{18}$$

as $n \to \infty$. Combining (17) and (18), we obtain that

$$\int_\Omega \Big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\Big)\Delta\big(u_n - u\big)\mathrm{d}x \to 0$$

as $n \to \infty$. Based on Clarkson's inequality [10], that is, there exists a positive constant L_p such that

$$|x-y|^p \le \begin{cases} L_p\big(|x|^{p-2}x - |y|^{p-2}y\big)\big(x-y\big), & p \ge 2, \\ L_p\Big(\big(|x|^{p-2}x - |y|^{p-2}y\big)\big(x-y\big)\Big)^{\frac{p}{2}}\big(|x|+|y|\big)^{\frac{(2-p)p}{2}}, & 1 < p < 2, \end{cases}$$

for all $x, y \in \mathbb{R}$. So, we have

$$\int_\Omega \big|\Delta u_n - \Delta u\big|^p \mathrm{d}x \to 0$$

as $n \to \infty$, which implies that $u_n \to u$ strongly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and the proof is completed. □

Lemma 2.8. *Assume that F satisfies assumptions (I_1), (I_5) and (I_6) , then the functional Φ satisfies the conditions (a) and (b) in Theorem 2.4.*

Proof. Define a symmetric closed subset of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ by

$$\mathcal{C}_{k+1} = \Big\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\colon \int_\Omega |\Delta u|^p \mathrm{d}x \ge \lambda_{k+1}\int_\Omega |u|^p \mathrm{d}x\Big\}.$$

For any $\varepsilon_2 > 0$, it follows from (I_1) and (I_6) that there is a positive constant $M_7 = M_7(\varepsilon_2)$ such that

$$F(x,t) \ge \frac{\lambda_k + 2\varepsilon_2}{p}|t|^p - M_7 \tag{19}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. By the definition of λ_k, there is $\Lambda \in \Sigma_k$ such that

$$\sup_{u \in \Lambda}\int_\Omega |\Delta u|^p \mathrm{d}x \le \lambda_k + \varepsilon_2. \tag{20}$$

For any $u \in \Lambda$ and $s > 0$, from (19) and (20), it holds that

$$\begin{aligned}\Phi(su) &\leq \frac{1}{p}\|su\|^p - \frac{\lambda_k + 2\varepsilon_2}{p}\|su\|_p^p + M_7|\Omega| \\ &\leq -\frac{\varepsilon_2}{p}s^p + M_7|\Omega|.\end{aligned} \tag{21}$$

Moreover, let $H(x,t) = F(x,t) - \frac{\lambda_{k+1}}{p}|t|^p$, we have

$$H'(x,t)t - pH(x,t) = f(x,t)t - pF(x,t).$$

It follows from (I_5) that for any $L > 0$, there exists a constant $M_8 = M_8(L) > 0$ such that

$$H'(x,t)t - pH(x,t) \geq L \tag{22}$$

for $x \in \Omega$ and $|t| > M_8$. Hence, as $t > M_8$, from (22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{H(x,t)}{t^p}\bigg) = \frac{H'(x,t)t - pH(x,t)}{t^{p+1}} \geq \frac{L}{t^{p+1}} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg(-\frac{L}{pt^p}\bigg). \tag{23}$$

Integrating (23) over the interval $[t,s] \subset [M_8, \infty)$, we obtain

$$\frac{H(x,t)}{t^p} \leq \frac{H(x,s)}{s^p} + \frac{L}{p}\bigg(\frac{1}{s^p} - \frac{1}{t^p}\bigg). \tag{24}$$

By (I_6), we have

$$\limsup_{s \to +\infty} \frac{H(x,s)}{s^p} \leq 0.$$

Letting $s \to +\infty$ in (24), we get

$$H(x,t) \leq -\frac{L}{p}$$

for $x \in \Omega$ and $t \geq M_8$. Similarly, we have $H(x,t) \leq -\frac{L}{p}$ for $x \in \Omega$ and $t \leq -M_8$. So, by the arbitrariness of L, we deduce that

$$\lim_{|t| \to \infty} H(x,t) = -\infty \quad \text{uniformly for } x \in \Omega. \tag{25}$$

Hence, it follows from (I_1) that there exists $M_9 > 0$ such that

$$H(x,t) < M_9 \tag{26}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Therefore, for any $u \in \mathcal{C}_{k+1}$, by (26), one has

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)\mathrm{d}x \\ &= \frac{1}{p}\big(\|u\|^p - \lambda_{k+1}\|u\|_p^p\big) - \int_\Omega H(x,u)\mathrm{d}x \\ &\geq -\int_\Omega H(x,u)\mathrm{d}x \\ &\geq -M_9|\Omega|.\end{aligned} \tag{27}$$

Therefore, for fixed $\Lambda \in \Sigma_k$, the combination of (21) and (27) implies that there exists $R > 0$ such that

$$\alpha := \max_{u \in \Lambda, s \geq R} \Phi(su) < -M_9|\Omega| =: \beta. \tag{28}$$

Moreover, we note $R\Lambda = \{su\colon u \in \Lambda, s \geq R\}$ and $G = \{h \in C(S^{k-1}, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))\colon h$ is odd and $h(S^{k-1}) \subset R\Lambda\}$, where $S^{k-1} = \partial B^k$ denotes the boundary of the closed unit ball B^k in $\mathbb{R}^k$. From (28), it follows that for any $h \in G$, we obtain

$$h\big(S^{k-1}\big) \cap \mathcal{C}_{k+1} = \emptyset,$$

which limplies that $G \subset C(S^{k-1}, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \mathcal{C}_{k+1})$.

Now, we claim that let

$$\Gamma = \{h \in C(B^k, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))\colon\ h|_{S^{k-1}} \in G\},$$

then Γ is nonempty, and

$$h\big(B^k\big) \cap \mathcal{C}_{k+1} \neq \emptyset$$

for all $h \in \Gamma$. If this claim is true, it means that $\mathcal{C}_{k+1}$ and S^{k-1} are G-link. On the other hand, from (28) and the compactness of B^k, it follows that (a) and (b) of the Theorem 2.4 is satisfied.

According to the definition of Σ_k, there exists a continuous odd surjection $h\colon S^{k-1} \to \Lambda$. Now, defining $T\colon B^k \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ as

$$T(su) = sRh(u) \qquad \text{for all } (u, s) \in \big(S^{k-1} \times [0, 1]\big).$$

Therefore, $T \in \Gamma$, which implies that Γ is nonempty. Moreover, taking $h \in \Gamma$, if there is a $u \in h(B^k)$ with $\|u\|_p = 0$, we have $h\big(B^k\big) \cap \mathcal{C}_{k+1} \neq \emptyset$. Otherwise we define $\eta\colon S^k \to \Sigma$ as

$$\eta(x_1, x_2, ..., x_{k+1}) = \begin{cases} \pi \circ h(x_1, x_2, ..., x_k) & \text{if } \ x_{k+1} \geq 0, \\ -\pi \circ h(-x_1, -x_2, ..., -x_k) & \text{if } \ x_{k+1} < 0, \end{cases}$$

where $\pi(u) = \frac{u}{\|u\|_p^p}$. Using the definition of η, it is easy to see that $\eta(S^k) \in \Sigma_{k+1}$. Thus, there is a $u_0 \in \eta(S^k)$ such that $\|u_0\|^p \geq \lambda_{k+1}$, that is, $u_0 \in \mathcal{C}_{k+1}$. We can conclude that $\pi \circ h(x) \in \mathcal{C}_{k+1}$, which means that $h(x) \in \mathcal{C}_{k+1}$, i.e., $h\big(B^k\big) \cap \mathcal{C}_{k+1} \neq \emptyset$. Now the proof of Lemma 2.8 is completed. □

Proof of Theorem 1.1. By Lemma 2.6, we obtain the coercivity of Φ on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and it satisfies (PS) condition. As Lemma 2.2 in [26], we can get that Φ has a local linking near the origin 0 (with respect to the decomposition $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = V \oplus W$) under (I_1) and (I_2). By Theorem 2.1 in [22], we get that 0 is a critical point of Φ and $C_1(\Phi, 0) \ncong \{0\}$. Combined with Theorem 2.2, we can finally deduce the existence of two nontrivial critical points for Φ within $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

Proof of Theorem 1.3. By Lemma 2.7, it is clear that Φ satisfies (C) condition in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Combined Lemma 2.8 and Theorem 2.4, then functional Φ has a critical point in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

Proof of Theorem 1.4. By Lemma 2.7, it is clear that Φ satisfies (C) condition in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Using a method similar to that in Lemma 2.8, we can obtain the same result under conditions (I_7) and (I_8). Therefore, Φ satisfies all assumptions in Theorem 2.4. Then Φ has a critical point in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

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