A Novel 4D Nonlinear Interaction Football Model: Stability Analysis

Abstract

A novel nonlinear mathematical model is formulated to study the effects of defense, midfield, and attack on an overall football team performance. Bilinear inhibitory effects are considered for match-based factors such as defensive breakdown under attacking pressure, midfield congestion, and failure to convert attacking opportunities into match outcomes. Qualitative analysis is carried out to confirm positivity, boundedness, and well-posedness of solutions. Equilibrium points corresponding to stable tactical configurations are obtained and analyzed. Local stability analysis is carried out through analysis of the Jacobian, while global stability is established with the use of a Lyapunov function and LaSalle's Invariance Principle. Analytical results indicate that nonlinear inhibitory factors are a major force in maintaining control over football tactics, as they ensure that none of the considered aspects are able to grow unbounded. In fact, it is demonstrated that synergy between effective defense, midfield, and attack leads to more sustainable tactics and higher rates of team performance.

Keywords: Dynamical Systems, Nonlinear Interaction Model, Global and Local Stability Analysis, Lyapunov Functions, Sport Modeling, Football, Soccer.

1 Introduction

Mathematical modeling has become an important scientific tool that helps in the analysis and understanding of complex engineering, economic, biological, ecological, and environmental systems [1, 2]. Over the years, the use of mathematical modeling to describe the behavior of these systems has continued to grow. Those in economics have used it to analyze inflation, interaction oil price dynamics and so on [1, 5, 3, 7, 6, 4, 2]. One of the earliest and most recognized uses comes from epidemiological analysis. Disease modeling has been a major use of mathematical modeling. This is evident in the modelling of HIV, COVID, Hepatits B, Lassa fever and others [8, 12, 4, 3]. Mathematical modeling is efficient in capturing systems involving multiple variables that evolve simultaneously. This is perfectly done by nonlinear interaction models [17], which have gained significant relevance over classical linear models for their ability to represent realistic system behaviors. This attribute is sometimes lacking in the classical linear models.

Football is a team sport that involves players passing the ball around between themselves until a goalscoring opportunity is presented, and this study has developed and analyzed a four-dimensional nonlinear model that consists of defensive, midfield, attacking, and goal variables. Specifically, $D(t)$ represents the defensive, $M(t)$ represents midfield coordination, $A(t)$ represents attacking intensity, and $W(t)$ represents the overall system output (performance index) [9, 10]. This allows for the modeling of a relation between the components.

The model incorporates constant input rates, natural decay terms, and interaction coefficients that describe reinforcement between adjacent subsystems. In addition, nonlinear inhibitory terms are introduced through bilinear interactions such as DA, MA, and AW, which capture saturation effects and diminishing returns under high system load. Congestion effects are captured by the nonlinearities, loss of efficiency, and competition for resources, which are commonly observed in real-world complex systems. In our study, the term $\eta_1 DA$ is a reflection of defensive degradation, which is often the case during excessive attacking pressure. This study considered the offensive overload caused by a congestion of the midfield.This was captured by the term $\eta_2 MA$. Inefficiency in converting attacking chances is represented by $\eta_3 AW$. This work is an extension of the mathematical model developed by Nwokike et al. (2026). Their study developed a four–dimensional football passing dynamics, which was used to analyze ball flow and possession play. It focused on the qualitative analysis, local and stability analysis, and theoretical analysis of structured ball movement [11]. The proposed model is a system of coupled nonlinear ordinary differential equations. These kinds of systems support qualitative analysis, such as positivity of the model solutions, solution boundedness, existence of equilibrium points, and stability analysis. This study took a cue from the theory of classical nonlinear systems (biological and ecological modeling) [13, 14, 18, 19, 20, 21, 22, 23, 24, 25].

This work is designed to investigate the stability and other qualitative properties of the system. It is consistent with known facts that the stability of equilibrium points and the positive results of the qualitative analysis are the conditions under which the system remains well defined. To understand the system's performance under different parameters, the interaction between linear reinforcement and nonlinear saturation effects is also studied.

1.1 Model Formulation: Nonlinear Interaction Model

The state variables are defined as follows:

- $D(t)$: Level of defensive structure (or base resource)
- $M(t)$: Level of midfield coordination (intermediate resource)
- $A(t)$: Level of attacking intensity (forward activity)
- $W(t)$: Overall system output or performance index

The nonlinear interaction model is given by:

$$\frac{dD}{dt} = \Lambda_1 - \mu_1 D + \gamma_1 M - \eta_1 DA, \tag{1}$$

$$\frac{dM}{dt} = \Lambda_2 + \gamma_2 D - \mu_2 M + \gamma_3 A - \eta_2 MA, \tag{2}$$

$$\frac{dA}{dt} = \Lambda_3 + \gamma_4 M - \mu_3 A - \eta_3 AW, \tag{3}$$

$$\frac{dW}{dt} = \Lambda_4 + \gamma_5 A - \mu_4 W. \tag{4}$$

Parameter Definitions

- $\Lambda_i > 0$: constant input rates ($i = 1, \ldots, 4$)
- $\mu_i > 0$: natural decay/removal rates
- $\gamma_i > 0$: interaction (enhancement) coefficients
- $\eta_i > 0$: nonlinear interaction (inhibitory/saturation) coefficients

Model Interpretation

The bilinear terms DA, MA, and AW introduce nonlinear inhibition effects:

- $\eta_1 DA$: defensive degradation under attacking pressure
- $\eta_2 MA$: midfield congestion due to attacking overload
- $\eta_3 AW$: inefficiency in converting attacking intensity into output

Such nonlinear incidence structures are widely used in biological and interaction systems [13, 14].

2 Qualitative Analysis of the Model

In this section, we shall analyze the fundamental properties of the system. This will include the positivity, invariance, boundedness, and existence of solutions.

2.1 Positivity and Boundedness of Model Solutions

In this section, we consider the system where all parameters are positive.

Theorem 1 (Positivity)**.** *Let the initial conditions satisfy*

$$D(0) > 0, \quad M(0) > 0, \quad A(0) > 0, \quad W(0) > 0.$$

Then the solutions of system (1)–(4) *remain positive for all* $t > 0$*.*

Proof. Suppose there exists a first time $t_1 > 0$ such that $D(t_1) = 0$. Then from (1),

$$\left.\frac{dD}{dt}\right|_{D=0} = \Lambda_1 + \gamma_1 M(t_1) > 0.$$

Hence the vector field points inward along the hyperplane $D = 0$, implying that $D(t)$ cannot cross into the negative region.

Similarly,

$$\left.\frac{dM}{dt}\right|_{M=0} = \Lambda_2 + \gamma_2 D + \gamma_3 A > 0,$$

$$\left.\frac{dA}{dt}\right|_{A=0} = \Lambda_3 + \gamma_4 M > 0,$$

and

$$\left.\frac{dW}{dt}\right|_{W=0} = \Lambda_4 + \gamma_5 A > 0.$$

Therefore all trajectories remain in the positive orthant

$$\mathbb{R}^4_+.$$

□

It can be said that the results of the positivity analysis is an indication that all state variables remain non-negative at all times, provided they bare subjected to non-negative initial conditions.This property ensures that the model remains physically meaningful. The model is structured in such a way that enables each compartment to receives sufficient inflow of the ball to prevent extinction in finite time [13, 18, 19, 20].

Theorem 2 (Boundedness)**.** *Solutions of system* (1)–(4) *are uniformly bounded in a positively invariant region of* $\mathbb{R}^4_+$*.*

Proof. Define

$$N(t) = D(t) + M(t) + A(t) + W(t).$$

Then

$$\begin{aligned}\frac{dN}{dt} &= \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 - \mu_1 D - \mu_2 M - \mu_3 A - \mu_4 W \\ &\quad + \gamma_1 M + \gamma_2 D + \gamma_3 A + \gamma_4 M + \gamma_5 A \\ &\quad - \eta_1 DA - \eta_2 MA - \eta_3 AW.\end{aligned}$$

Ignoring the negative nonlinear terms,

$$\frac{dN}{dt} \leq \Lambda - \kappa N,$$

where

$$\Lambda = \sum_{i=1}^{4} \Lambda_i,$$

and

$$\kappa = \min\{\mu_1 - \gamma_2, \mu_2 - (\gamma_1 + \gamma_4), \mu_3 - (\gamma_3 + \gamma_5), \mu_4\} > 0.$$

By the comparison theorem,

$$N(t) \leq \frac{\Lambda}{\kappa} + \left(N(0) - \frac{\Lambda}{\kappa}\right) e^{-\kappa t}.$$

Hence all solutions are bounded. □

Thus, there exists a compact invariant region $\Omega \subset \mathbb{R}^4_+$.

The boundedness result indicates that the system curves will remain within a compact region of the phase space. This is caused by the nonlinear inhibitory terms DA, MA, and AW. These terms act as self-regulating mechanisms.

Unlike in most linear models where unbounded growth is prevalent, this study introduced effects in the like of saturation. This was done by the bilinear interaction as shown in the study. This action acts to suppress excessive expansion. That said, it is important to note that the term $\eta_1 DA$ means that when attacking intensity A is increased, the effective level of D is reduced,¿ This mechanism consequently prevents uncontrolled growth. The same kind of interpretations apply to $\eta_2 MA$ and $\eta_3 AW$.

3 Equilibrium and Stability Analysis of the System

This section contains the equilibrium and stability analysis of the nonlinear interaction football model.

3.1 Equilibrium Points

The equilibrium $E^* = (D^*, M^*, A^*, W^*)$ satisfies:

$$\Lambda_1 - \mu_1 D^* + \gamma_1 M^* - \eta_1 D^* A^* = 0, \tag{5}$$

$$\Lambda_2 + \gamma_2 D^* - \mu_2 M^* + \gamma_3 A^* - \eta_2 M^* A^* = 0, \tag{6}$$

$$\Lambda_3 + \gamma_4 M^* - \mu_3 A^* - \eta_3 A^* W^* = 0, \tag{7}$$

$$W^* = \frac{\Lambda_4 + \gamma_5 A^*}{\mu_4},. \tag{8}$$

3.2 Existence of the Positive Equilibrium

From (8),

$$W^* = \frac{\Lambda_4 + \gamma_5 A^*}{\mu_4}.$$

Substituting into (7),

$$\Lambda_3 + \gamma_4 M^* - \mu_3 A^* - \eta_3 A^* \left(\frac{\Lambda_4 + \gamma_5 A^*}{\mu_4} \right) = 0.$$

Rearranging,

$$\eta_3 \gamma_5 (A^*)^2 + (\mu_3 \mu_4 + \eta_3 \Lambda_4) A^* - \mu_4 (\Lambda_3 + \gamma_4 M^*) = 0.$$

Hence,

$$A^* = \frac{-(\mu_3 \mu_4 + \eta_3 \Lambda_4) + \sqrt{(\mu_3 \mu_4 + \eta_3 \Lambda_4)^2 + 4 \eta_3 \gamma_5 \mu_4 (\Lambda_3 + \gamma_4 M^*)}}{2 \eta_3 \gamma_5}.$$

Similarly, from (5),

$$D^* = \frac{\Lambda_1 + \gamma_1 M^*}{\mu_1 + \eta_1 A^*},$$

and from (6),

$$M^* = \frac{\Lambda_2 + \gamma_2 D^* + \gamma_3 A^*}{\mu_2 + \eta_2 A^*}.$$

Therefore the system admits a unique positive equilibrium

$$E^* = (D^*, M^*, A^*, W^*).$$

3.3 Local Stability of the System

Let

$$E^* = (D^*, M^*, A^*, W^*)$$

be the positive equilibrium of system (1)–(4). To investigate its local stability, we linearize the system about E^*.

3.3.1 Jacobian Matrix

The Jacobian matrix associated with system (1)–(4) is

$$J(D, M, A, W) = \begin{pmatrix} -\mu_1 - \eta_1 A & \gamma_1 & -\eta_1 D & 0 \\ \gamma_2 & -\mu_2 - \eta_2 A & \gamma_3 - \eta_2 M & 0 \\ 0 & \gamma_4 & -\mu_3 - \eta_3 W & -\eta_3 A \\ 0 & 0 & \gamma_5 & -\mu_4 \end{pmatrix}.$$

Evaluating at the equilibrium E^* gives

$$J^* = J(E^*) = \begin{pmatrix} -\mu_1 - \eta_1 A^* & \gamma_1 & -\eta_1 D^* & 0 \\ \gamma_2 & -\mu_2 - \eta_2 A^* & \gamma_3 - \eta_2 M^* & 0 \\ 0 & \gamma_4 & -\mu_3 - \eta_3 W^* & -\eta_3 A^* \\ 0 & 0 & \gamma_5 & -\mu_4 \end{pmatrix}.$$

The characteristic equation is

$$\det(\lambda I - J^*) = 0.$$

After expansion, one obtains the quartic polynomial

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where the coefficients a_i depend on the model parameters and the equilibrium coordinates.

3.3.2 Routh–Hurwitz Stability Criterion

According to the Routh–Hurwitz theorem, all eigenvalues of J^* have negative real parts if and only if

$$a_1 > 0, \tag{9}$$
$$a_2 > 0, \tag{10}$$
$$a_3 > 0, \tag{11}$$
$$a_4 > 0, \tag{12}$$
$$a_1 a_2 - a_3 > 0, \tag{13}$$
$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0. \tag{14}$$

These conditions are necessary and sufficient for local asymptotic stability of the positive equilibrium.

Theorem 3. *Let*

$$E^* = (D^*, M^*, A^*, W^*)$$

be the positive equilibrium of system (1)–(4)*. If the coefficients of the characteristic polynomial satisfy conditions* (9)–(14)*, then E^* is locally asymptotically stable.*

Proof. The linearized system near E^* is

$$\frac{dX}{dt} = J^* X,$$

where

$$X = \begin{pmatrix} D - D^* \\ M - M^* \\ A - A^* \\ W - W^* \end{pmatrix}.$$

The eigenvalues of J^* are precisely the roots of

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0.$$

By the Routh–Hurwitz theorem, conditions (9)–(14) guarantee that all roots possess negative real parts. Therefore every sufficiently small perturbation from E^* decays exponentially with time. Hence the equilibrium E^* is locally asymptotically stable.

□

Theorem 4. *Suppose that*

$$\mu_1 + \eta_1 A^* > \gamma_1 + \eta_1 D^*, \tag{15}$$

$$\mu_2 + \eta_2 A^* > \gamma_2 + \gamma_3 + \eta_2 M^*, \tag{16}$$

$$\mu_3 + \eta_3 W^* > \gamma_4 + \eta_3 A^*, \tag{17}$$

$$\mu_4 > \gamma_5. \tag{18}$$

Then the positive equilibrium E^* is locally asymptotically stable.

Proof. Under conditions (15)–(18), the Jacobian matrix J^* is strictly diagonally dominant with negative diagonal entries. Indeed,

$$|J_{ii}| > \sum_{j \neq i} |J_{ij}|, \qquad i = 1, 2, 3, 4.$$

By the Gershgorin Circle Theorem, every eigenvalue of J^* lies entirely in the open left-half complex plane. Consequently,

$$\Re(\lambda) < 0$$

for every eigenvalue λ of J^*.

Therefore the equilibrium E^* is locally asymptotically stable.

□

3.4 Global Stability of the System

Let

$$E^* = (D^*, M^*, A^*, W^*)$$

denote the unique positive equilibrium of system (1)–(4), satisfying

$$0 = \Lambda_1 - \mu_1 D^* + \gamma_1 M^* - \eta_1 D^* A^*, \tag{19}$$
$$0 = \Lambda_2 + \gamma_2 D^* - \mu_2 M^* + \gamma_3 A^* - \eta_2 M^* A^*, \tag{20}$$
$$0 = \Lambda_3 + \gamma_4 M^* - \mu_3 A^* - \eta_3 A^* W^*, \tag{21}$$
$$0 = \Lambda_4 + \gamma_5 A^* - \mu_4 W^*. \tag{22}$$

We establish conditions under which this equilibrium is globally asymptotically stable in the positively invariant region Ω.

Theorem 5. *Assume that the positive equilibrium*

$$E^* = (D^*, M^*, A^*, W^*)$$

exists uniquely in the feasible region Ω. Furthermore, suppose that

$$\mu_1 > \gamma_2, \tag{23}$$
$$\mu_2 > \gamma_1 + \gamma_4, \tag{24}$$
$$\mu_3 > \gamma_3 + \gamma_5, \tag{25}$$
$$\mu_4 > \eta_3 A^*. \tag{26}$$

Then the equilibrium E^* is globally asymptotically stable in Ω.

Proof. Consider the Lyapunov function

$$V(D, M, A, W) = \sum_{X \in \{D, M, A, W\}} \left(X - X^* - X^* \ln \frac{X}{X^*} \right),$$

that is,

$$V = \left(D - D^* - D^* \ln \frac{D}{D^*} \right) + \left(M - M^* - M^* \ln \frac{M}{M^*} \right) + \left(A - A^* - A^* \ln \frac{A}{A^*} \right) + \left(W - W^* - W^* \ln \frac{W}{W^*} \right).$$

Since

$$x - x^* - x^* \ln\left(\frac{x}{x^*}\right) \geq 0, \qquad x > 0,$$

it follows that

$$V \geq 0,$$

with equality if and only if

$$(D, M, A, W) = (D^*, M^*, A^*, W^*).$$

Differentiating V along solutions of system (1)–(4) gives

$$\dot{V} = \left(1 - \frac{D^*}{D}\right)\dot{D} + \left(1 - \frac{M^*}{M}\right)\dot{M} + \left(1 - \frac{A^*}{A}\right)\dot{A} + \left(1 - \frac{W^*}{W}\right)\dot{W}.$$

Substituting from (1)–(4) yields

$$\begin{aligned}\dot{V} = &\left(1 - \frac{D^*}{D}\right)(\Lambda_1 - \mu_1 D + \gamma_1 M - \eta_1 DA) \\ &+ \left(1 - \frac{M^*}{M}\right)(\Lambda_2 + \gamma_2 D - \mu_2 M + \gamma_3 A - \eta_2 MA) \\ &+ \left(1 - \frac{A^*}{A}\right)(\Lambda_3 + \gamma_4 M - \mu_3 A - \eta_3 AW) \\ &+ \left(1 - \frac{W^*}{W}\right)(\Lambda_4 + \gamma_5 A - \mu_4 W).\end{aligned}$$

Using the equilibrium relations (19)–(22) and simplifying, we obtain

$$\dot{V} = -\mu_1 \frac{(D - D^*)^2}{D} - \mu_2 \frac{(M - M^*)^2}{M} - \mu_3 \frac{(A - A^*)^2}{A} - \mu_4 \frac{(W - W^*)^2}{W} + R,$$

where R denotes the collection of mixed interaction terms.

Applying Young's inequality

$$ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}, \qquad \varepsilon > 0,$$

to each mixed term gives

$$\begin{aligned}R \leq &\gamma_1 |D - D^*||M - M^*| + \gamma_2 |D - D^*||M - M^*| \\ &+ (\gamma_3 + \gamma_4)|M - M^*||A - A^*| + \gamma_5 |A - A^*||W - W^*|.\end{aligned}$$

Consequently,

$$\dot{V} \leq -c_1(D - D^*)^2 - c_2(M - M^*)^2 - c_3(A - A^*)^2 - c_4(W - W^*)^2,$$

where

$$c_1 = \mu_1 - \gamma_2,$$

$$c_2 = \mu_2 - (\gamma_1 + \gamma_4),$$

$$c_3 = \mu_3 - (\gamma_3 + \gamma_5),$$

$$c_4 = \mu_4 - \eta_3 A^*.$$

Under assumptions (23)–(26), we have

$$c_i > 0, \qquad i = 1, 2, 3, 4.$$

Therefore,

$$\dot{V} \leq 0,$$

with equality if and only if

$$D = D^*, \qquad M = M^*, \qquad A = A^*, \qquad W = W^*.$$

Define

$$\mathcal{E} = \left\{ (D, M, A, W) \in \Omega : \dot{V} = 0 \right\}.$$

The largest invariant subset of $\mathcal{E}$ is the singleton set

$$\mathcal{E} = \{E^*\}.$$

Since Ω is positively invariant and bounded, LaSalle's Invariance Principle implies that every solution of system (1)–(4) converges to E^* as $t \to \infty$. Hence

$$(D(t), M(t), A(t), W(t)) \longrightarrow (D^*, M^*, A^*, W^*) \qquad \text{as } t \to \infty.$$

Therefore, the positive equilibrium E^* is globally asymptotically stable in Ω.

□

This reveals that under certain conditions, the equilibrium may be globally asymptotically stable. This is demonstrated by the Lyapunov-based argument. This is an indication that the system's long-term behavior does not depend on initial conditions. A quality that ensures robustness against perturbations.

4 Discussion and Conclusion

The nonlinear interaction model shown by way of presentation in this study is an extension of classical linear and weak nonlinear models. This was achieved by the incorporation of bilinear inhibitions of some interactions. Realistic phenomena such as competition, congestion, and diminishing returns were captured by these interactions. A common scene in complex systems. The integration of multiple layers of interaction is a key strength of the model. The fact that system components cannot be analyzed in isolation encouraged the idea of coupling between compartments. This can be seen in epidemiological and ecological models. The mathematical well-posedness of the model is confirmed by the qualitative analysis. We have shown that the nonlinear interactions in preventing unwanted behavior from the system were highlighted by the boundedness result [14, 23, 25].

The stability analysis shows how a change in parameter values can influence system behavior. The growth of the system is determined by the balance between input rates and decay rates. This also determines whether the system declines or stabilizes. The system can be stabilized/destabilized by the interaction coefficients η_i, which act as regulatory controls. In other words, managing interaction can strengthen the system effectively instead of controlling individual components. Not accounting for stochastic variations or time delays can be assumed to be one of the weaknesses of the model. This is because it assumes constant parameters. Another limitation of the study is the lack of empirical validation. This would have confirmed the model's predictive capabilities.Future works will include extending this work to consider stochastic effects, time delays, or fractional-order dynamics. The consideration of numerical simulations, bifurcation analysis, and parameter estimation would make meaningful research.

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