Closed-Form Solutions of Second-Order Leonardo-Type Sequences: Homogeneous Counterparts in Jacobsthal and Mersenne Numbers

Yüksel Soykan

Department of Mathematics,
Faculty of Science,
Zonguldak Bülent Ecevit University,
67100, Zonguldak, Turkey
e-mail: yuksel_soykan@hotmail.com

Abstract. The objective of this study is to derive explicit closed-form solutions for second-order nonhomogeneous linear recurrence relations with polynomial inputs, formulated as generalized Leonardo-type sequences. A central aspect of the framework is the multiplicity parameter r, which measures the occurrence of the root 1 in the characteristic equation. This parameter determines whether the recurrence falls into the non-resonant case ($r = 0$, no root equal to 1) or the resonant case ($r = 1$, unity as a simple root), with corresponding adjustments in the construction of particular solutions.

Within this setting, we highlight two principal families: the generalized Jacobsthal sequences, where the characteristic roots are $\{2, -1\}$ and thus $r = 0$, and the generalized Mersenne sequences, where the roots are $\{2, 1\}$ and hence $r = 1$. In both cases, closed-form solutions are obtained under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$, covering constant through septic forcing terms. These results clarify how root multiplicity and polynomial degree jointly shape the explicit formulas, while the homogeneous counterparts (Jacobsthal, Jacobsthal–Lucas, Mersenne, and Mersenne–Lucas sequences) emerge naturally when the input polynomial is suppressed.

The study thus provides a unified framework that connects classical integer sequences with their nonhomogeneous extensions, offering resonance-aware closed forms that are both theoretically significant and pedagogically accessible.

2020 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Jacobsthal numbers, Mersenne numbers, Leonardo numbers, nonhomogeneous recurrence relations, homogeneous recurrence relations, closed-form solutions, particular solutions.

1. Introduction

Recurrence relations have long formed a cornerstone of mathematics, extending their reach into physics, engineering, architecture, biology, computer science, and even artistic inquiry. Though their definitions often appear elementary, they conceal remarkable depth, modeling processes of growth, oscillation, and symbolic construction. Classical second-order families—most prominently the Fibonacci, Lucas, Pell, and Jacobsthal sequences—remain enduring exemplars of this tradition.

The framework, however, extends far beyond second-order cases. Higher-order recurrence sequences enrich both theory and practice, broadening the classical foundation while uncovering intricate algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences illustrate this expansion, each governed by characteristic polynomials whose root configurations determine closed-form formulas. Homogeneous recurrences emphasize the interplay between characteristic polynomials and root multiplicities, while nonhomogeneous forms introduce symbolic terms that interact with root structures to produce resonance phenomena. Collectively, these families establish a coherent framework that unites classical recurrence identities with the evolving discipline of symbolic recurrence theory.

This work aims to present closed-form solutions for second-order nonhomogeneous linear recurrence relations, expressed as generalized Leonardo-type sequences with polynomial inputs. In this setting, the homogeneous counterparts arise naturally when the input term $p(n)$ is removed, leading to generalized Fibonacci-type sequences. Our study focuses on several classical examples, most notably the Jacobsthal and Jacobsthal–Lucas sequences, together with the Mersenne and Mersenne–Lucas families, each representing a distinct case within recurrence theory.

The classical Leonardo numbers are introduced through the non-homogeneous recurrence relation

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with initial conditions $l_0 = 1$ and $l_1 = 1$.

The historical background of the Leonardo sequence is somewhat elusive. Its evolution appears to have unfolded gradually: extensions and generalizations were examined before the sequence itself acquired a formal designation, and its modern resurgence has been closely linked to the study of special cases and their applications. Over the years, the Leonardo sequence has drawn renewed interest, not only for its inherent mathematical appeal but also for its versatility in modeling hybrid recurrence systems that blend homogeneous dynamics with non-homogeneous inputs. This dual character has made it a rich subject for symbolic investigation, connecting classical recurrence theory with contemporary applications.

From a pedagogical perspective, the simplicity of its defining recurrence and the availability of explicit examples render the Leonardo sequence particularly suitable for teaching materials. It offers students a clear demonstration of how non-homogeneous recurrence relations function, while simultaneously opening the door to advanced symbolic techniques and resonance phenomena. Thus, the Leonardo sequence serves both as a continuing topic of scholarly research and as a valuable instructional resource (see, for example, [1, 2, 3, 5, 6, 10, 11, 19, 20, 21, 12, 15, 16, 17, 13, 14]).

To the best of our knowledge, the first systematic extension of the Leonardo numbers was undertaken by J. A. Jeske in a trilogy of papers published in *The Fibonacci Quarterly* during 1963–1964 (see [7, 8, 9]). For a concise literature review, particularly from *The Fibonacci Quarterly*, as well as an overview of selected contributions outside the journal that advance the study of Leonardo-type recurrence relations, see Soykan [22, Section 5].

Let the second order nonhomogeneous linear recurrence relation be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + p(n) \tag{1.1}$$

with initial conditions $W_0 = u_0, W_1 = u_1$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i$$

and the recurrence coefficients a_1, a_2 are complex scalars or polynomials in $\mathbb{C}[x]$.

By a homogeneous counterpart we refer to the classical recurrence sequence that emerges once the non-homogeneous input term $p(n)$ is suppressed. In other words, the homogeneous case corresponds to the underlying recurrence relation stripped of external forcing, thereby revealing its intrinsic dynamics. For instance, within the generalized Jacobsthal family, the introduction of a polynomial input $p(n)$ produces a non-homogeneous extension; however, setting $p(n) = 0$ recovers the standard Jacobsthal sequence in its familiar form. Likewise, the generalized Mersenne constructions admit homogeneous reductions: when the non-homogeneous component is removed, one obtains the classical Mersenne sequence together with its Lucas-type analogue. This perspective highlights the structural role of homogeneous counterparts as the foundational archetypes upon which non-homogeneous generalizations are built. The homogeneous sequences serve as the baseline models, while the non-homogeneous variants enrich the theory by incorporating symbolic inputs and resonance phenomena. Thus, the relationship between homogeneous and non-homogeneous forms is not merely technical but conceptual, illustrating how classical integer sequences can be extended into broader families without losing their essential identity.

A sequence $\{W_n\}_{n\geq 0}$ defined by (1.1) is called a generalized Leonardo-type sequence with input function $p(n)$. For more information on generalized Leonardo-type sequences, see Soykan [23] and [22].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1.$$

Suppose that θ_1 and θ_2 are the roots of the characteristic equation of (1.2)

$$z^2 - a_1 z - a_2 = 0, \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^2 - a_1 z - a_2 = (z-1)^2 = z^2 - 2z + 1 = 0$$

so that $a_1 = 2,\ a_2 = -1$ and (1.2) reduces to

$$V_n = 2V_{n-1} - V_{n-2}. \tag{1.4}$$

In earlier work, particular solutions of second-order nonhomogeneous recurrences (1.1) with polynomial inputs were obtained for degrees $s = 0,1,2,3,4,5,6,7$; see Soykan [24]. Here, we extend those results to unified closed-form solutions by applying Theorem 1.1. Each recurrence is expressed as the sum of its homogeneous and particular components, with the particular solution determined through iterative coefficient relations. The framework is organized by two parameters: the multiplicity r of 1 as a root of the characteristic equation (1.3), and the polynomial degree s. Explicit formulas are obtained for all cases $r = 0,1,2$ and $s = 0,1,2,3,4,5,6,7$, showing how resonance and multiplicity corrections arise. Since our examples—generalized Jacobsthal numbers ($r = 0$) and Mersenne numbers ($r = 1$)—fall into these categories, we present the theorem in detail only for $r = 0$ and $r = 1$, noting that the double root case $r = 2$ can be derived analogously.

Theorem 1.1.

(a): *[22, Theorem 7.6. (a)] The case $r = 0$, i.e., all two roots of the characteristic equation of (1.3) is distinct from 1. The solution of (1.1) is given by*

$$\begin{aligned} W_n(W_0, W_1) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

is the particular solution of (1.1). For each $0 \le i \le s$, A_i can be calculated with the iteration

$$A_s = -\frac{c_s}{a_1 + a_2 - 1},\ \textit{for } n = s$$

and

$$A_n = -\frac{1}{a_1 + a_2 - 1}\Big(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+1} \binom{k}{n} (a_1 + 2^{k-n} a_2) A_k\Big),\ \textit{for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i) \\ &= \frac{(W_1 - a_1 W_0)\sum_{i=0}^{s} A_i - a_2 W_0 A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_n(W_0, W_1) + \frac{-a_2(W_0 \sum_{i=0}^{s} A_i - A_0 W_1)}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n-1}(W_0, W_1) \\ &= -\frac{W_0 \sum_{i=0}^{s} A_i - A_0 W_1}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n+1}(W_0, W_1) + \frac{W_1 \sum_{i=0}^{s} A_i - (W_0 a_2 + W_1 a_1) A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_n(W_0, W_1). \end{aligned}$$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1) = \left(-\frac{(W_1 - a_1 W_0)\sum_{i=0}^{s} A_i - a_2 W_0 A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} + 1 \right) V_n(W_0, W_1) + \frac{a_2(W_0 \sum_{i=0}^{s} A_i - A_0 W_1)}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n-1}(W_0, W_1) + \sum_{i=0}^{s} A_i n^i$$

i.e.,

$$W_n(W_0, W_1) = \frac{W_0 \sum_{i=0}^{s} A_i - A_0 W_1}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n+1}(W_0, W_1) + \left(-\frac{W_1 \sum_{i=0}^{s} A_i - (W_0 a_2 + W_1 a_1) A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} + 1 \right) V_n(W_0, W_1) + \sum_{i=0}^{s} A_i n^i$$

(b): *The case $r = 1$, i.e., 1 is a simple root of the characteristic equation of (1.3). The solution of (1.1) is given by*

$$\begin{aligned} W_n(W_0, W_1) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n \sum_{i=0}^{s} A_i n^i = n(A_0 + \sum_{i=1}^{s} A_i n^i)$$

is the particular solution of (1.1). For each $0 \le i \le s$, A_i can be calculated with the iteration

$$A_s = (-1)^{1+1} \frac{c_s}{(\sum_{j=1}^{2} j a_j)\binom{s+1}{1}} = \frac{c_s}{(a_1 + 2a_2)(s+1)}, \quad \textit{for } n = s$$

and

$$A_n = \frac{1}{(a_1 + 2a_2)(n+1)} (c_n - \sum_{k=n+1}^{s} (-1)^{k-n+2} \binom{k+1}{n} (a_1 + 2^{k-n+1} a_2) A_k), \ \textit{for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= 0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \end{aligned}$$

and

$$\begin{aligned}
V_n(W_0^{(p)},W_1^{(p)}) &= V_n(0,\sum_{i=0}^{s}A_i)\\
&= \frac{(W_1-a_1W_0)\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_n(W_0,W_1)+\frac{-a_2W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n-1}(W_0,W_1)\\
&= -\frac{W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n+1}(W_0,W_1)+\frac{W_1\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_n(W_0,W_1).
\end{aligned}$$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1)=\left(-\frac{(W_1-a_1W_0)\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+\frac{a_2W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n-1}(W_0,W_1)+n\sum_{i=0}^{s}A_in^i$$

i.e.,

$$W_n(W_0,W_1)=\frac{W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n+1}(W_0,W_1)+\left(-\frac{W_1\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+n\sum_{i=0}^{s}A_in^i$$

(c): *The case $r=2$, i.e., 1 is a double root of the characteristic equation of (1.3). The construction follows the same method as in case (a) and (b), with appropriate multiplicity corrections.*

Proof. The solution is obtained by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104-105), and Theorem 3.1 (pp. 88-89) for $m=2$, as presented in Soykan [22]. In general, the theorem applies to all multiplicities $r=0,1,2$: the case $r=0$ corresponds to the non-resonant situation where both roots are distinct from 1, while $r=1$ and $r=2$ represent simple and double roots of 1 in the characteristic equation (1.3). Since our applications involve only the non-resonant case $r=0$ and the simple root case $r=1$, the double root case $r=2$ is omitted. It can be obtained in the same manner with multiplicity adjustments, but is not needed for the examples considered here.

2. Examples: Closed-Form Solutions of Second Order nonhomogeneous Linear Recurrence Relations

In this section, we present illustrative applications of Theorem 1.1 to selected second-order nonhomogeneous recurrence relations. Two distinct situations arise depending on the multiplicity of the root 1 in the characteristic equation (1.3). For the generalized Jacobsthal family, the characteristic polynomial

$$z^2-z-2=0$$

has roots $\{2,-1\}$, neither equal to unity. Hence, the non-resonant case $r=0$ applies, and Theorem 1.1 (a) yields the closed-form solutions. By contrast, in the generalized Mersenne family the characteristic polynomial

$$z^2-3z+2=0$$

has roots $\{2,1\}$, so that 1 appears as a simple root. In this setting, resonance corrections are required, and Theorem 1.1(b) governs the construction of solutions. Thus, the parameter r precisely encodes the multiplicity of the root 1: $r = 0$ for Jacobsthal-type sequences, and $r = 1$ for Mersenne-type sequences. This distinction clarifies why different parts of the theorem are invoked in each case.

For both families, explicit closed-form solutions are derived under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$. The analysis systematically incorporates constant ($s = 0$), linear ($s = 1$), quadratic ($s = 2$), cubic ($s = 3$), quartic ($s = 4$), quintic ($s = 5$), sextic ($s = 6$), and septic ($s = 7$) inputs, thereby illustrating the breadth of the method and its adaptability across classical recurrence structures. The role of root multiplicities is central: when unity is absent from the spectrum, the homogeneous dynamics remain non-resonant; when unity occurs as a simple root, resonance phenomena must be accounted for. In this way, the framework unifies the treatment of Jacobsthal and Mersenne sequences while respecting the algebraic subtleties of their characteristic equations.

To recall the general setting, the second-order nonhomogeneous recurrence relation is given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + p(n), \quad n \geq 2,$$

with initial conditions $W_0 = k_0$, $W_1 = k_1$, where the input polynomial takes the form

$$p(n) = \sum_{i=0}^{s} c_i n^i, \qquad c_i \in \mathbb{C} \text{ or } c_i \in \mathbb{C}[x].$$

Here, the coefficients a_1, a_2 are arbitrary complex scalars (or polynomials in $\mathbb{C}[x]$), and the choice of (k_0, k_1) specifies the initial configuration of the sequence. In the examples that follow, we concentrate on two distinguished parameter sets:

$$(a_1, a_2) = (1, 2), \qquad (a_1, a_2) = (3, -2),$$

corresponding respectively to the generalized Jacobsthal and generalized Mersenne families. For each case, closed-form solutions are obtained under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$, illustrating how the general framework adapts to different initial values and forcing terms while unifying the treatment of these classical sequences.

REMARK 2.1. *Although the following examples are presented case by case, they are all unified under the same formal framework determined by the characteristic equation and the polynomial input degree s. The closed-form solutions are systematically derived from Theorem 1.1, with the multiplicity parameter r distinguishing between non-resonant cases ($r = 0$) and resonant cases ($r = 1$). This unified formalism ensures that each example—whether Jacobsthal-type ($a_1 = 1, a_2 = 2$) or Mersenne-type ($a_1 = 3, a_2 = -2$)—is a direct specialization of the general recurrence relation. The detailed presentation is retained intentionally to serve as a clear reference for students and researchers, allowing each case to be consulted independently while remaining consistent with the overarching theory.*

2.1. Generalized Jacobsthal Numbers: The case $a_1 = 1$ and $a_2 = 2$. Now, we consider the case $a_1 = 1, a_2 = 2$. A generalized Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$V_n = V_{n-1} + 2V_{n-2}, \tag{2.1}$$

with the initial values $V_0 = v_0, V_1 = v_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

The Binet formula of generalized Jacobsthal numbers can be written as

$$\begin{aligned} V_n &= \frac{V_1 - \beta V_0}{\alpha - \beta}\alpha^n - \frac{V_1 - \alpha V_0}{\alpha - \beta}\beta^n \\ &= \frac{p_1\alpha^n - p_2\beta^n}{\alpha - \beta} \end{aligned} \tag{2.2}$$

where α and β are the roots of the quadratic equation $z^2 - z - 2 = 0$ and

$$\begin{aligned} p_1 &= V_1 - \beta V_0 \\ p_2 &= V_1 - \alpha V_0. \end{aligned}$$

Moreover

$$\begin{aligned} \alpha &= 2 \\ \beta &= -1 \end{aligned}$$

So

$$V_n = \frac{(V_1 - \beta V_0) \times 2^n - (V_1 - \alpha V_0) \times (-1)^n}{3}.$$

Now we define two special cases of the sequence $\{V_n\}$. Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$J_n = J_{n-1} + 2J_{n-2}, \qquad J_0 = 0, J_1 = 1, \tag{2.3}$$

$$j_n = j_{n-1} + 2j_{n-2}, \qquad j_0 = 2, j_1 = 1, \tag{2.4}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} + \frac{1}{2}J_{-(n-2)} \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} + \frac{1}{2}j_{-(n-2)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.4) hold for all integer n.

For all integers n, Jacobsthal and Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} J_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{3}, \\ j_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 2, \\ W_0 &= 0, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = J_n$ (Jacobsthal numbers).

EXAMPLE 2.2. *In this example, we consider the case $a_1 = 1, a_2 = 2, W_0 = 0, W_1 = 1$ in Theorem 1.1 (a). So $V_n(0,1) = V_n$ represents nth Jacobshal number i.e., $V_n = J_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_0,$$

is given by

$$W_n(0,1) = \frac{1}{2}(c_0+2)J_n + c_0J_{n-1} - \frac{1}{2}c_0,$$

i.e.,

$$W_n(0,1) = \frac{1}{2}c_0J_{n+1} + J_n - \frac{1}{2}c_0.$$

(b): *The case $s = 1$. For*

$$W_n = W_{n-1} + 2W_{n-2} + c_1n + c_0,$$

the closed-form solution is

$$W_n(0,1) = \frac{1}{4}(2c_0+7c_1+4)J_n + \frac{1}{2}(2c_0+5c_1)J_{n-1} + A_1n + A_0,$$

i.e.,

$$W_n(0,1) = \frac{1}{4}(2c_0+5c_1)J_{n+1} + \frac{1}{2}(c_1+2)J_n + A_1n + A_0,$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0+5c_1). \end{aligned}$$

(c): *The case $s = 2$. For*

$$W_n = W_{n-1} + 2W_{n-2} + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$$W_n(0,1) = \frac{1}{4}(2c_0+7c_1+28c_2+4)J_n + \frac{1}{2}(2c_0+5c_1+16c_2)J_{n-1} + A_2n^2 + A_1n + A_0,$$

i.e.,

$$W_n(0,1) = \frac{1}{4}(2c_0 + 5c_1 + 16c_2)J_{n+1} + \frac{1}{2}(c_1 + 6c_2 + 2)J_n + A_2n^2 + A_1n + A_0,$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1 + 16c_2). \end{aligned}$$

(d): *The case* $s = 3$. *For*

$$W_n = W_{n-1} + 2W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3)J_{n-1} + A_3n^3 + A_2n^2 + A_1n + A_0$,

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3)J_{n+1} + \frac{1}{4}\left(2c_1 + 12c_2 + 65c_3 + 4\right)J_n + A_3n^3 + A_2n^2 + A_1n + A_0$,

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3). \end{aligned}$$

(e): *The case* $s = 4$. *For*

$$W_n = W_{n-1} + 2W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 1592c_4 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4)J_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$,

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4)J_{n+1} + \frac{1}{4}(2c_1 + 12c_2 + 65c_3 + 396c_4 + 4)J_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$,

where

$$\begin{aligned} A_4 &= -\frac{1}{2}c_4, \\ A_3 &= -\frac{1}{2}(c_3 + 10c_4), \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4). \end{aligned}$$

(f): *The case* $s = 5$. *For*

$$W_n = W_{n-1} + 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 1592c_4 + 11\,534c_5 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5)J_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5)J_{n+1} + \frac{1}{4}(2c_1 + 12c_2 + 65c_3 + 396c_4 + 2882c_5 + 4)J_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

where

$$\begin{aligned} A_5 &= -\frac{1}{2}c_5, \\ A_4 &= -\frac{1}{4}(2c_4 + 25c_5), \\ A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5), \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5). \end{aligned}$$

(g): *The case* $s = 6$. *For*

$$W_n = W_{n-1} + 2W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 1592c_4 + 11\,534c_5 + 99\,896c_6 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49\,952c_6)J_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49\,952c_6)J_{n+1} + \frac{1}{4}(2c_1 + 12c_2 + 65c_3 + 396c_4 + 2882c_5 + 24\,972c_6 + 4)J_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

where

$$\begin{aligned}
A_6 &= -\frac{1}{2}c_6, \\
A_5 &= -\frac{1}{2}(c_5+15c_6), \\
A_4 &= -\frac{1}{4}(2c_4+25c_5+240c_6), \\
A_3 &= -\frac{1}{2}(c_3+10c_4+80c_5+695c_6), \\
A_2 &= -\frac{1}{4}(2c_2+15c_3+96c_4+695c_5+6000c_6), \\
A_1 &= -\frac{1}{2}(c_1+5c_2+24c_3+139c_4+1000c_5+8655c_6), \\
A_0 &= -\frac{1}{8}(4c_0+10c_1+32c_2+139c_3+800c_4+5770c_5+49952c_6).
\end{aligned}$$

(h): *The case $s=7$. For*

$$W_n = W_{n-1}+2W_{n-2}+c_7n^7+c_6n^6+c_5n^5+c_4n^4+c_3n^3+c_2n^2+c_1n+c_0,$$

the closed-form solution is

$W_n(0,1) = \frac{1}{16}(8c_0+28c_1+112c_2+538c_3+3184c_4+23068c_5+199792c_6+2017813c_7+16)J_n + \frac{1}{8}(8c_0+20c_1+64c_2+278c_3+1600c_4+11\,540c_5+99904c_6+1008923c_7)J_{n-1}+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0,$

i.e.,

$W_n(0,1) = \frac{1}{16}(8c_0+20c_1+64c_2+278c_3+1600c_4+11\,540c_5+99904c_6+1008923c_7)J_{n+1} + \frac{1}{8}(4c_1+24c_2+130c_3+792c_4+5764c_5+49944c_6+504445c_7+8)J_n+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0,$

where

$$\begin{aligned}
A_7 &= -\frac{1}{2}c_7, \\
A_6 &= -\frac{1}{4}(2c_6+35c_7), \\
A_5 &= -\frac{1}{2}(c_5+15c_6+168c_7), \\
A_4 &= -\frac{1}{8}(4c_4+50c_5+480c_6+4865c_7), \\
A_3 &= -\frac{1}{2}(c_3+10c_4+80c_5+695c_6+7000c_7), \\
A_2 &= -\frac{1}{4}(2c_2+15c_3+96c_4+695c_5+6000c_6+60585c_7), \\
A_1 &= -\frac{1}{2}(c_1+5c_2+24c_3+139c_4+1000c_5+8655c_6+87416c_7), \\
A_0 &= -\frac{1}{16}(8c_0+20c_1+64c_2+278c_3+1600c_4+11540c_5+99904c_6+1008923c_7).
\end{aligned}$$

In the example below, we do not treat $p(n)$ in its general form but instead select specific coefficients c_i to construct concrete forcing terms of degrees $s = 0, 1, 2, 3$. This approach highlights how the closed-form solutions adapt to constant, linear, quadratic, and cubic inputs.

Example 2.3. *In this example, we consider the case $a_1 = 1, a_2 = 2, W_0 = 0, W_1 = 1$ with specific input polynomials of degree $s = 0, 1, 2, 3$ (see Example 2.2).*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + 3,$$

is given by

$$W_n(0,1) = \frac{5}{2}J_n + 3J_{n-1} - \frac{3}{2},$$

i.e.,

$$W_n(0,1) = \frac{3}{2}J_{n+1} + J_n - \frac{3}{2}.$$

(b): *The case $s = 1$. For*

$$W_n = W_{n-1} + 2W_{n-2} - 2n + 1,$$

the closed-form solution is

$$W_n(0,1) = -2J_n - 4J_{n-1} + n + 2,$$

i.e.,

$$W_n(0,1) = -2J_{n+1} + n + 2.$$

(c): *The case $s = 2$. For*

$$W_n = W_{n-1} + 2W_{n-2} + 3n^2 + n + 2,$$

the closed-form solution is

$$W_n(0,1) = \frac{99}{4}J_n + \frac{57}{2}J_{n-1} - \frac{3}{2}n^2 - 8n - \frac{57}{4},$$

i.e.,

$$W_n(0,1) = \frac{57}{4}J_{n+1} + \frac{21}{2}J_n - \frac{3}{2}n^2 - 8n - \frac{57}{4}.$$

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ -8 &= -\frac{1}{2}(c_1 + 15), \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1 + 16c_2). \end{aligned}$$

(d): *The case $s=3$. For*

$$W_n = W_{n-1} + 2W_{n-2} + \frac{3}{2}n^3 - \frac{2}{5}n^2 + 4n + \frac{1}{2},$$

the closed-form solution is

$$W_n(0,1) = \frac{4471}{80}J_n + \frac{2377}{40}J_{n-1} - \frac{3}{4}n^3 - \frac{217}{40}n^2 - 19n - \frac{2377}{80},$$

i.e.,

$$W_n(0,1) = \frac{2377}{80}J_{n+1} + \frac{1047}{40}J_n - \frac{3}{4}n^3 - \frac{217}{40}n^2 - 19n - \frac{2377}{80}.$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 2, \\ W_0 &= 2, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = j_n$ (Jacobsthal-Lucas numbers).

Example 2.4. *In this example, we consider the case $a_1 = 1, a_2 = 2, W_0 = 2, W_1 = 1$ in Theorem 1.1 (a). So $V_n(2,1) = V_n$ represents nth Jacobsthal-Lucas number, i.e., $V_n = j_n$.*

(a): *The case $s=0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_0,$$

is given by

$$W_n(2,1) = \frac{1}{18}(5c_0 + 18)j_n + \frac{1}{9}c_0 j_{n-1} - \frac{1}{2}c_0,$$

i.e.,

$$W_n(2,1) = \frac{1}{18}c_0 j_{n+1} + \frac{1}{9}(2c_0 + 9)j_n - \frac{1}{2}c_0.$$

(b): *The case $s=1$. For*

$$W_n = W_{n-1} + 2W_{n-2} + c_1 n + c_0,$$

the closed-form solution is

$$W_n(2,1) = \frac{1}{36}(10c_0 + 27c_1 + 36)j_n + \frac{1}{18}(2c_0 + 9c_1)j_{n-1} + A_1 n + A_0,$$

i.e.,

$$W_n(2,1) = \frac{1}{36}(2c_0 + 9c_1)j_{n+1} + \frac{1}{18}(4c_0 + 9c_1 + 18)j_n + A_1 n + A_0,$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case* $s = 2$. *For*

$$W_n = W_{n-1} + 2W_{n-2} + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$$W_n(2,1) = \frac{1}{36}(10c_0 + 27c_1 + 92c_2 + 36)j_n + \frac{1}{18}(2c_0 + 9c_1 + 40c_2)j_{n-1} + A_2 n^2 + A_1 n + A_0,$$

i.e.,

$$W_n(2,1) = \frac{1}{36}(2c_0 + 9c_1 + 40c_2)j_{n+1} + \frac{1}{18}(4c_0 + 9c_1 + 26c_2 + 18)j_n + A_2 n^2 + A_1 n + A_0,$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1 + 16c_2), \end{aligned}$$

(d): *The case* $s = 3$. *For*

$$W_n = W_{n-1} + 2W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$$W_n(2,1) = \frac{1}{72}(20c_0+54c_1+184c_2+825c_3+72)j_n+\frac{1}{36}(4c_0+18c_1+80c_2+399c_3)j_{n-1}+A_3n^3+A_2n^2+A_1n+A_0,$$

i.e.,

$$W_n(2,1) = \frac{1}{72}(4c_0+18c_1+80c_2+399c_3)j_{n+1}+\frac{1}{36}(8c_0+18c_1+52c_2+213c_3+36)j_n+A_3n^3+A_2n^2+A_1n+A_0,$$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3). \end{aligned}$$

(e): *The case $s=4$. For*

$$W_n = W_{n-1} + 2W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(2,1) = \frac{1}{72}(20c_0 + 54c_1 + 184c_2 + 825c_3 + 4792c_4 + 72)j_n + \frac{1}{36}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4)j_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

i.e.,

$W_n(2,1) = \frac{1}{72}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4)j_{n+1} + \frac{1}{36}(8c_0 + 18c_1 + 52c_2 + 213c_3 + 1204c_4 + 36)j_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

where

$$\begin{aligned}
A_4 &= -\frac{1}{2}c_4, \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4), \\
A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4).
\end{aligned}$$

(f): *The case $s=5$. For*

$$W_n = W_{n-1} + 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

the closed-form solution is

$W_n(2,1) = \frac{1}{72}(20c_0 + 54c_1 + 184c_2 + 825c_3 + 4792c_4 + 34614c_5 + 72)j_n + \frac{1}{36}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5)j_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(2,1) = \frac{1}{72}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5)j_{n+1} + \frac{1}{36}(8c_0 + 18c_1 + 52c_2 + 213c_3 + 1204c_4 + 8658c_5 + 36)j_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned}
A_5 &= -\frac{1}{2}c_5, \\
A_4 &= -\frac{1}{4}(2c_4 + 25c_5), \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5), \\
A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5).
\end{aligned}$$

(g): *The case $s = 6$. For*

$$W_n = W_{n-1} + 2W_{n-2} + c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$W_n(2,1) = \frac{1}{72}(20c_0 + 54c_1 + 184c_2 + 825c_3 + 4792c_4 + 34614c_5 + 299704c_6 + 72)j_n + \frac{1}{36}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5 + 149840c_6)j_{n-1} + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0,$

i.e.,

$W_n(2,1) = \frac{1}{72}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5 + 149840c_6)j_{n+1} + \frac{1}{36}(8c_0 + 18c_1 + 52c_2 + 213c_3 + 1204c_4 + 8658c_5 + 74932c_6 + 36)j_n + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0,$

where

$$\begin{aligned}
A_6 &= -\frac{1}{2}c_6, \\
A_5 &= -\frac{1}{2}(c_5 + 15c_6), \\
A_4 &= -\frac{1}{4}(2c_4 + 25c_5 + 240c_6), \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5 + 695c_6), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5 + 6000c_6), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5 + 8655c_6), \\
A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49952c_6).
\end{aligned}$$

(h): *The case $s = 7$. For*

$$W_n = -W_{n-1} + 2W_{n-2} + c_7 n^7 + c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$W_n(2,1) = \frac{1}{144}(40c_0 + 108c_1 + 368c_2 + 1650c_3 + 9584c_4 + 69228c_5 + 599408c_6 + 6053505c_7 + 144)j_n + \frac{1}{72}(8c_0 + 36c_1 + 160c_2 + 798c_3 + 4768c_4 + 34596c_5 + 299680c_6 + 3026703c_7)j_{n-1} + A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0,$

i.e.,

$W_n(2,1) = \frac{1}{144}(8c_0 + 36c_1 + 160c_2 + 798c_3 + 4768c_4 + 34596c_5 + 299680c_6 + 3026703c_7)j_{n+1} + \frac{1}{72}(16c_0 + 36c_1 + 104c_2 + 426c_3 + 2408c_4 + 17\,316c_5 + 149864c_6 + 1513401c_7 + 72)j_n + A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0,$

where

$$
\begin{aligned}
A_7 &= -\frac{1}{2}c_7, \\
A_6 &= -\frac{1}{4}(2c_6+35c_7), \\
A_5 &= -\frac{1}{2}(c_5+15c_6+168c_7), \\
A_4 &= -\frac{1}{8}(4c_4+50c_5+480c_6+4865c_7), \\
A_3 &= -\frac{1}{2}(c_3+10c_4+80c_5+695c_6+7000c_7), \\
A_2 &= -\frac{1}{4}(2c_2+15c_3+96c_4+695c_5+6000c_6+60585c_7), \\
A_1 &= -\frac{1}{2}(c_1+5c_2+24c_3+139c_4+1000c_5+8655c_6+87416c_7), \\
A_0 &= -\frac{1}{16}(8c_0+20c_1+64c_2+278c_3+1600c_4+11540c_5+99904c_6+1008923c_7).
\end{aligned}
$$

2.2. Generalized Mersenne Numbers: The case $a_1 = 3$ and $a_2 = -2$. In this subsection, we consider the case $a_1 = 3, a_2 = -2$. A generalized Mersenne sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$V_n = 3V_{n-1} - 2V_{n-2} \tag{2.5}$$

with the initial values $V_0 = c_0, V_1 = c_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = \frac{3}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (2.5) holds for all integer n. For more information on generalized Mersenne numbers, see Soykan [18].

The Binet formula of generalized Mersenne numbers can be written as

$$V_n = \frac{V_1 - \beta V_0}{\alpha - \beta}\alpha^n - \frac{V_1 - \alpha V_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation

$$z^2 - 3z + 2 = 0. \tag{2.6}$$

Moreover

$$
\begin{aligned}
\alpha &= 2 \\
\beta &= 1
\end{aligned}
$$

So

$$V_n = (V_1 - V_0)2^n - (V_1 - 2V_0). \tag{2.7}$$

Now, we define two special cases of the sequence $\{V_n\}$. Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$M_n = 3M_{n-1} - 2M_{n-2}, \qquad M_0 = 0, M_1 = 1, \tag{2.8}$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \qquad H_0 = 2, H_1 = 3, \tag{2.9}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.8)-(2.9) hold for all integer n.

For all integers n, Mersenne and Mersenne-Lucas can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (b) for

$$\begin{aligned} a_1 &= 3, \ a_2 = -2, \\ W_0 &= 0, \ W_1 = 1, \end{aligned}$$

so that we have the case $V_n = M_n$ (Mersenne numbers).

EXAMPLE 2.5. *In this example, we consider the case $a_1 = 3$, $a_2 = -2, W_0 = 0$, $W_1 = 1$ in Theorem 1.1 (b). So $V_n(0,1) = V_n$ represents nth Mersenne number, i.e., $V_n = M_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_0,$$

is given by

$$W_n(0,1) = (c_0 + 1)M_n - c_0 n.$$

(b): *The case $s = 1$. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_1 n + c_0,$$

the closed-form solution is

$$W_n(0,1) = (c_0 + 3c_1 + 1)M_n + A_1 n^2 + A_0 n,$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{2}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case* $s = 2$. *For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 1)M_n + A_2 n^3 + A_1 n^2 + A_0 n,$$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2). \end{aligned}$$

(d): *The case* $s = 3$. *For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 1)M_n + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$$

where

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{6}(2c_2 + 15c_3), \\ A_1 &= -\frac{1}{4}(2c_1 + 10c_2 + 49c_3), \\ A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2 + 216c_3). \end{aligned}$$

(e): *The case* $s = 4$. *For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 1)M_n + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$$

where

$$\begin{aligned}
A_4 &= -\frac{1}{5}c_4, \\
A_3 &= -\frac{1}{4}(c_3 + 10c_4), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4), \\
A_1 &= -\frac{1}{4}(2c_1 + 10c_2 + 49c_3 + 288c_4), \\
A_0 &= -\frac{1}{30}(30c_0 + 75c_1 + 245c_2 + 1080c_3 + 6239c_4).
\end{aligned}$$

(f): *The case $s = 5$. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 1)M_n + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n,$$

where

$$\begin{aligned}
A_5 &= -\frac{1}{6}c_5, \\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5), \\
A_3 &= -\frac{1}{12}(3c_3 + 30c_4 + 245c_5), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5), \\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5), \\
A_0 &= -\frac{1}{30}(30c_0 + 75c_1 + 245c_2 + 1080c_3 + 6239c_4 + 45000c_5).
\end{aligned}$$

(g): *The case $s = 6$. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 1)M_n + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n,$

where

$$\begin{aligned}
A_6 &= -\frac{1}{7}c_6,\\
A_5 &= -\frac{1}{6}(c_5+15c_6),\\
A_4 &= -\frac{1}{10}(2c_4+25c_5+245c_6),\\
A_3 &= -\frac{1}{12}(3c_3+30c_4+245c_5+2160c_6),\\
A_2 &= -\frac{1}{6}(2c_2+15c_3+98c_4+720c_5+6239c_6),\\
A_1 &= -\frac{1}{12}(6c_1+30c_2+147c_3+864c_4+6239c_5+54000c_6),\\
A_0 &= -\frac{1}{210}(210c_0+525c_1+1715c_2+7560c_3+43673c_4+315000c_5+2726645c_6).
\end{aligned}$$

(h): *The case $s=7$. For*

$$W_n = 3W_{n-1}-2W_{n-2}+c_7n^7+c_6n^6+c_5n^5+c_4n^4+c_3n^3+c_2n^2+c_1n+c_0,$$

the closed-form solution is

$W_n(0,1) = (c_0+3c_1+11c_2+51c_3+299c_4+2163c_5+18731c_6+189171c_7+1)M_n + A_7n^8 + A_6n^7+A_5n^6+A_4n^5+A_3n^4+A_2n^3+A_1n^2+A_0n,$

where

$$\begin{aligned}
A_7 &= -\frac{1}{8}c_7,\\
A_6 &= -\frac{1}{14}(2c_6+35c_7),\\
A_5 &= -\frac{1}{12}(2c_5+30c_6+343c_7),\\
A_4 &= -\frac{1}{10}(2c_4+25c_5+245c_6+2520c_7),\\
A_3 &= -\frac{1}{24}(6c_3+60c_4+490c_5+4320c_6+43673c_7),\\
A_2 &= -\frac{1}{6}(2c_2+15c_3+98c_4+720c_5+6239c_6+63000c_7),\\
A_1 &= -\frac{1}{12}(6c_1+30c_2+147c_3+864c_4+6239c_5+54000c_6+545329c_7),\\
A_0 &= -\frac{1}{210}(210c_0+525c_1+1715c_2+7560c_3+43673c_4+315000c_5+2726645c_6+27536040c_7).
\end{aligned}$$

In the following Example, we consider Theorem 1.1 (b) for

$$\begin{aligned}
a_1 &= 3,\ a_2=-2,\\
W_0 &= 2,\ W_1=3,
\end{aligned}$$

so that we have the case $V_n = H_n$ (Mersenne-Lucas numbers).

EXAMPLE 2.6. *In this example, we consider the case* $a_1 = 3,\ a_2 = -2, W_0 = 2,\ W_1 = 3$ *in Theorem 1.1 (b). So* $V_n(2,3) = V_n$ *represents nth Mersenne-Lucas number, i.e.,* $V_n = H_n$*).*

(a): *The case* $s = 0$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_0,$$

is given by

$$W_n(2,3) = (3c_0 + 1)H_n - 4c_0 H_{n-1} - c_0 n,$$

i.e.,

$$W_n(2,3) = 2c_0 H_{n+1} - (3c_0 - 1)\, H_n - c_0 n.$$

(b): *The case* $s = 1$*. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_1 n + c_0,$$

the closed-form solution is

$$W_n(2,3) = (3c_0 + 9c_1 + 1)H_n - 4(c_0 + 3c_1)H_{n-1} + A_1 n^2 + A_0 n,$$

i.e.,

$$W_n(2,3) = 2(c_0 + 3c_1)H_{n+1} - (3c_0 + 9c_1 - 1)H_n + A_1 n^2 + A_0 n,$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{2}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case* $s = 2$*. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 1)H_n - 4(c_0 + 3c_1 + 11c_2)H_{n-1} + A_2 n^3 + A_1 n^2 + A_0 n,$$

i.e.,

$$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2)H_{n+1} - (3c_0 + 9c_1 + 33c_2 - 1)H_n + A_2 n^3 + A_1 n^2 + A_0 n,$$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2). \end{aligned}$$

(d): *The case* $s=3$. *For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3)H_{n-1} + A_3n^4 + A_2n^3 + A_1n^2 + A_0n,$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 - 1)H_n + A_3n^4 + A_2n^3 + A_1n^2 + A_0n,$

where

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{6}(2c_2 + 15c_3), \\ A_1 &= -\frac{1}{4}(2c_1 + 10c_2 + 49c_3), \\ A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2 + 216c_3). \end{aligned}$$

(e): *The case* $s=4$. *For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

the closed-form solution is

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4)H_{n-1} + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 - 1)H_n + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$\begin{aligned} A_4 &= -\frac{1}{5}c_4, \\ A_3 &= -\frac{1}{4}(c_3 + 10c_4), \\ A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4), \\ A_1 &= -\frac{1}{4}(2c_1 + 10c_2 + 49c_3 + 288c_4), \\ A_0 &= -\frac{1}{30}(30c_0 + 75c_1 + 245c_2 + 1080c_3 + 6239c_4). \end{aligned}$$

(f): *The case* $s=5$. *For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0,$$

the closed-form solution is

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5)H_{n-1} + A_5 n^6 + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 - 1)H_n + A_5 n^6 + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$

where

$$\begin{aligned}
A_5 &= -\frac{1}{6}c_5,\\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5),\\
A_3 &= -\frac{1}{12}(3c_3 + 30c_4 + 245c_5),\\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5),\\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5),\\
A_0 &= -\frac{1}{30}(30c_0 + 75c_1 + 245c_2 + 1080c_3 + 6239c_4 + 45000c_5).
\end{aligned}$$

(g): *The case $s = 6$. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 56\,193c_6 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18\,731c_6)H_{n-1} + A_6 n^7 + A_5 n^6 + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18\,731c_6)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 56\,193c_6 - 1)H_n + A_6 n^7 + A_5 n^6 + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$

where

$$\begin{aligned}
A_6 &= -\frac{1}{7}c_6,\\
A_5 &= -\frac{1}{6}(c_5 + 15c_6),\\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5 + 245c_6),\\
A_3 &= -\frac{1}{12}(3c_3 + 30c_4 + 245c_5 + 2160c_6),\\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5 + 6239c_6),\\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5 + 54000c_6),\\
A_0 &= -\frac{1}{210}(210c_0 + 525c_1 + 1715c_2 + 7560c_3 + 43673c_4 + 315000c_5 + 2726645c_6).
\end{aligned}$$

(h): *The case $s = 7$. For*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_7 n^7 + c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0,$$

the closed-form solution is

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 56193c_6 + 567513c_7 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 189171c_7)H_{n-1} + A_7 n^8 + A_6 n^7 + A_5 n^6 + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 189171c_7)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 56193c_6 + 567513c_7 - 1)H_n + A_7 n^8 + A_6 n^7 + A_5 n^6 + A_4 n^5 + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n,$

where

$$\begin{aligned}
A_7 &= -\frac{1}{8}c_7, \\
A_6 &= -\frac{1}{14}(2c_6 + 35c_7), \\
A_5 &= -\frac{1}{12}(2c_5 + 30c_6 + 343c_7), \\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5 + 245c_6 + 2520c_7), \\
A_3 &= -\frac{1}{24}(6c_3 + 60c_4 + 490c_5 + 4320c_6 + 43673c_7), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5 + 6239c_6 + 63000c_7), \\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5 + 54000c_6 + 545329c_7), \\
A_0 &= -\frac{1}{210}(210c_0 + 525c_1 + 1715c_2 + 7560c_3 + 43673c_4 + 315000c_5 + 2726645c_6 + 27536040c_7).
\end{aligned}$$

Conclusion

This paper has established explicit closed-form expressions for second-order nonhomogeneous linear recurrence relations, formulated as generalized Leonardo-type sequences with polynomial inputs. The analysis was organized around the multiplicity parameter r, which measures the occurrence of the root 1 in the characteristic equation. Two principal cases were emphasized: the non-resonant case $r = 0$, in which both characteristic roots differ from unity, and the resonant case $r = 1$, in which unity appears as a simple root. Solutions were obtained for input polynomials of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$, thereby clarifying how root multiplicity and polynomial degree jointly determine the structure of the particular solution, while the homogeneous component remains governed by the same recurrence relation.

Applications were carried out for two distinguished families: the generalized Jacobsthal and generalized Mersenne sequences. In the Jacobsthal case, the characteristic roots $\{2, -1\}$ exclude unity, so Theorem 1.1(a) applies with $r = 0$. Classical Jacobsthal and Jacobsthal–Lucas numbers were recovered from the choices

$(W_0, W_1) = (0, 1)$ and $(2, 1)$, respectively. In contrast, the Mersenne case involves roots $\{2, 1\}$, where unity occurs as a simple root; hence Theorem 1.1(b) applies with $r = 1$. Classical Mersenne and Mersenne–Lucas numbers were obtained from the initial values $(W_0, W_1) = (0, 1)$ and $(2, 3)$. For each family, closed-form solutions were derived under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$, illustrating how the framework adapts to constant, linear, quadratic, cubic, quartic, quintic, sextic, and septic forcing terms.

Beyond theoretical contributions, the results carry both pedagogical and practical significance. For teaching, they provide accessible models that allow students to explore nonhomogeneous recurrences with polynomial inputs of varying degrees without excessive computation. For research, they supply resonance-aware formulas and explicit derivations that can be applied in mathematics, computer science, engineering, and physics. In this way, the study strengthens instructional practice while extending the reach of recurrence theory across multiple disciplines.

The unified framework consolidates recurrence theory and opens pathways for modern applications. By supplying closed-form solutions that account for resonance phenomena, the method can be adapted to optimization problems in transportation, coding theory, and discrete modeling. Exact formulas are particularly valuable because they yield predictable outcomes for complex systems, avoiding the computational burden of iterative evaluation. The framework developed here extends recurrence theory, clarifies the role of root multiplicity, and provides tools relevant for algorithm analysis, numerical modeling, and optimization problems where efficiency and precision are essential.

At the same time, the iterative scheme introduced in this paper produces explicit polynomial-type particular solutions, but its scope is currently limited to polynomial inputs. Extending the framework to accommodate non-polynomial functions—such as trigonometric or other analytic inputs—would require further methodological refinements and may introduce new resonance effects. Moreover, while the approach successfully clarifies the role of root multiplicity and resonance corrections, computational complexity grows rapidly with higher-order recurrences. This suggests that practical implementation for advanced cases will benefit from integration with computer algebra systems or symbolic computation platforms. In this respect, the current framework provides a solid foundation while pointing toward natural directions for further generalization and computational support.

Finally, the methodology opens promising avenues for future research. It can be incorporated into symbolic computation platforms, applied to verify classical identities in recurrence theory, and extended to interdisciplinary contexts such as cryptography, coding theory, and discrete modeling in physics and biology. By acknowledging current limitations while outlining directions for extension, the study offers a balanced perspective: consolidating the contribution of the present results while pointing toward further exploration and development.

References

[1] Abd-Elhameed, W.M., Alqubori, O.M., Alluhaybi, A.A., Amin, A.K., Novel Expressions for Certain Generalized Leonardo Polynomials and Their Associated Numbers, Axioms, 14, 286, 2025. https://doi.org/10.3390/

[2] Catarino, P., Borges, A., On Leonardo Numbers, Acta Mathematica Universitatis Comenianae, 89(1), 75–86, 2020. Available online at: http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005/650.

[3] Dikmen, C.D., Properties of Gaussian Generalized Leonardo Numbers, Karaelmas Science and Engineering Journal, 15(1), 134-145, 2025. DOI: 10.7212/karaelmasfen.1578154

[4] Göcen, M., Soykan, Y., On Generalized Avicenna Numbers, Mathematical Methods in the Applied Sciences, 0:1-17, 2025. https://doi.org/10.1002/mma.11103

[5] Gökbaş, H., k-Leonardo Numbers, Palestine Journal of Mathematics, 13(4), 1427-1435, 2024.

[6] İşbilir, Z., Akyiğit, M., Tosun, M., Pauli–Leonardo Quaternions, Notes on Number Theory and Discrete Mathematics, 29(1), 1-16, 2023. DOI: 10.7546/nntdm.2023.29.1.1-16

[7] Jeske, J.A., Linear Recurrence Relations, Part I, The Fibonacci Quarterly, 1(2), 69-74, 1963.

[8] Jeske, J.A., Linear Recurrence Relations, Part II, The Fibonacci Quarterly, 1(4), 34-39, 1963.

[9] Jeske, J.A., Linear Recurrence Relations, Part III, The Fibonacci Quarterly, 2(3), 197-203, 1964.

[10] Kuhapatanakul, K., Chobsorn, J, On the Generalized Leonardo Numbers, Integers 22, 2022, #A48.

[11] Kuhapatanakul, K., Ruankong, P., On Generalized Leonardo p-numbers, Journal of Integer Sequences, 27, Article 24.4.6, 2024.

[12] Prasad, K., Kumari, M., The Leonardo Polynomials and Their Algebraic Properties. Proceedings of the Indian National Science Academy, 2024. https://doi.org/10.1007/s43538-024-00348-0

[13] Özimamoğlu, H., On Leonardo Sedenions, Afrika Matematika (2023) 34:26, 2023. https://doi.org/10.1007/s13370-023-01065-5

[14] Özkan, E., Akkuş, H., Generalized Bronze Leonardo Sequence, Notes on Number Theory and Discrete Mathematics, 30(4), 811-824, 2024. DOI: 10.7546/nntdm.2024.30.4.811-824

[15] Shannon, A.G., A Note On Generalized Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 25(3), 97–101, 2019. DOI: 10.7546/nntdm.2019.25.3.97-101

[16] Shannon, A.G., Deveci, Ö., A Note on Generalized and Extended Leonardo Sequences, Notes on Number Theory and Discrete Mathematics, 28(1), 109–114, 2022. DOI: 10.7546/nntdm.2022.28.1.109-114

[17] Shannon, A.G., Shiue, P.J.S., Huang, S.C., Notes on Generalized and Extended Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 29(4), 752–773, 2023. DOI: 10.7546/nntdm.2023.29.4.752-773

[18] Soykan, Y., A Study On Generalized Mersenne Numbers, Journal of Progressive Research in Mathematics, 18(3), 90-112, 2021.

[19] Soykan, Y., Generalized Horadam-Leonardo Numbers and Polynomials, Asian Journal of Advanced Research and Reports, 17(8), 128-169, 2023. https://doi.org/10.9734/ajarr/2023/v17i8511

[20] Soykan, Y., Interrelations between Horadam and Generalized Horadam-Leonardo Polynomials via Identities, International Journal of Advances in Applied Mathematics and Mechanics, 11(1), 42-55, 2023. ISSN: 2347-2529

[21] Soykan, Y., Generalized Leonardo Numbers, Journal of Progressive Research in Mathematics, 18(4), 58-84, 2021.

[22] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026.

[23] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

[24] Soykan, Y., Explicit Formulas for Polynomial-type Particular Solutions of Generalized Leonardo-type Sequences, Asian Journal of Advanced Research and Reports, 20(5), 164-208, 2026. https://doi.org/10.9734/ajarr/2026/v20i51361