Counting methods of area integrals on the unit disk

Abstract

This paper considers selected methods for evaluating area integrals associated with domains in the complex plane, with emphasis on the unit disk, exterior conformal mappings, ellipses, and symmetric lemniscates. The study begins with the classical representation of the area enclosed by a simple closed curve through a line integral and uses this representation to discuss Gronwall's area formula for mappings defined outside a disk. The formula is then applied to elementary geometric regions, including circles and ellipses, and to regions related to symmetric lemniscates through suitable Laurent-series representations of conformal mappings. In addition, the paper reviews the use of Green's and Stokes' formula for converting certain planar area integrals into contour integrals. This approach is further connected with expansions involving Tchebychev polynomials and with orthogonality relations for polynomials of the second kind over an elliptical domain with respect to the two-dimensional Lebesgue measure. The manuscript also derives coefficient relations for functions represented by Tchebychev polynomial expansions and discusses primitive functions expressed in the same basis. The results presented illustrate how classical tools from complex analysis, conformal mapping theory, and orthogonal polynomial theory can be combined to compute or represent area integrals over specific planar regions. The treatment is mainly theoretical and focuses on formula derivation rather than numerical computation. Overall, the work aims to provide a unified exposition of several area-counting techniques on the unit disk and related domains, using conformal mappings, boundary integrals, and polynomial expansions as the principal analytical tools.

Keywords: Unit disk, Conformal mapping ,Area integral,Gronwall's area formula, Univalents functions,regular functions ,Circles, Ellipses ,Lemniscates, Differentiation.

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1 Introduction

The paper will be structured as follows : in section 1 we present some useful terminology as well as some necessary definitions regarding the counting methods of area integrals on the unit disc with respect to the two-dimensional Lebesgue planar measure $dm(x,y) = dxdy$,where $z = x + iy$. In section 2,first we present the Gronwall's area formula in the case of general simple closed curve C is defined parametrically in the complex w-plane

$$u = u(\theta) \quad ,v = v(\theta) \quad ,0 \leq \theta \leq 2\pi$$

Then the area A enclosed by C is given by,

$$A = \frac{1}{2}\int_{0}^{2\pi}\left(u\frac{dv}{d\theta} - v\frac{du}{d\theta}\right)d\theta$$

we prove If there is a conformal mapping Ψ_r,from the exterior of D_r ,$r > 1$ to the exterior of C_r of the form

$$\Psi_r(z) = z + \sum_{n=0}^{\infty}\frac{b_n}{z^n}$$

then by Gronwall's area formula, the area A_r of the region B_r enclosed by C_r is given by,

$$A_r = \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right)$$

Second,we use Gronwall's area formula to find the area of some differents regions as circles with radius r centred ot the origin lying in the complex w-plane,ellipses and lemniscates.

A lemniscate shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2,3,4.....$$

We use Laurent and Taylor series expansions of conformal mapping from the exterior of the unit disk to either of these regions to compute the area of theses bounded regions.

One of other counting methods of area integrals on the unit disc and ellipses is so called Green's and Stokes formula,[7],[18],[16],[3],[4],[5]

$$\iint_D f(z)\overline{g'(z)}dxdy = \frac{1}{2i}\int_C f(z)\overline{g(z)}dz$$

where the domain D is enclosed by closed curve C in the complex z-plane .We apply this formula to get some results concerning counting methods of area integrals on the unit disc and ellipses.As application of Green's and Stokes formula on the ellipse D : $b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.

We consider a new system of orthogonal Tchebychev polynomials of second kind $\{U_n(z)\}_{n=0,1,2...}$,with respect to the Lebesgue planar measure $dm(x,y) = dxdy$,where $z = x + iy$ concentrated on the ellipse D: $b^2x^2 + a^2y^2 < a^2b^2$ where $a > b$,a system of orthogonal polynomials, given by :

$$U_n(z) = \frac{T'_{n+1}(z)}{n+1} = \frac{\sin\left((n+1)\cos^{-1}z\right)}{\sqrt{1-z^2}} \quad , \quad n = 0,1,2....$$

where $T_n(z) = \cos\left(n\cos^{-1} z\right), n = 0, 1, 2, 3....$is a polynomial of degree n . $T_n(z)$ is called the Tchebychev polynomial of degree n of first kind .

They satisfies

$$\iint\limits_{D} U_n(z)\overline{U_m(z)}dxdy = \frac{4(n+1)}{\pi\left(\rho^{n+1}-\rho^{-n-1}\right)}\delta_{n,m} \qquad ,n, m = 0, 1, 2....$$

where $\delta_{n,m}$, is the symbol of Kronecker and $(a+b)^2 = \rho$.

Let $f(z)$ be regular in a domain D.If

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$$

which converges uniformly and absolutely in any closed subdomain of ellipse D.By applying the Green's and Stokes formula,we show that

$$a_{n+1} = \frac{i\pi\left(\rho^{n+1}-\rho^{-n-1}\right)}{8(n+1)^2}\int\limits_{C} f(z)\overline{U_n(z)}d\overline{z} \quad , \rho = (a+b)^2 , \; n = 0, 1, 2, ...$$

where C: is the ellipse :$b^2x^2 + a^2y^2 = a^2b^2$.

Suppose $G \subset \mathbb{C}$ is an arbitrary domain ,f is analytic in G and set,[7],[18],[16],[1],[10]

$$I(f) = \iint\limits_{G} |f(z)|^2 \, dm \qquad ,z = x + iy \tag{1}$$

where $dm(x, y) = dxdy$ is the two-dimensional area Lebesgue measure.$I(f)$ can be represented explicitily in terms of the coefficients of f .

Let us compute $I(f)$ for a special case .Let $G = \{z : r < |z| < R\}, 0 \le r < R < \infty$,and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad ,z \in G$$

it follows that,

$$I(f) = \int\limits_{\rho=r}^{R}\int\limits_{\phi=0}^{2\pi}\left(\sum_{n=-\infty}^{\infty} a_n\rho^n e^{in\phi}\right)\left(\sum_{m=-\infty}^{\infty} \overline{a_m}\rho^m e^{-im\phi}\right)\rho d\rho d\phi$$

$$= 2\pi\int\limits_{\rho=r}^{R}\sum_{n=-\infty}^{\infty} |a_n|^2 \rho^{2n+1}d\rho$$

$$= 2\pi\sum_{n=-\infty}^{\infty} |a_n|^2\int\limits_{\rho=r}^{R}\rho^{2n+1}d\rho$$

i-e

$$I(f) = \pi\sum_{n=-\infty}^{\infty}\frac{R^{2n+2}-r^{2n+2}}{n+1}|a_n|^2 \tag{2}$$

Since

$$a_0 = f(0), a_1 = f'(0) \ldots\ldots\ldots a_k = \frac{f^{(k)}(0)}{k!}$$

We can state

$$|f(0)|^2 \leq \frac{1}{R^2 - r^2} I(f) \tag{3}$$

and

$$\left|f'(0)\right|^2 \leq \frac{2}{R^4 - r^4} I(f) \tag{4}$$

Hence

$$\left|f^{(k)}(0)\right| \leq \frac{(k+1)!}{R^{2k+2} - r^{2k+2}} I(f) \tag{5}$$

Definition 1

$$L^2(G) = \{f : f \ \textit{analytic in } G \textit{ and } \ I(f) < \infty\} \tag{6}$$

in this case,if $d_z = dist(z, \partial G)$

$$|f(z)|^2 \leq \frac{I(f)}{\pi d_z^2} \tag{7}$$

Theorem 2 ***Caratheodory*** *theorem [17],[16],[13],[1],[7],[2].*

The conformal mapping φ *of* G *onto the disk* $D_R = \{w, |w| < R\}$ *such that*

$$\varphi(\xi) = 0, \qquad \textit{and} \quad \varphi'(\xi) = 1$$

can be conformally continued by ***Caratheodory*** *theorem,[17],[16],[13],[1],[7],[2] If we suppose that*

$$\psi : D_R = \{w, |w| < R\} \rightarrow G$$

is its inverse map, then we have,

$$\varphi(t) = u \Longleftrightarrow t = \psi(u) \qquad\qquad (t \in E)$$

where

$$\varphi(t) = \lim_{\substack{z \longrightarrow t \\ z \in G}} \varphi(z) \qquad\qquad (\ t \in E)$$

and

$$\psi\left(e^{i\theta}\right) = \lim_{r \longrightarrow 1^-} \psi\left(re^{i\theta}\right) \qquad\qquad (0 \leq \theta \leq 2\pi)$$

These limit boundary values of φ *and* ψ *are respectively bijections and inversely from the contour* E *onto unit circle* $\{u : |u| = 1\}$.

If $w = u + iv = f(z)$, $z = x + iy \in D$, whose boundary is the closed curve C ,the Jacobien of the transformation , $u, v \longrightarrow x, y$ is $\left|f'(z)\right|^2$.

Hence the area of $f(D)$ can be expressed as integral **:**

$$A = \iint\limits_{f(D)} dudv = \iint\limits_{D} \left|f'(z)\right|^2 dxdy \tag{8}$$

By Green's formula,[7],[18],[16],[3],[4],[5]

$$\iint\limits_{D} f(z)\, \overline{g'(z)} dxdy = \frac{1}{2i} \int\limits_{C} f(z)\, \overline{g(z)} dz \tag{9}$$

Notice that

$$A=\iint\limits_{D}\left|f'(z)\right|^{2}dxdy=\iint\limits_{D}f(z)\,\overline{f'(z)}dxdy \tag{10}$$

ie

$$A=\iint\limits_{D}\left|f'(z)\right|^{2}dxdy=\frac{1}{2i}\int\limits_{C}f'(z)\,\overline{f(z)}dz \tag{11}$$

If f is regular and univalent in a domain D and satisfies $|f(z)|\leq 1$ then ,

$$\iint\limits_{D}\left|f'(z)\right|^{2}dxdy\leq\pi \tag{12}$$

Suppose that we are given the function analytique on the unit disk $D=\{|z|\leq 1\}$ function $f(z)=\sum\limits_{n=0}^{\infty}a_{n}z^{n}$ .Regarding the magnitude of the coefficients can be obtanined by means of Parseval's identity,,[7],[18],[16],[6]

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{2}d\theta=\sum_{n=0}^{\infty}|a_{n}|^{2}r^{2n} \tag{13}$$

which has also many other useful applications in the theory of univalents and holomorphics functions.To prove (13),we observe that

$$\left|f\left(re^{i\theta}\right)\right|^{2}=f\left(re^{i\theta}\right)\overline{f\left(re^{i\theta}\right)}=\sum_{n=0}^{\infty}a_{n}r^{n}e^{in\theta}\sum_{n=0}^{\infty}\overline{a_{n}}r^{n}e^{-in\theta}$$

i-e

$$\left|f\left(re^{i\theta}\right)\right|^{2}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{n}\overline{a_{m}}r^{n+m}e^{i(n-m)\theta}$$

The rearrangement of the terms are permissible .Hence,

$$\int\limits_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{2}d\theta=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{n}\overline{a_{m}}r^{n+m}\int\limits_{0}^{2\pi}e^{i(n-m)\theta}d\theta$$

reduces to

$$\int\limits_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{2}d\theta=2\pi\sum_{n=0}^{\infty}|a_{n}|^{2}r^{2n} \tag{14}$$

If $|f(z)|\leq M(r)$ for $|z|=r$, then obviously,[7],[18],[16],[10]

$$\sum_{n=0}^{\infty}|a_{n}|^{2}r^{2n}\leq M^{2}(r)$$

We known that If

$$f(z)=\sum_{n=0}^{\infty}a_{n}z^{n}\qquad ,|z|\leq r$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \qquad , |z| \le \rho$$

then,[7],[18],[16],[11],[12],[13],[15]

$$\frac{1}{2\pi i} \int\limits_{|w|=r} f(w)\, g\left(\frac{z}{w}\right) \frac{dw}{w} = \sum_{n=0}^{\infty} a_n b_n z^n \qquad , |z| < \rho r \tag{15}$$

In polar coordinates r, θ of $|z| = r$,expressed as R, ϕ in the w-plane ,when f maps $|z| = r$, $r < 1$ in the positive direction , to the unit circle,$f\left(re^{i\theta}\right) = R(r,\theta) + i\phi(r,\theta)$

$$A = -\frac{1}{2} \int_{0}^{2\pi} R^2(r,\theta) \frac{\partial \phi}{\partial \theta}(r,\theta)\, d\theta \tag{16}$$

Since by the Cauchy Riemann equations in polar coordinates r, θ

$$\frac{\partial \phi}{\partial \theta}(r,\theta) = \frac{r}{R} \frac{\partial R}{\partial r}(r,\theta) \tag{17}$$

it follows that,[7],[18],[16],[21]

$$A = -\frac{r}{4} \frac{\partial}{\partial r} \left(\int_{0}^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \right) \tag{18}$$

Example 3 *Let M is a region whose boundary is the cardioid has a parametric polar equation,for $0 \le \theta \le 2\pi$*

$$r = \frac{1}{2}(1 + \cos\theta)$$

The area $A_{cardioid}$ can be found by integration .We have

$$A_{cardioid} = \int_{0}^{\pi} r^2 d\theta = \frac{1}{4} \int_{0}^{\pi} (1 + \cos\theta)^2\, d\theta$$

$$= \frac{1}{4} \int_{0}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \right) d\theta = \frac{3\pi}{8}$$

Proposition 4 *The class of univalents functions in $|z| < 1$ which have a Laurent expansion,[7],[18],[16],,*

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \tag{19}$$

The coefficients $b_0, b_1, b_2 .....b_n ....$ are subject to the inequality

$$\sum_{n=0}^{\infty} n\, |b_n|^2 \le 1 \tag{20}$$

Proof.

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta = \int_0^{2\pi} f\left(re^{i\theta}\right) \overline{f\left(re^{i\theta}\right)} d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{re^{i\theta}} + \sum_{n=0}^{\infty} b_n r^n e^{in\theta} \right] \left[\frac{1}{re^{-i\theta}} + \sum_{n=0}^{\infty} b_n r^n e^{-in\theta} \right] d\theta$$

$$= 2\pi \left[\frac{1}{r^2} + \sum_{n=0}^{\infty} |b_n|^2 r^{2n} \right]$$

Hence,

$$A = -\frac{r}{4}\frac{\partial}{\partial r} \left(\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \right)$$

then

$$A = -\frac{\pi r}{2}\frac{\partial}{\partial r} \left[\frac{1}{r^2} + \sum_{n=0}^{\infty} |b_n|^2 r^{2n} \right]$$

and thus

$$\frac{1}{\pi} A = \frac{1}{r^2} - \sum_{n=1}^{\infty} n\,|b_n|^2 r^{2n}$$

Since the area can not negative it follows that

$$\sum_{n=1}^{\infty} n\,|b_n|^2 r^{2n} \leq \frac{1}{r^2}$$

holds for all values of r between 0 and 1,see,[7],[18],[16],[20],[8].It therefore must also be true for $r \longrightarrow 1$.This proves ,(20). ■

Our first result is to prove and apply Gronwall's area formula in the next theorem,we show that If there is a conformal mapping Ψ_r ,from the exterior of D_r ,$r > 1$ to the exterior of C_r of the form

Theorem 5

$$\Psi_r(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

then the area A_r of the region B_r enclosed by C_r is given by,

$$A_r = \pi \left(r^2 - \sum_{n=1}^{\infty} n\,|b_n|^2 r^{-2n} \right)$$

the orthogonality of Tchebychev polynomials of the second kind ,$\{T_n(z)\}_{n=0,1,2,3,...}$ with respect to the Lebesgue planar measure concentrated on the ellipse : $D : b^2x^2+a^2y^2 < a^2b^2$ where $a > b$,we show that

$$\iint_D U_n(z)\overline{U_m(z)}dxdy = \frac{4(n+1)}{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}\delta_{n,m} \qquad ,n, m = 0, 1, 2, 3, ...$$

where $\delta_{n,m}$, is the symbol of Kronecker and $(a+b)^2 = \rho$.

2 Gronwall's area formula

Lemma 6 *Suppose that a simple closed curve C is defined parametrically in the complex w-plane*

$$u = u(\theta) \quad ,v = v(\theta) \quad ,0 \leq \theta \leq 2\pi \tag{21}$$

Then the area A enclosed by C is given by,[7],[18],[6],[19],[20].

$$A = \frac{1}{2}\int_{0}^{2\pi}\left(u\frac{dv}{d\theta} - v\frac{du}{d\theta}\right)d\theta \tag{22}$$

Proof. By Green's theorem ,if $p = p(u,v)$ and $q = q(u,v)$ are functions with continuous partial derivatives on a region B enclosed by C ,[7],[18],[16]then

$$A = \iint_{B}\left(\frac{\partial p}{\partial u} + \frac{\partial q}{\partial v}\right)dudv = \int_{C}(-qdu + pdv)$$

Choosing ,$p = u$,$q = v$ gives

$$A = \iint_{B} dudv = \frac{1}{2}\int_{C}(-vdu + udv) = \frac{1}{2}\int_{0}^{2\pi}\left(u\frac{dv}{d\theta} - v\frac{du}{d\theta}\right)d\theta$$

We shall use this result to prove Gronwall's area theorem. ■

Theorem 7 *If there is a conformal mapping Ψ_r ,from the exterior of D_r ,$r > 1$ to the exterior of C_r of the form*

$$\Psi_r(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + ...\frac{b_n}{z^n} + ... \tag{23}$$

then the area A_r of the region B_r enclosed by C_r is given by,[7],[18],[16],[20],[15],

$$A_r = \pi\left(r^2 - \sum_{n=1}^{\infty} n\left|b_n\right|^2 r^{-2n}\right) \tag{24}$$

Proof. Firstly,we write the mapping Ψ_r as

$$w = \Psi_r(z) = u(r,\theta) + iv(r,\theta) \qquad , \quad z = re^{i\theta}$$

where u and v are real-valued functions.By (22) and since r is constant, the area enclosed by C_r is given by

$$A_r = \frac{1}{2}\int_{0}^{2\pi}\left(u\frac{dv}{d\theta} - v\frac{du}{d\theta}\right)d\theta \tag{25}$$

We now want to prove that

$$A_r = \operatorname{Im}\left\{\frac{1}{2}\int_{0}^{2\pi}\overline{\Psi_r(z)}\Psi_r'(z)\,dz\right\} \tag{26}$$

where Im denote the imaginary part .Since

$$\Psi_r'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{e^{-i\theta}}{r}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right) \tag{27}$$

on using the Cauchy Riemann equations in polar coordinates form,.In fact $dz = ire^{i\theta}d\theta$,we see that

$$\mathrm{Im}\left\{\frac{1}{2}\int_0^{2\pi}\overline{\Psi_r(z)}\Psi_r'(z)\,dz\right\} = \mathrm{Im}\left\{\frac{1}{2}\int_0^{2\pi}(u-iv)\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)d\theta\right\}$$

$$= \frac{1}{2}\int_0^{2\pi}\left(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}\right)d\theta = A_r$$

We can now rewrite equation (23) as

$$\Psi_r(z) = \sum_{n=-1}^{\infty}\frac{b_n}{z^n} \qquad , b_{-1} = 1 \tag{28}$$

Using (28) and (26) we have

$$A_r = \mathrm{Im}\left\{\frac{1}{2}\int_0^{2\pi}\left(\sum_{n=-1}^{\infty}\frac{\overline{b}_n}{\overline{z}^n}\sum_{m=-1}^{\infty}\frac{-mb_m}{z^{m+1}}\right)dz\right\}$$

$$= \mathrm{Im}\left\{-\frac{1}{2}\sum_{m=-1}^{\infty}\sum_{n=-1}^{\infty}mb_m\overline{b}_n\int_{|z=r|}\overline{z}^{-n}z^{m-1}dz\right\}$$

By writing $z = re^{i\theta}$,we find

$$\int_{|z=r|}\overline{z}^{-n}z^{m-1}dz = ir^{-m-n}\int_0^{2\pi}e^{i(n-m)}d\theta = 0 \qquad , n \neq m$$

and

$$\int_{|z=r|}\overline{z}^{-n}z^{n-1}dz = 2\pi i r^{-2n}$$

therefore

$$A_r = \mathrm{Im}\left\{-i\pi\sum_{n=-1}^{\infty}n\left|b_n\right|^2 r^{-2n}\right\} = -\pi\sum_{n=-1}^{\infty}n\left|b_n\right|^2 r^{-2n}$$

$$= \pi\left(r^2 - \sum_{n=1}^{\infty}n\left|b_n\right|^2 r^{-2n}\right)$$

The last relation proves the theorem. ■

The second result of paper is applying Gronwall's area formula to find the area of some differents regions as circles with radius r centred ot the origin lying in the complex

w-plane , ellipses and lemniscates A lemniscate shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2, 3, 4.....$$

We use Laurent and Taylor series expansions of conformal mapping from the exterior of the unit disk to either of these regions to compute the area of them.We obtain the following corollary of Gronwall's area formula property.

3 Circles, ellipses and lemniscates

In this case C_r is the circle with radius r centred ot the origin lying in the complex w-plane .The conformal mapping from the exterior of D_r onto the exterior of C_r is given by :

$$w = \Psi_r(z) = z \tag{29}$$

Thus we see that all the b_n's from equation(23) are zero.By Gronwall's area theorem,the area A_r of the circle C_r is given by

$$A_r = \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right) = \pi r^2 \tag{30}$$

The conformal mapping from the exterior of the unit circle centred at the origin of the complex $z-$plane D_r onto an ellipse C_r in the complex w-plane with semi axis $a = r + \frac{1}{r}$,$b = r - \frac{1}{r}$ is
given by

$$w = \Psi_r(z) = z + \frac{1}{z} \tag{31}$$

Thus we see that all the b_n's from equation(23) are zero except for $b_1 = 1$.By Gronwall's area theorem,the area A_r of the circle C_r is given by

$$A_r = \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right) = \pi\left(r + \frac{1}{r}\right)\left(r - \frac{1}{r}\right)$$

i-e

$$A_r = \pi ab \tag{32}$$

which agrees with the classical formula for the area of an ellipse.

A lemniscate is shapped with any number of leaves .The boundary of m-leafed symmetric lemniscate is the set

$$\{w \in \mathbb{C}, |w^m - 1| = 0\}, m = 2, 3, 4..... \tag{33}$$

The conformal mapping from the exterior of the uniit circle centred at the origin of the complex $z-$plane D_r onto the exterior of the m-leafed symmetric lemniscate in the complex $w-$plane is given by

$$w = \Psi(z) = z\left(1 + \frac{1}{z^m}\right)^{\frac{1}{m}} \tag{34}$$

The image of this mapping to include the boundary of the lemniscate ,we look at the image on the unit circle in the the complex $z-$plane, that is, take $r = 1$.In this case

$$w^m = z^m + 1$$

so that

$$|w^m - 1| = |z|^m = 1$$

Theorem 8 *If z describe the unit circle.Then the area A of an m-leafed symetric lemniscate is given by*

$$A = \pi\left(\sum_{n=0}^{\infty}\left(C_n^{\frac{1}{m}}\right)^2 - m\sum_{n=1}^{\infty} n\left(C_n^{\frac{1}{m}}\right)^2\right) \tag{35}$$

Hence

$$A = 2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\sqrt{\pi} \tag{36}$$

Proof. If z describe the unit circle.Then by (34) for large $|z|$

$$\Psi(z) = z\sum_{n=0}^{\infty} C_n^{\frac{1}{m}}\frac{1}{z^{mn-1}} \tag{37}$$

According to Gronwall's area formula,(24) the area A of an m-leafed symetric lemniscate is given by

$$A = \pi\left[1 - \sum_{n=1}^{\infty} n\left|b_n\right|^2\right]$$

$$= \pi\left[1 - \sum_{n=1}^{\infty}(mn-1)\left|b_{mn-1}\right|^2\right]$$

by (37)

$$A = \pi m\sum_{n=1}^{\infty}\left(\frac{1}{m} - n\right)\left(C_n^{\frac{1}{m}}\right)^2$$

$$= \pi\left(\sum_{n=0}^{\infty}\left(C_n^{\frac{1}{m}}\right)^2 - m\sum_{n=1}^{\infty} n\left(C_n^{\frac{1}{m}}\right)^2\right)$$

and (35) is proved.To prove (36), we known ([6] page 5) ,we known that

$$\sum_{k=0}^{\infty}\left(C_k^{\alpha}\right)^2 = \frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2} \tag{38}$$

and

$$\sum_{k=0}^{\infty} k\left(C_k^{\alpha}\right)^2 = \frac{\Gamma(2\alpha)}{\left(\Gamma(\alpha)\right)^2} \tag{39}$$

Using these assumptions and proprietes of the Gamma function we find

$$A=\pi\left(\frac{\Gamma\left(\frac{2}{m}+1\right)}{\left(\Gamma\left(\frac{1}{m}+1\right)\right)^{2}}-m\frac{\Gamma\left(\frac{2}{m}\right)}{\left(\Gamma\left(\frac{1}{m}\right)\right)^{2}}\right)$$

By the Duplicate formula ([6] page 946),we get

$$A=2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\sqrt{\pi}$$

We will now confirm this result by using polar coordinates. Writing $w=\rho e^{i\phi}$,we have

$$|w^{m}-1|=1$$

or

$$\left(w^{m}-1\right)\left(\overline{w}^{m}-1\right)$$

i-e

$$\rho^{2m}-2\rho^{m}\cos m\phi=0$$

so that

$$\rho=2^{\frac{1}{m}}\cos^{\frac{1}{m}}\left(m\phi\right)$$

The area of one leaf of the m-leafed lemniscate is given by,[6]

$$\frac{1}{m}A=\frac{1}{2}\int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}}\rho^{2}d\phi$$

and thus the area of the m-leafed lemniscate is given by,[6]

$$A=m\int_{0}^{\frac{\pi}{2m}}\rho^{2}d\phi$$

$$=2^{\frac{2}{m}}m\int_{0}^{\frac{\pi}{2m}}\cos^{\frac{2}{m}}\left(m\phi\right)d\phi$$

If $u=\cos m\phi$,then

$$A=2^{\frac{2}{m}}\int_{0}^{1}\frac{u^{\frac{2}{m}}}{\sqrt{1-u^{2}}}du$$

If $v=u^{2}$,then

$$A=2^{\frac{2}{m}-1}\int_{0}^{1}v^{\frac{1}{m}-\frac{1}{2}}\left(1-v\right)^{-\frac{1}{2}}dv$$

and therefore,by the Beta function,

$$A=2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+\frac{1}{2}+\frac{1}{2}\right)}$$

$$= 2^{\frac{2}{m}-1}\frac{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\sqrt{\pi}$$

and the theorem is proved. ■

In some cases to be considered ,we will take the limit $r \overset{>}{\longrightarrow} 1$ from above ,and assume that the image of the unit circle is the boundary of the region for which we are trying to find the area.

Proposition 9 *Consider the area integral*

$$\Im(f) = \int_0^{2\pi}\int_0^{2\pi} \frac{\left|f\left(e^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i\varphi} - e^{i\theta}\right|^2} d\theta d\varphi \tag{40}$$

If

$$\Im_r(f) = \int_{C_r}\int_{C_1} \frac{\left|f(y) - f(x)\right|^2}{\left|y - x\right|^2} dxdy \tag{41}$$

where ,$C_r = \{|z| = r, 0 \le r \le 1\}$ *,oriented in the positive direction then*

$$Lim_{r \overset{<}{\longrightarrow} 1} \Im_r(f) = \Im(f) \tag{42}$$

if $r < 1$*. Then,*

$$\int_{C_r}\int_{C_1} \frac{\left|f(y)\right|^2}{(y-x)^2} dxdy = 0 \qquad ,r < 1 \tag{43}$$

then

$$\int_{C_r}\int_{C_1} \frac{\left|f(x)\right|^2}{(y-x)^2} dxdy = 0 \tag{44}$$

$$\int_{C_r}\int_{C_r} \frac{\overline{f(x)}f(y)}{(y-x)^2} dx\ dy\ = 0 \qquad (|z| = 1, |w| < 1) \tag{45}$$

If

$$K_r(f) = \int_{C_r}\int_{C_1} \frac{\overline{f(y)}f(x)}{(y-x)^2} dxdy \tag{46}$$

then

$$\Im(f) = 4\pi^2 \sum_{p=0}^{\infty} p\left|a_p\right|^2 \tag{47}$$

and

$$A = \frac{1}{4\pi}\Im(f) \tag{48}$$

Proof. The function f is holomorphe on the disk $|z| \le 1$,then the function

$$F(\theta, \varphi) = \frac{\left|f\left(e^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i\varphi} - e^{i\theta}\right|^2}$$

is continued on $[0 \quad 2\pi] \times [0 \quad 2\pi]$,then $\Im(f)$ exist.Setting,

$$x = e^{i\theta}, y = e^{i\varphi}$$

we have

$$\left|e^{i\varphi} - e^{i\theta}\right|^2 = \left(e^{i\varphi} - e^{i\theta}\right)\left(e^{-i\varphi} - e^{-i\theta}\right) = -e^{-i(\theta+\varphi)}(y-x)^2$$

$$dx = ie^{i\theta}d\theta, dy = ie^{i\varphi}d\varphi$$

then

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\left|f\left(e^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i\varphi} - e^{i\theta}\right|^2}d\theta d\varphi = \int_{C_1}\int_{C_1}\frac{|f(y) - f(x)|^2}{|y-x|^2}dxdy$$

If

$$\Im_r(f) = -\int_0^{2\pi}d\varphi\int_0^{2\pi}\frac{\left|f\left(re^{i\varphi}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|re^{i\varphi} - e^{i\theta}\right|^2}re^{i(\theta+\varphi)}d\theta$$

then

$$Lim_{r \overset{<}{\longrightarrow} 1}\int_{C_r}dy\int_{C_1}\frac{|f(y) - f(x)|^2}{|y-x|^2}dx = \int_{C_1}dy\int_{C_1}\frac{|f(y) - f(x)|^2}{|y-x|^2}dx$$

i-e

$$Lim_{r \overset{<}{\longrightarrow} 1}\Im_r(f) = \Im(f)$$

we have

$$\int_{C_1}\frac{dx}{(y-x)^2} = 0 \qquad , |y| < 1$$

and

$$\int_{C_r}\frac{dy}{(y-x)^2} = 0 \qquad , |x| = 1$$

also we have

$$\int_{C_r}f(y)\frac{dy}{(y-x)^2} = 0 \qquad , |x| = 1$$

thus proprietes (43),(44),(45) are proved.We have,

$$|f(y) - f(x)|^2 = |f(y)|^2 + |f(x)|^2 - \overline{f(y)}f(x) - \overline{f(x)}f(y)$$

then,in view (43),(44),(45) we get

$$\Im_r(f) = -\int_{C_r}dy\int_{C_1}\overline{f(y)}f(x)\frac{dx}{(y-x)^2} = -K_r(f)$$

By residu's theorem at the pole $x = y$, we have

$$\int_{C_1}f(x)\frac{dx}{(y-x)^2} = 2i\pi f'(y)$$

then if $|y| = r < 1$,we get

$$\Im_r(f) = -K_r(f) = -2i\pi \int_{C_r} \overline{f(y)} f'(y)\, dy \tag{49}$$

$$= -2i\pi \int_{C_r} \overline{\sum_{q=0}^{\infty} r^q e^{i\varphi}} \sum_{p=0}^{\infty} p a_p r^p e^{i\theta} d\theta$$

$$= 2\pi \int_0^{2\pi} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} p a_p \overline{a_q} r^{p+q} e^{i(p-q)\varphi} d\varphi$$

$$= 4\pi^2 \sum_{p=0}^{\infty} p \left|a_p\right|^2 r^{2p}$$

Thus

$$Lim_{r \xrightarrow{<} 1} \Im_r(f) = 4\pi^2 \sum_{p=0}^{\infty} p \left|a_p\right|^2$$

By (49),and (**??**),we get

$$Lim_{r \xrightarrow{<} 1} \Im_r(f) = \frac{4\pi}{2i} \int_{C_1} f'(y)\, \overline{f(y)} dy = 4\pi A = \Im(f)$$

i-e

$$A = \frac{1}{4\pi} \Im(f)$$

The proposition is proved. ■

Proposition 10 *Let $z_1, z_2, z_3 .... z_p$ be n distincts points which are situated in the interior of $D = \{|z| < 1\}$,therefore*

$$\iint_D \left(\sum_{p=1}^{m} \left| \frac{1}{z - z_p} \right|^2 \right) dxdy = \pi \log \left| \prod_{p=1}^{m} \frac{1 - |z_p|^2}{|z_p|^2} \right| \tag{50}$$

and

$$\iint_D \left(\sum_{p=1}^{m} \left| \frac{z + z_p}{z - z_p} \right|^2 \right) dxdy = m\pi + 2\pi \log \left| \prod_{p=1}^{m} \frac{1 - |z_p|^2}{|z_p|^2} \right|^{2|z_p|^2} \tag{51}$$

Proof. If,

$$I_p = \frac{1}{\pi} \iint_D \left| \frac{1}{z - z_p} \right|^2 dxdy \quad , \quad z = x + iy$$

suppose $z = re^{i\theta} = x + iy$

$$dxdy = rdrd\theta, \quad 0 \le r < 1, \; 0 \le \theta \le 2\pi$$

then

$$I_p = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \frac{rdrd\theta}{\left(re^{i\theta}-z_p\right)\left(re^{-i\theta}-\overline{z_p}\right)}$$

$$= \frac{1}{\pi}\int_0^1 rdr\int_0^{2\pi} \frac{d\theta}{\left(re^{i\theta}-z_p\right)\left(re^{-i\theta}-\overline{z_p}\right)}$$

Setting

$$w = e^{i\theta}, dw = iwd\theta, 0 \le \theta \le 2\pi$$

In fact

$$|w| = 1$$

and

$$I_p = \frac{1}{\pi}\int_0^1 rdr\int_{C_1} \frac{\frac{dw}{iw}}{\left(rw-z_p\right)\left(\frac{r}{w}-\overline{z_p}\right)}$$

i-e

$$I_p = \frac{1}{\pi i}\int_0^1 dr\int_{C_1} \frac{dw}{\left(w-\frac{z_p}{r}\right)\left(r-w\overline{z_p}\right)}$$

now we state

$$\frac{1}{\pi i}\int_{C_1} \frac{dw}{\left(w-\frac{z_p}{r}\right)\left(r-w\overline{z_p}\right)} = \frac{2r}{r^2-|z_p|^2}$$

hence

$$I_p = \int_0^1 \frac{2r}{r^2-|z_p|^2}dr = \log\left|\frac{1-|z_p|^2}{|z_p|^2}\right|$$

then

$$\iint_D \left(\sum_{p=1}^m \left|\frac{1}{z-z_p}\right|^2\right) dxdy = \pi \log\left|\prod_{p=1}^m \frac{1-|z_p|^2}{|z_p|^2}\right|$$

To prove (51),setting

$$J_p = \frac{1}{2\pi}\iint_D \left|\frac{z+z_p}{z-z_p}\right|^2 dxdy \quad , \quad z = x+iy$$

suppose $z = re^{i\theta} = x+iy$

$$dxdy = rdrd\theta, \quad 0 \le r < 1, \; 0 \le \theta \le 2\pi$$

then

$$J_p = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{\left(re^{i\theta}+z_p\right)\left(re^{-i\theta}+\overline{z_p}\right)}{\left(re^{i\theta}-z_p\right)\left(re^{-i\theta}-\overline{z_p}\right)}rdrd\theta$$

$$= \frac{1}{2\pi}\int_0^1 r dr \int_0^{2\pi} \frac{\left(re^{i\theta}+z_p\right)\left(re^{-i\theta}+\overline{z_p}\right)}{\left(re^{i\theta}-z_p\right)\left(re^{-i\theta}-\overline{z_p}\right)} d\theta$$

Setting

$$w = e^{i\theta}, dw = iw d\theta, 0 \leq \theta \leq 2\pi$$

in fact

$$|w| = 1 \, |w| = 1$$

and

$$J_p = \frac{1}{2\pi}\int_0^1 r dr \int_{C_1} \frac{\left(rw+z_p\right)\left(r+w\overline{z_p}\right)}{\left(rw-z_p\right)\left(r-w\overline{z_p}\right)} \frac{dw}{iw}$$

i-e

$$J_p = \frac{1}{2\pi i}\int_0^1 dr \int_{C_1} \frac{\left(rw+z_p\right)\left(r+w\overline{z_p}\right)}{\left(w-\frac{z_p}{r}\right)\left(r-w\overline{z_p}\right)} \frac{dw}{w} = \frac{2r\left(r^2+|z_p|^2\right)-r\left(r^2-|z_p|^2\right)}{r^2-|z_p|^2}$$

now we state

$$J_p = \int_0^1 r\frac{r^2+3\,|z_p|^2}{r^2-|z_p|^2}dr = \int_0^1 \left(r + \frac{4r\,|z_p|^2}{r^2-|z_p|^2}\right) dr$$

hence

$$J_p = \frac{1}{2} + 2\,|z_p|^2 \log\left|\frac{1-|z_p|^2}{|z_p|^2}\right|$$

then

$$\iint_D \left(\sum_{p=1}^{m}\left|\frac{z+z_p}{z-z_p}\right|^2\right) dxdy = m\pi + 2\pi \log\left|\prod_{p=1}^{m}\frac{1-|z_p|^2}{|z_p|^2}\right|^{2|z_p|^2}$$

and the proposition is proved. ■

The third result of paper is applying Green's and Stokes formula,[7],[18],[16],[3],[4],[5],[19]

$$\iint_D f(z)\,\overline{g'(z)}dxdy = \frac{1}{2i}\int_C f(z)\,\overline{g(z)}dz$$

where the domain D is he ellipse : $b^2x^2+a^2y^2 < a^2b^2$enclosed by closed curve C in the complex z-plane on orthogonal Tchebychev polynomials of the second kind ,$\{U_n(z)\}_{n=0,1,2,3,...}$ with respect to the Lebesgue planar measure $dm(x,y) = dxdy$ concentrated on the ellipse $D : b^2x^2+a^2y^2 < a^2b^2$ where $a > b$,$z = x+iy$ given by :

$$U_n(z) = \frac{T'_{n+1}(z)}{n+1} = \frac{\sin\left((n+1)\cos^{-1} z\right)}{\sqrt{1-z^2}} \quad , \quad n = 0,1,2....$$

where $T_n(z) = \cos\left(n\cos^{-1} z\right)$,$n = 0,1,2,3....$is a polynomial of degree n . $T_n(z)$ is called the Tchebychev polynomial of degree n of first kind .

We obtain the following two corollaries of Green's and Stokes formula property.

Corollary 11 *If the ellipse :*$D: b^2x^2 + a^2y^2 < a^2b^2$ *and* C *is the closed contour by which* D *is bounded.Let* $f(z)$ *be regular in a domain* D*.If*

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z) \tag{52}$$

which converges uniformly and absolutely in any closed subdomain of ellipse D*.Then*

$$a_{n+1} = \frac{i\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8(n+1)^2} \int_C f(z)\overline{U_n(z)}d\overline{z} \quad , \rho = (a+b)^2, \ n = 0, 1, 2, ... \tag{53}$$

where C*:*$b^2x^2 + a^2y^2 = a^2b^2$.

Proof. The orthogonality of Tchebychev polynomials of the second kind ,$\{U_n(z)\}_{n=0,1,2,3,...}$ with respect to the Lebesgue planar measure $dxdy$ concentrated on the ellipse : D : $b^2x^2 + a^2y^2 < a^2b^2$ where $a > b$,we permits to show that

$$\iint_D U_n(z)\overline{U_m(z)}dxdy = \frac{4(n+1)}{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}\delta_{n,m} \qquad ,n, m = 0, 1, 2, 3, ...$$

where $z = x + iy$.The coefficients a_n are the Fourier coefficients of the function $f'(z)$ corresponding to the orthogonal Tchebychev polynomials of second kind on the ellipse D.

$$(n+1)\,a_{n+1} = \frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{4(n+1)} \iint_D f'(z)\overline{U_n(z)}dxdy \quad , \quad n = 0, 1, 2, 3, ...$$

According to Eq.(9)this may be replaced,

$$a_{n+1} = -\frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8i(n+1)^2} \overline{\int_C \overline{f(z)}U_n(z)\,dz} \quad , z = x + iy \ , n = 0, 1, 2, 3, ...$$

i-e

$$a_{n+1} = -\frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8i(n+1)^2} \int_C f(z)\overline{U_n(z)}d\overline{z} \quad , \quad n = 0, 1, 2, 3....$$

and the corollary is proved. ■

Corollary 12 *If the ellipse* $D: b^2x^2 + a^2y^2 < a^2b^2$ *and* C *is the closed contour by which* D *is bounded.Let* $f(z)$ *be regular in a domain* D*.If*

$$\int_a^z f(w)\,dw = \sum_{n=1}^{\infty} b_n T_n(z) \tag{54}$$

which converges uniformly and absolutely in any closed subdomain of ellipse D*.Then*

$$b_1 = a_0 - \frac{a_2}{2}$$

and

$$b_2 = \frac{a_1 - a_3}{4}$$

with

$$b_n = \frac{a_{n-1-}a_{n+1}}{2n} \qquad ,n = 3, 4, 5, ...$$

where

$$a_{n+1} = \frac{i\pi \left(\rho^{n+1} - \rho^{-n-1}\right)}{8\left(n+1\right)^2} \int\limits_C f\left(z\right) \overline{U_n\left(z\right)} d\overline{z} \quad , \ \rho = \left(a+b\right)^2 \qquad , \ n = 0, 1, 2, ... \tag{55}$$

Hence

$$b_n = \frac{i\pi}{16n} \int\limits_C f\left(z\right) \left(\frac{\rho^n - \rho^{-n}}{n^2} \overline{U_{n-1}\left(z\right)} - \frac{\rho^{n+2} - \rho^{-n-2}}{\left(n+2\right)^2} \overline{U_{n+1}\left(z\right)} \right) d\overline{z} \qquad , \ n = 0, 1, 2, ... \tag{56}$$

Proof. We have

$$\int T_n\left(z\right) dz = \frac{1}{2\left(n+1\right)} T_{n+1}\left(z\right) - \frac{1}{2\left(n-1\right)} T_{n-1}\left(z\right) + C \tag{57}$$

for $n \in \{0, 1\}$, we treat these cases individually

$$\int T_0\left(z\right) dz = T_1\left(z\right) + C, \quad \text{and} \int T_1\left(z\right) dz = \frac{1}{4} T_2\left(z\right) + C$$

If the ellipse :$D : b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.Let $f\left(z\right)$ be regular in a domain D.If

$$f\left(z\right) = \sum_{n=0}^{\infty} a_n T_n\left(z\right)$$

Becomes

$$\int\limits_a^z f\left(w\right) dw = a_0 T_1\left(z\right) + \frac{a_1}{4} T_2\left(z\right) + \sum_{n=2}^{\infty} a_n \left(\frac{1}{2\left(n+1\right)} T_{n+1}\left(z\right) - \frac{1}{2\left(n-1\right)} T_{n-1}\left(z\right) \right)$$

Because the inifinite sum equal

$$\sum_{n=3}^{\infty} \frac{a_{n-1-}a_{n+1}}{2n} T_n\left(z\right) - \frac{a_2}{2} T_1\left(z\right) - \frac{a_3}{4} T_2\left(z\right)$$

yields

$$\int\limits_a^z f\left(w\right) dw = \left(a_0 - \frac{a_2}{2}\right) T_1\left(z\right) + \frac{a_1 - a_3}{4} T_2\left(z\right) + \sum_{n=3}^{\infty} \frac{a_{n-1-}a_{n+1}}{2n} T_n\left(z\right)$$

and the corollary is proved. ■

Conclusion

This work examined several analytical approaches for evaluating area integrals in planar domains related to the unit disk. The discussion centered on the use of Gronwall's area formula for exterior conformal mappings and its application to standard regions such as circles, ellipses, and symmetric lemniscate-type domains. The paper also considered the use of Green's and Stokes' formula to transform area integrals into contour integrals, providing an alternative method for studying integrals over domains bounded by smooth curves. In the elliptical case, the analysis was connected with Tchebychev polynomials of the second kind and their orthogonality with respect to the planar Lebesgue measure. Coefficient formulas were obtained for functions represented through Tchebychev polynomial expansions, and related expressions were derived for primitive functions in the same polynomial basis. These results show that conformal mapping methods and orthogonal polynomial expansions provide useful frameworks for representing and evaluating certain classes of area integrals. The formulas discussed in the paper may support further theoretical study of area integral identities, boundary representations, and polynomial approximation problems in complex domains.

Limitations

The results in this paper are limited to theoretical settings in which the required conformal mappings, Laurent expansions, Taylor expansions, and boundary regularity assumptions are available. The methods are developed for selected domains, mainly the unit disk, ellipses, and symmetric lemniscate-type regions, and therefore they do not directly cover arbitrary planar domains. The derivations also depend on convergence assumptions for the relevant series expansions and on the validity of applying Green's, Stokes', and Gronwall's formulas under the stated analytical conditions. No numerical examples or computational comparisons are provided, so the practical performance of the formulas for approximation or numerical integration is not assessed. In addition, the discussion does not treat domains with nonsmooth boundaries, multiply connected regions beyond the considered cases, or possible error estimates for truncated polynomial expansions. Further work may extend these methods to broader classes of domains and include numerical validation.

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