On Computing the Lower and Upper Bounds of -Graphs via Some Topological Indices.

Abstract

Topological indices are mathematical invariants that provide numerical correlations between the chemical structure of a molecule and its physical properties, chemical reactivity, or biological activity. In this paper, we study the lower and upper bounds of several significant topological indices and illustrate the derived results with appropriate examples. In particular, we consider the first Zagreb index, the Sombor index, the Nirmala index, the Harmonic index, the Y-coindex, and the VL index.

Keywords: Corona product, Topological index and Degree based topological indices.

2010 Mathematics Subject classification: 05C90, 05C12.

1 Introduction

A topological descriptor is a numerical invariant derived from a molecular graph that captures specific structural characteristics of the associated chemical compound. Such descriptors are widely employed in mathematical chemistry, particularly in QSPR/QSAR studies, where they play a crucial role in the modeling and prediction of molecular properties. Among these descriptors, topological indices form an important subclass and are especially effective in estimating the physicochemical behavior of chemical compounds.

Throughout this work, we consider $G = (V(G), E(G))$ as a simple, connected, and undirected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The use of topological indices in chemical research can be traced back to 1947, when Harold Wiener introduced one of the earliest and most influential topological descriptors, now known as the Wiener index, and demonstrated its effectiveness in correlating molecular structure with the physical properties of paraffin alkanes.

Now, we recall some well-known topological indices. Gutman and Trinajstić [6] introduced the first Zagreb index in 1972, which is defined as

$$M_1(G) = \sum_{uv \in E(G)} \big[d(u) + d(v)\big].$$

The concept of the Sombor index was recently introduced by Gutman [?] in chemical graph theory. It is a vertex-degree-based topological index and is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}.$$

Motivated by the study of Sombor indices, V. R. Kulli introduced the Nirmala index of a graph G [?], defined as

$$N(G) = \sum_{uv \in E(G)} \sqrt{d_u + d_v}.$$

Alameri *et al.* introduced the Y-index in 2020 [?], which is defined as

$$Y(G) = \sum_{uv \in E(G)} \big(d^3(u) + d^3(v)\big).$$

A new version of the Zagreb indices, called the Hyper-Zagreb index, was introduced by Shirdel *et al.* in 2013 [15] and is defined as

$$HM(G) = \sum_{uv \in E(G)} \big[d(u) + d(v)\big]^2.$$

Inspired by the Zagreb indices, Deepika T. introduced the VL index of a graph G [?], which is defined as

$$VL(G) = \frac{1}{2} \sum_{uv \in E(G)} \big[d_e + d_f + 4\big],$$

where, $d_e = d_u + d_v - 2$ and $d_f = (d_u \times d_v) - 2$, with d_u and d_v denoting the degrees of the vertices u and v in G, respectively.

Definition 1.1. *[12] The corona product $G \odot H$ of two graphs is obtained by taking one copy of G and n_1 copies of H, and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, where $1 \leq i \leq n_1$.*

In order to study bounds on the Γ-graph, the paper is organized into several sections. Section 1 presents the preliminaries and definitions of well-known topological indices that are useful for proving the main results. Section 2 introduces a new class of operator graphs along with their properties. Section 3 is devoted to results concerning the lower and upper bounds of the defined class of graphs using the recalled topological indices. Finally, the paper concludes with a conclusion and references.

2 Basic Preliminaries

Throughout this article, we consider graphs that are finite, simple, and connected. In this section, we introduce the fundamental definitions, concepts, and notations that will be used throughout the paper.

Let G and H be graphs with vertex sets $V(G)$ and $V(H)$, and edge sets $E(G)$ and $E(H)$, respectively. The degree of a vertex $v \in V(G)$ (or $v \in V(H)$) is defined as the number of vertices adjacent to v.

We assume that the vertex sets of G and H are disjoint, that is, $V(G) \cap V(H) = \emptyset$, where $g \in V(G)$ and $h \in V(H)$. Let $V_1 = |V(G)|$ and $V_2 = |V(H)|$ denote the number of vertices, and let $E_1 = |E(G)|$ and $E_2 = |E(H)|$ denote the number of edges in the graphs G and H, respectively. Accordingly, the degrees of vertices g and h satisfy the inequalities

$$\Delta_G \geq \deg_G(g), \qquad \delta_G \leq \deg_G(g),$$
$$\Delta_H \geq \deg_H(h), \qquad \delta_H \leq \deg_H(h),$$

where Δ and δ represent the maximum and minimum vertex degrees of the corresponding graphs.

Bounds for several topological indices of graphs have been established by many researchers (see, for instance, [8]). Motivated by these studies, we introduce a new class of operator graphs, namely the semi-total point graph of the vertex corona of G and H. This graph is referred to as the Γ-graph throughout this work.

Definition 2.1. *The graph $G \odot_R H = \Gamma$ is obtained from one copy of the graph G and E_1 copies of the graph H by joining each vertex of $V(G)$—that is, the i^{th} vertex of G—to every vertex of the i^{th} copy of H.*

The bounds of the Γ–graph indices are derived using an edge decomposition of the graph, where each distinct edge type represents a specific form of connectivity between the vertices of G and their corresponding copies in H. These edge types characterize the structural interactions within the Γ–graph and are systematically summarized in Table 1.

Table 1. Edge partition of the Γ-graph

Edge type	$(2d_G + V_2,\, 2d_G + V_2)$	$(2d_G + V_2,\, 2)$	$(d_H + 1,\, d_H + 1)$	$(d_H + 1,\, 2d_G + V_2)$
Frequency	E_1	$2E_1$	V_1E_2	V_1V_2

3 Lower and Upper Bounds of Topological Indices for the Γ-Graph

In this section, we recall only the statements of bounds for the M_1, SO, VL, N, Y, and HM indices of the Γ-graph [11] [10] and provide illustrative examples for each result.

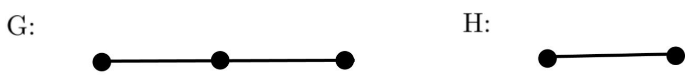
Statement 1. *For two finite, simple, and connected graphs G and H, the bounds of the first Zagreb index for the Γ–graph are given by*

$$M_1[\Gamma] \leq 2E_1(2\Delta_G+V_2)+2E_1(2\Delta_G+V_2+2)+2V_1E_2(\Delta_H+1)+V_1V_2(2\Delta_G+\Delta_H+V_2+1),$$

and

$$M_1[\Gamma] \geq 2E_1(2\delta_G+V_2)+2E_1(2\delta_G+V_2+2)+2V_1E_2(\delta_H+1)+V_1V_2(2\delta_G+\delta_H+V_2+1).$$

Illustration of Statement 1



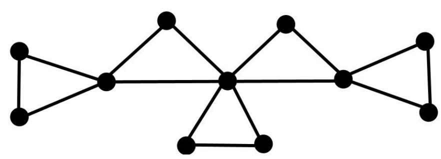


Figure 1: Representation of the $M_1[\Gamma]$–graph for $G(P_3)$ and $H(P_2)$

$G(P_3)$	$H(P_2)$
$V_G = 3$	$V_H = 2$
$E_G = 2$	$E_H = 1$
$\Delta_G = 2$	$\Delta_H = 1$
$\delta_G = 1$	$\delta_H = 1$

Table 1: Parameters of $G(P_3)$ and $H(P_2)$[1].

Computation of the First Zagreb Index

$$M_1(G) = \sum_{uv\in E(G)} \big[d(u)+d(v)\big],$$

By applying Table 1 and the definition of the first Zagreb index, we obtain

$M_1[\Gamma] = 3(2+2) + 6(2+4) + 2(6+4) + 4(6+2) = 100.$

Now, we apply the structural properties of these graphs to establish bounds for the first Zagreb index. This involves leveraging vertex degree distributions and graph invariants to derive rigorous upper and lower limits for this important topological index.

Upper Bound Calculation

$$\begin{aligned} M_1[\Gamma] &\leq 2E_1(2\Delta_G + V_2) + 2E_1(2\Delta_G + V_2 + 2) + 2V_1E_2(\Delta_H + 1) + V_1V_2(2\Delta_G + \Delta_H + V_2 + 1) \\ &\leq 2 \cdot 2(2 \cdot 2 + 2) + 2 \cdot 2(2 \cdot 2 + 2 + 2) + 2 \cdot 3 \cdot 1(1+1) + 3 \cdot 2(2 \cdot 2 + 1 + 2 + 1) \\ &\leq 24 + 32 + 12 + 48 \\ M_1[\Gamma] &\leq 116. \end{aligned}$$

Lower Bound Calculation

$$\begin{aligned} M_1[\Gamma] &\geq 2E_1(2\delta_G + V_2) + 2E_1(2\delta_G + V_2 + 2) + 2V_1E_2(\delta_H + 1) + V_1V_2(2\delta_G + \delta_H + V_2 + 1) \\ &\geq 2 \cdot 2(2 \cdot 1 + 2) + 2 \cdot 2(2 \cdot 1 + 2 + 2) + 2 \cdot 3 \cdot 1(1+1) + 3 \cdot 2(2 \cdot 1 + 1 + 2 + 1) \\ &= 16 + 24 + 12 + 36 \\ M_1[\Gamma] &\geq 88. \end{aligned}$$

Thus, the first Zagreb index of the graph Γ satisfies a significant inequality that establishes important bounds on this topological descriptor.

$$88 \leq 100 \leq 116.$$

Statement 2. *Let G and H be two simple connected graphs. Then, the Nirmala index of the graph Γ satisfies the following bounds[7]. As established in [1], we have*

$$\begin{aligned} N[\Gamma] \leq{} & E_1\sqrt{4\Delta_G + 2V_2} + 2E_1\sqrt{2\Delta_G + V_2 + 2} + V_1E_2\sqrt{2\Delta_H + 2} \\ & + V_1V_2\sqrt{\Delta_H + 2\Delta_G + V_2 + 1}, \end{aligned}$$

and

$$\begin{aligned} N[\Gamma] \geq{} & E_1\sqrt{4\delta_G + 2V_2} + 2E_1\sqrt{2\delta_G + V_2 + 2} + V_1E_2\sqrt{2\delta_H + 2} \\ & + V_1V_2\sqrt{\delta_H + 2\delta_G + V_2 + 1}. \end{aligned}$$

Illustration of Statement 2

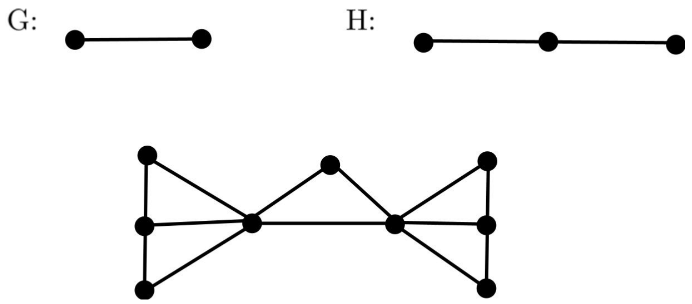


Figure 2: $N[\Gamma]$-Graph for $G(P_2)$ and $H(P_3)$ [5].

$G(P_2)$	$H(P_3)$
$V_G = 2$	$V_H = 3$
$E_G = 1$	$E_H = 2$
$\Delta_G = 1$	$\Delta_H = 2$
$\delta_G = 1$	$\delta_H = 1$

Table 2: Tabulated characteristics of the graphs $G(P_2)$ and $H(P_3)$ [5].

Calculation of the Nirmala index

$$N(G) = \sum_{uv \in E(G)} \sqrt{d_u + d_v},$$

By applying Table 2 and the definition of the Nirmala index, we obtain

$$N[\Gamma] = 4\sqrt{2+3} + 6\sqrt{2+5} + 2\sqrt{3+5} + \sqrt{5+5},$$

$$N[\Gamma] \approx 8.94 + 15.87 + 5.66 + 3.16 \;\approx\; 33.63.$$

Upper Bound Estimation

The upper bound for the Nirmala index of Γ can be evaluated as follows:

$$\begin{aligned} N[\Gamma] &\leq E_1\sqrt{4\Delta_G + 2V_2} + 2E_1\sqrt{2\Delta_G + V_2 + 2} + V_1E_2\sqrt{2\Delta_H + 2} \\ &\quad + V_1V_2\sqrt{\Delta_H + 2\Delta_G + V_2 + 1} \\ &\leq 1\sqrt{4\cdot 1 + 2\cdot 3} + 2\cdot 1\sqrt{2\cdot 1 + 3 + 2} + 2\cdot 2\sqrt{2\cdot 2 + 2} \\ &\quad + 2\cdot 3\sqrt{2 + 2\cdot 1 + 3 + 1} \\ &\approx 3.16 + 5.29 + 9.80 + 16.97 \;\approx\; 35.22. \end{aligned}$$

Lower Bound Calculation

$$\begin{aligned} N[\Gamma] &\geq E_1\sqrt{4\delta_G + 2V_2} + 2E_1\sqrt{2\delta_G + V_2 + 2} + V_1E_2\sqrt{2\delta_H + 2} + V_1V_2\sqrt{\delta_H + 2\delta_G + V_2 + 1} \\ &\geq 1\sqrt{4\cdot 1 + 2\cdot 3} + 2\cdot 1\sqrt{2\cdot 1 + 3 + 2} + 2\cdot 2\sqrt{2\cdot 1 + 2} + 2\cdot 3\sqrt{1 + 2\cdot 1 + 3 + 1} \\ &\approx 3.16 + 5.29 + 8.00 + 15.87 \;\approx\; 32.32. \end{aligned}$$

Thus, the Nirmala index $N[\Gamma]$ satisfies the following bounds, reflecting the combined influence of vertex degrees and edge connectivity:

$$32.32 \;\leq\; N[\Gamma] \;\approx\; 33.63 \;\leq\; 35.22.$$

Statement 3. *Suppose G and H are finite, simple, and connected graphs[3] . Then the Sombor index of the associated Γ–graph satisfies the following bounds:*

$$\begin{aligned} SO[\Gamma] \leq{}& \sqrt{2}\,E_1(2\Delta_G + V_2) + 2E_1\,\sqrt{(2\Delta_G + V_2)^2 + 4} \\ &+ V_1E_2\,\sqrt{2}(\Delta_H + 1) + V_1V_2\,\sqrt{(\Delta_H + 1)^2 + (2\Delta_G + V_2)^2}, \end{aligned}$$

and

$$\begin{aligned} SO[\Gamma] \geq{}& \sqrt{2}\,E_1(2\delta_G + V_2) + 2E_1\,\sqrt{(2\delta_G + V_2)^2 + 4} \\ &+ V_1E_2\,\sqrt{2}(\delta_H + 1) + V_1V_2\,\sqrt{(\delta_H + 1)^2 + (2\delta_G + V_2)^2}. \end{aligned}$$

Illustration of Statement 3

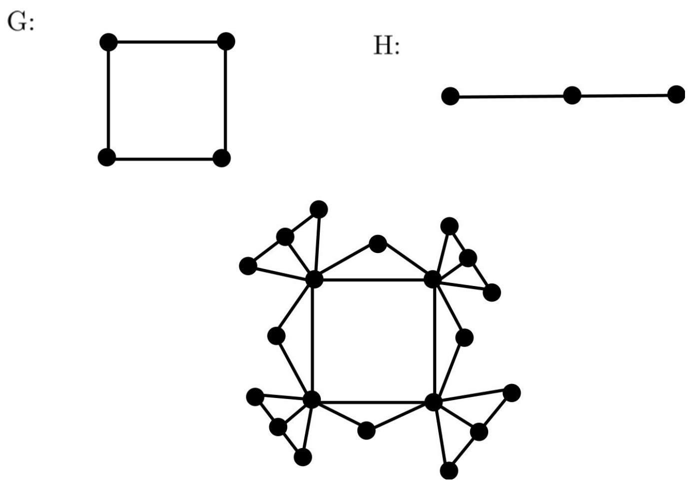


Figure 3: $SO[\Gamma]$–graph of $G(C_4)$ and $H(P_3)$ [3].

$G(C_4)$	$H(P_3)$
$V_G = 4$	$V_H = 3$
$E_G = 4$	$E_H = 2$
$\Delta_G = 2$	$\Delta_H = 2$
$\delta_G = 2$	$\delta_H = 1$

Table 3: Graph invariants of $G(C_4)$ and $H(P_3)$ [1].

Formulation of the Sombor Index

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

By applying Table 3 and the definition of the Sombor index, we obtain

$$SO(\Gamma) = 8\sqrt{3^2 + 2^2} + 4\sqrt{3^2 + 7^2} + 16\sqrt{2^2 + 7^2} + 4\sqrt{7^2 + 7^2},$$

$$SO(\Gamma) \approx 28.84 + 30.46 + 116.48 + 39.60 = 215.38.$$

Upper Bound Calculation

$$\begin{aligned}
SO[\Gamma] &\leq \sqrt{2}E_1(2\Delta_G + V_2) + 2E_1\sqrt{(2\Delta_G + V_2)^2 + 4} \\
&\quad + V_1E_2\sqrt{2}(\Delta_H + 1) + V_1V_2\sqrt{(\Delta_H + 1)^2 + (2\Delta_G + V_2)^2} \\
&\leq \sqrt{2} \cdot 4(2 \cdot 2 + 3) + 2 \cdot 4\sqrt{(2 \cdot 2 + 3)^2 + 4} + 4 \cdot 2\sqrt{2}(2 + 1) \\
&\quad + 4 \cdot 3\sqrt{(2 + 1)^2 + (2 \cdot 2 + 3)^2} \\
&\leq 39.60 + 58.24 + 33.94 + 91.39 \\
SO[\Gamma] &\leq 223.17.
\end{aligned}$$

Lower Bound Calculation

$$\begin{aligned}
SO[\Gamma] &\geq \sqrt{2}E_1(2\delta_G + V_2) + 2E_1\sqrt{(2\delta_G + V_2)^2 + 4} \\
&\quad + V_1E_2\sqrt{2}(\delta_H + 1) + V_1V_2\sqrt{(\delta_H + 1)^2 + (2\delta_G + V_2)^2} \\
&\geq \sqrt{2} \cdot 4(2 \cdot 2 + 3) + 2 \cdot 4\sqrt{(2 \cdot 2 + 3)^2 + 4} + 4 \cdot 2\sqrt{2}(1 + 1) \\
&\quad + 4 \cdot 3\sqrt{(1 + 1)^2 + (2 \cdot 2 + 3)^2} \\
&\geq 39.60 + 58.24 + 22.63 + 87.36 \\
SO[\Gamma] &\geq 207.83.
\end{aligned}$$

Thus, the Sombor index of the graph Γ satisfies a key inequality that establishes bounds on this topological invariant, which reflects the combined effects of vertex degrees and edge connections within the graph structure.

$$207.83 \ \leq \ SO[\Gamma] \ = \ 215.38 \ \leq \ 223.17.$$

Statement 4. *Let G and H be two finite, connected graphs. For the corresponding graph Γ, the hyper-Zagreb index admits the following bounds [1, 14]:*

$$\begin{aligned} M_{HZ}(\Gamma) \ &\leq E_1\big(4\Delta_G + 2V_2\big)^2 + 2E_1\big(2\Delta_G + V_2 + 2\big)^2 \\ &\quad + V_1E_2\big(2\Delta_H + 2\big)^2 + V_1V_2\big(\Delta_H + 2\Delta_G + V_2 + 1\big)^2. \end{aligned}$$

while the corresponding lower bound is given by

$$\begin{aligned} M_{HZ}(\Gamma) \ &\geq E_1\big(4\delta_G + 2V_2\big)^2 + 2E_1\big(2\delta_G + V_2 + 2\big)^2 \\ &\quad + V_1E_2\big(2\delta_H + 2\big)^2 + V_1V_2\big(\delta_H + 2\delta_G + V_2 + 1\big)^2. \end{aligned}$$

Illustrative of Statement 4

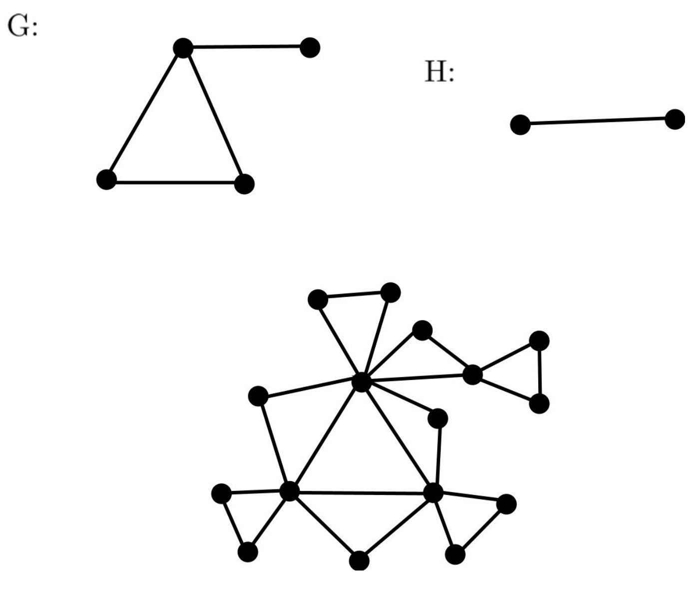


Figure 4: $M_{HZ}[\Gamma]$- graph of $G(T_{3,1})$ and $H(P_2)$[1].

$\mathbf{G}(T_{3,1})$	$\mathbf{H}(P_2)$
$V_G = 4$	$V_H = 2$
$E_G = 4$	$E_H = 1$
$\Delta_G = 3$	$\Delta_H = 1$
$\delta_G = 1$	$\delta_H = 1$

Table 4: Parameters for the graphs $G(T_{3,1})$ and $H(P_2)$[1].

$M_{HZ}[\Gamma]$**- Graph Calculation [14]**

$$M_{HZ}[\Gamma] = \sum_{uv \in E[G]} [d(u) + d(v)]^2$$

By applying Table 4 and the definition of the hyper-Zagreb index, we obtain

$$\begin{aligned} &= 1(6+6)^2 + 8(6+2)^2 + 2(8+6)^2 + 5(8+2)^2 \\ &\quad + 1(8+4)^2 + 3(4+2)^2 + 4(2+2)^2 \\ &= 144 + 512 + 392 + 500 + 144 + 108 + 64 \\ M_{HZ}[\Gamma] &= 1864. \end{aligned}$$

Upper Bound Estimation

The upper bound for the hyper-Zagreb index of Γ can be evaluated as follows:

$$\begin{aligned} M_{HZ}[\Gamma] &\leq E_1(4\Delta_G + 2V_2)^2 + 2E_1(2\Delta_G + V_2 + 2)^2 + V_1E_2(2\Delta_H + 2)^2 \\ &\quad + V_1V_2(\Delta_H + 2\Delta_G + V_2 + 1)^2 \\ &\leq 4(4\cdot 3 + 2\cdot 2)^2 + 2\cdot 4(2\cdot 3 + 2 + 2)^2 + 4\cdot 1(2\cdot 1 + 2)^2 + 4\cdot 2(1 + 2\cdot 3 + 2 + 1)^2 \\ &\leq 1024 + 800 + 64 + 800 \\ M_{HZ}[\Gamma] &\leq 2688. \end{aligned}$$

Lower Bound Calculation

$$\begin{aligned} M_{HZ}[\Gamma] &\geq E_1(4\delta_G + 2V_2)^2 + 2E_1(2\delta_G + V_2 + 2)^2 + V_1E_2(2\delta_H + 2)^2 \\ &\quad + V_1V_2(\delta_H + 2\delta_G + V_2 + 1)^2 \\ &\geq 4(4\cdot 1 + 2\cdot 2)^2 + 2\cdot 4(2\cdot 1 + 2 + 2)^2 + 4\cdot 1(2\cdot 1 + 2)^2 + 4\cdot 2(1 + 2\cdot 1 + 2 + 1)^2 \\ &\geq 256 + 288 + 64 + 288 \\ M_{HZ}[\Gamma] &\geq 896. \end{aligned}$$

$$896 \ \leq \ 1864 \ \leq \ 2688.$$

Statement 5. *Consider two finite, simple, and connected graphs G and H[1]. For the associated graph Γ, the Y-coindex satisfies the following inequalities:*

$$Y(\Gamma) \leq E_1\Big(2(2\Delta_G + V_2)^3\Big) + 2E_1\Big((2\Delta_G + V_2)^3 + 8\Big) + V_1E_2\Big(2(\Delta_H + 1)^3\Big) + V_1V_2\Big((\Delta_H + 1)^3 + (2\Delta_G + V_2)^3\Big).$$

and

$$Y(\Gamma) \geq E_1\Big(2(2\delta_G + V_2)^3\Big) + 2E_1\Big((2\delta_G + V_2)^3 + 8\Big) + V_1E_2\Big(2(\delta_H + 1)^3\Big) + V_1V_2\Big((\delta_H + 1)^3 + (2\delta_G + V_2)^3\Big).$$

Illustrative of Statement 5

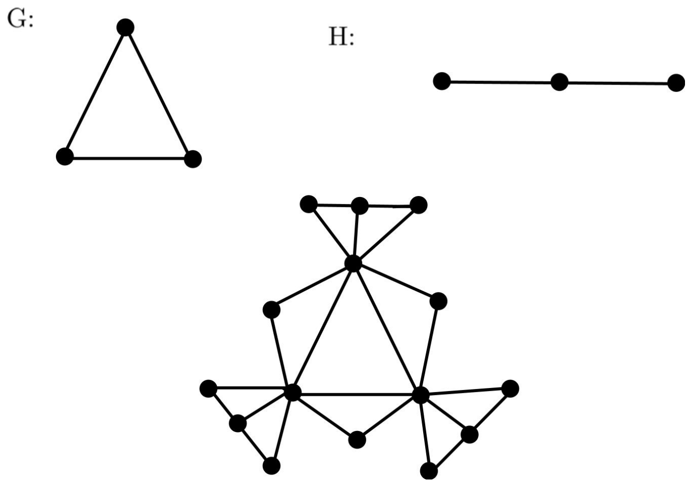


Figure 5: $Y[\Gamma]$- graph of $G(C_3)$ and $H(P_3)$.

$\mathbf{G}(C_3)$	$\mathbf{H}(P_3)$
$V_G = 3$	$V_H = 3$
$E_G = 3$	$E_H = 2$
$\Delta_G = 2$	$\Delta_H = 2$
$\delta_G = 2$	$\delta_H = 1$

Table 5: Graph parameters of $G(C_3)$ and $H(P_3)$[1].

Computation of the Y-coindex

$$Y[G] = \sum_{uv \in E(G)} \left(d_u^3 + d_v^3\right),$$

By applying Table 5 and the definition of the Y-coindex, we obtain

$$\begin{aligned}
Y[\Gamma] &= 6\,(2^3 + 3^3) \;+\; 12\,(2^3 + 7^3) \;+\; 3\,(3^3 + 7^3) \;+\; 3\,(7^3 + 7^3) \\
&= 6(8 + 27) \;+\; 12(8 + 343) \;+\; 3(27 + 343) \;+\; 3(343 + 343) \\
&= 6(35) + 12(351) + 3(370) + 3(686) \\
&= 210 + 4212 + 1110 + 2058 \\
Y[\Gamma] &= 7590.
\end{aligned}$$

Upper Bound Calculation

$$\begin{aligned}
Y[\Gamma] &\leq E_1\left[2(2\Delta_G + V_2)^3\right] + 2E_1\left[(2\Delta_G + V_2)^3 + 8\right] + V_1E_2\left[2(\Delta_H + 1)^3\right] \\
&\quad + V_1V_2\left[(\Delta_H + 1)^3 + (2\Delta_G + V_2)^3\right] \\
&\Rightarrow 3\big[2(2 \cdot 2 + 3)^3\big] + 2 \cdot 3\big[(2 \cdot 2 + 3)^3 + 8\big] + 3 \cdot 2\big[2(2 + 1)^3\big] \\
&\quad + 3 \cdot 3\big[(2 + 1)^3 + (2 \cdot 2 + 3)^3\big] \\
&= 2058 + 2106 + 324 + 3330 \\
Y[\Gamma] &\leq 7818.
\end{aligned}$$

Lower Bound Calculation

$$\begin{aligned}
Y[\Gamma] &\geq E_1\left[2(2\delta_G + V_2)^3\right] + 2E_1\left[(2\delta_G + V_2)^3 + 8\right] + V_1E_2\left[2(\delta_H + 1)^3\right] \\
&\quad + V_1V_2\left[(\delta_H + 1)^3 + (2\delta_G + V_2)^3\right] \\
&\Rightarrow 3\big[2(2 \cdot 2 + 3)^3\big] + 2 \cdot 3\big[(2 \cdot 2 + 3)^3 + 8\big] + 3 \cdot 2\big[2(1 + 1)^3\big] \\
&\quad + 3 \cdot 3\big[(1 + 1)^3 + (2 \cdot 2 + 3)^3\big] \\
&= 2058 + 2106 + 96 + 3159 \\
Y[\Gamma] &\geq 7419.
\end{aligned}$$

Thus, the Y-index of the graph Γ satisfies the following bounds, reflecting the combined influence of vertex degrees and edge connectivity:

$$7419 \;\leq\; Y[\Gamma] \;=\; 7590 \;\leq\; 7818.$$

Statement 6. *Let G and H be finite, simple, connected graphs[3]. For the graph Γ, the extremal bounds of its VL-index are:*

$$VL[\Gamma] \leq \frac{E_1}{2}\big(4\Delta_G + 2V_2 + (2\Delta_G + V_2)^2\big) + E_1(6\Delta_G + 3V_2 + 2) + \frac{V_1E_1}{2}\big(2\Delta_H + 2 + (\Delta_H + 1)^2\big) + \frac{V_1V_2}{2}(2\Delta_G + \Delta_H + V_2 + 1 + (\Delta_H + 1)(2\Delta_G + V_2)) \, .$$

and

$$VL[\Gamma] \geq \frac{E_1}{2}\big(4\delta_G + 2V_2 + (2\delta_G + V_2)^2\big) + E_1(6\delta_G + 3V_2 + 2) + \frac{V_1E_1}{2}\big(2\delta_H + 2 + (\delta_H + 1)^2\big) + \frac{V_1V_2}{2}(2\delta_G + \delta_H + V_2 + 1 + (\delta_H + 1)(2\delta_G + V_2)) \, .$$

Illustration of Statement 6:

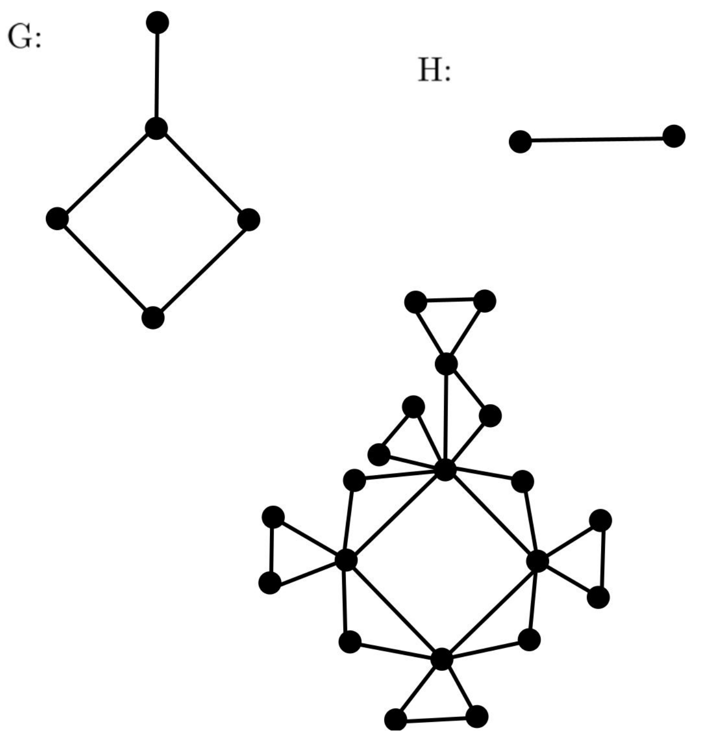


Figure 6: $VL[\Gamma]$- graph of $G(T_{4,1})$ and $H(P_2)$.

$G(T_{4,1})$	$H(P_2)$
$V_G = 5$	$V_H = 2$
$E_G = 5$	$E_H = 1$
$\Delta_G = 3$	$\Delta_H = 1$
$\delta_G = 1$	$\delta_H = 1$

Table 6: Parameters of graphs $G(T_{4,1})$ and $H(P_2)$ [1].

$VL[\Gamma]$-Graph Calculation [2]

$$VL(G) = \tfrac{1}{2} \sum_{uv \in E(G)} \big[d_u + d_v + d_u \cdot d_v\big],$$

By applying Table 6 and the definition of the VL-index, we obtain

$$\begin{aligned}
VL[\Gamma] &= \tfrac{1}{2}\Big[5(2+2+2\cdot 2) + 3(4+2+4\cdot 2) + 5(2+8+2\cdot 8) + 12(2+6+2\cdot 6)\\
&\qquad + 2(6+6+6\cdot 6) + 2(6+8+6\cdot 8) + 1(4+8+4\cdot 8)\Big],\\
&= \tfrac{1}{2}\big[40 + 42 + 130 + 240 + 96 + 124 + 44\big],\\
VL[\Gamma] &= 358.
\end{aligned}$$

Upper Bound Calculation

$$\begin{aligned}
VL[\Gamma] \leq{} & \frac{E_1}{2}\big(4\Delta_G + 2V_2 + (2\Delta_G + V_2)^2\big) + E_1\big(6\Delta_G + 3V_2 + 2\big)\\
& + \frac{V_1E_1}{2}\big(2\Delta_H + 2 + (\Delta_H + 1)^2\big) + \frac{V_1V_2}{2}\big(2\Delta_G + \Delta_H + V_2 + 1 + (\Delta_H + 1)(2\Delta_G + V_2)\big),\\
\leq{} & \tfrac{5}{2}\big(4\cdot 3 + 2\cdot 2 + (2\cdot 3 + 2)^2\big) + 5\big(6\cdot 3 + 3\cdot 2 + 2\big)\\
& + \tfrac{5\cdot 5}{2}\big(2\cdot 1 + 2 + (1+1)^2\big) + \tfrac{5\cdot 2}{2}\big(2\cdot 3 + 1 + 2 + 1 + (1+1)(2\cdot 3 + 2)\big),\\
={} & 200 + 130 + 100 + 130,\\
VL[\Gamma] \leq{} & 560.
\end{aligned}$$

Lower Bound Calculation

$$
\begin{aligned}
VL[\Gamma] \geq{} & \frac{E_1}{2}\big(4\delta_G + 2V_2 + (2\delta_G + V_2)^2\big) + E_1\big(6\delta_G + 3V_2 + 2\big) \\
& + \frac{V_1E_1}{2}\big(2\delta_H + 2 + (\delta_H + 1)^2\big) + \frac{V_1V_2}{2}\big(2\delta_G + \delta_H + V_2 + 1 + (\delta_H + 1)(2\delta_G + V_2)\big), \\
\geq{} & \tfrac{5}{2}\big(4 \cdot 1 + 2 \cdot 2 + (2 \cdot 1 + 2)^2\big) + 5\big(6 \cdot 1 + 3 \cdot 2 + 2\big) \\
& + \tfrac{5 \cdot 5}{2}\big(2 \cdot 1 + 2 + (1 + 1)^2\big) + \tfrac{5 \cdot 2}{2}\big(2 \cdot 1 + 1 + 2 + 1 + (1 + 1)(2 \cdot 1 + 2)\big), \\
={} & 60 + 70 + 100 + 70, \\
VL[\Gamma] \geq{} & 300.
\end{aligned}
$$

Thus, the VL-index of the graph Γ satisfies a fundamental inequality that provides crucial bounds on this topological invariant, reflecting significant structural characteristics of the graph.

$$300 \ \leq \ 358 \ \leq \ 560.$$

4 Conclusion

In this work, we considered the Γ-graph and focused on six important topological indices, for which we determined the corresponding bounds. In a similar manner, researchers may consider other classes of topological indices and establish their respective bounds for the Γ-graph.

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