Existence of solutions for a class of p-biharmonic equations

Abstract

By using Morse theory and critical point theory, we obtain the existence and multiplicity of solutions for a class of p-biharmonic equations with Navier boundary value conditions.

Key words: p-biharmonic; G-link Theorem; Navier boundary value conditions; noncrossing condition

1 Introduction and main results

We consider the following problem

$$\begin{cases} \Delta_p^2 u = f(x,u) & \text{in } \ \Omega, \\ u = \Delta u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2p+1)$ with smooth boundary $\partial\Omega$, $p > 1$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ denotes the p-biharmonic operator, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the following assumption:

(I_1) There exist $C > 0$ and $q \in (p, p_2^*)$ such that

$$|f(x,t)| \leq C\big(1 + |t|^{q-1}\big)$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where p_2^* is the critical Sobolev exponent defined as

$$p_2^* = \frac{Np}{N-2p}.$$

Let $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ be the Sobolev space with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

According to the Sobolev embedding theorem (see [1]), for all $1 \leq \gamma \leq p_2^*$, the embedding $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ is continuous, if $1 \leq \gamma < p_2^*$, the embedding $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ is compact. So there exists a positive constant C_0 such that

$$\|u\|_q \leq C_0 \|u\|, \tag{2}$$

where $\|u\|_\gamma = \left(\int_\Omega |u|^\gamma \mathrm{d}x\right)^{\frac{1}{\gamma}}$ is the usual norm in $L^\gamma(\Omega)$. In what follows, we first recall some concepts related to the eigenvalues of $\Delta_p^2 u$ (see [21]). Consider the following manifold:

$$\Sigma = \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \colon \int_\Omega |u|^p \mathrm{d}x = 1 \right\},$$

then Σ is a symmetric nonempty C^1- manifold in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Let

$$\Sigma_k = \left\{ \Lambda \subset \Sigma \colon \text{there exists a continuous, odd and surjective } h\colon S^{k-1} \to \Lambda \right\},$$

where S^{k-1} denotes the unit sphere in $\mathbb{R}^k$. We say λ is an eigenvalue of the nonlinear eigenvalue problem if

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

has nontrivial solutions. By the Ljusternik-Schnirelman theory on C^1- manifold (see [25]), we know that $\{\lambda_k\}_{k\in\mathbb{N}}$ is the eigenvalues sequence of (3), with the variational characterization:

$$\lambda_k = \inf_{\Lambda \in \Sigma_k} \sup_{u \in \Lambda} \int_\Omega |\Delta u|^p \mathrm{d}x.$$

In the past few decades, the study of the nonlinear fourth order differential equations was an interesting topic which appear in various branches of mathematical physics (see [2, 4, 6, 7, 13, 14, 16, 27, 28] and references therein). Many authors have widely developed various methods and techniques, such as critical point theory, More theory to look for multiple solutions of elliptic equations involving p-biharmonic type operators (see [5, 9, 12, 15, 20, 21, 23, 26] and references therein). In [12], Drábek and Ôtani proved the first eigenvalue λ_1 of (3) is simple, positive and isolated. Then, Liu and Squassina [21] divided the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into two complementary subspaces, i.e., $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = V \oplus W$, where $V = \mathrm{span}\{\phi_1\}$ be the eigenspace associated with λ_1 and W be a subspace of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ complementing V. By using Morse Theory and critical point theory, they studied the existence and multiplicity of nontrivial solutions for problem (1) and proved the following results.

Theorem A. *[21]. Assume that* (I_1) *holds,* $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$ *satisfies the following conditions:*

(I_2) There exist $\delta > 0$ and $\bar{\lambda} \in \left]\lambda_1, \hat{\lambda}\right[$ such that

$$\lambda_1 |t|^p \le pF(x,t) \le \bar{\lambda}|t|^p$$

for $x \in \Omega$ and $|t| \le \delta$, where $\hat{\lambda} > \lambda_1$ with $\|u\|^p \ge \hat{\lambda}\|u\|_p^p$ for any $u \in W$.

(I_3) $\lim\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} < \lambda_1$ *uniformly for $x \in \Omega$.*

Then problem (1) *admits two nontrivial solutions in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Theorem B. *[21]. Assume that* (I_1) *and* (I_2) *hold,* $F(x,t)$ *satisfies the following conditions:*

(I_4) $\lim\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} = \lambda_1$ *uniformly for $x \in \Omega$.*

(I_5) $\lim\limits_{|t|\to\infty} \big(f(x,t)t - pF(x,t)\big) = +\infty$ *uniformly for $x \in \Omega$.*

Then problem (1) *admits two nontrivial solutions in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Motivated by [21], in this paper, we prove the existence and multiplicity of solutions for problem (1) by using Morse theory (see [10,22]) and some abstract critical point theorems (see [17, 24]). Moreover, we investigate the existence of weak solutions for problem (1) using the G-link Theorem (see [24]), where $F(x,t)$ lies between two consecutive higher eigenvalues $\lambda_k < \lambda_{k+1} (k \geq 1)$ of (3). Our main results are the following theorems.

Theorem 1.1. *Assume that* (I_1), (I_2) *and* (I_4) *hold,* $F(x,t)$ *satisfies the following condition:*

(I_5') $\lim\limits_{|t|\to\infty} \left(F(x,t) - \frac{1}{p}\lambda_1 |t|^p\right) = -\infty$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits two nontrivial solutions in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Remark 1.2. *In fact, conditions* (I_4) *and* (I_5) *imply condition* (I_5') *by direct calculation. Hence, Theorem 1.1 generalizes Theorem B.*

Theorem 1.3. *Assume that* $\lambda_k < \lambda_{k+1}$ $(k \geq 1)$ *are two consecutive eigenvalues of* (3), (I_1) *and* (I_5) *hold,* $F(x,t)$ *satisfies the following condition:*

(I_6) $\lambda_k < \liminf\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} \leq \limsup\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} \leq \lambda_{k+1}$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits at least one weak solution in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Theorem 1.4. *Assume that* $\lambda_k < \lambda_{k+1}$ $(k \geq 1)$ *are two consecutive eigenvalues of* (3) *and* (I_1) *holds,* $F(x,t)$ *satisfies the following conditions:*

(I_7) $\lim\limits_{|t|\to\infty} \left(f(x,t)t - pF(x,t)\right) = -\infty$ *uniformly for* $x \in \Omega$.

(I_8) $\lambda_k \leq \liminf\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} \leq \limsup\limits_{|t|\to\infty} \frac{pF(x,\ t)}{|t|^p} < \lambda_{k+1}$ *uniformly for* $x \in \Omega$.

Then problem (1) *admits at least one weak solution in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Remark 1.5. *We note that the nonquadratic type conditions such as* (I_5) *(or* (I_7)*) and the asymptotic noncrossing conditions such as* (I_6) *(or* (I_8)*) were originally introduced by Costa and* Magalhães *[11] for studying a class of semilinear elliptic equations. To the best of our knowledge, it is the first time that these assumptions are shown in p-biharmonic equations with Navier boundary value conditions.*

2 Proof of main results

Firstly, we define the energy functional Φ associated to problem (1) as

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x$$

for every $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. It is well known that Φ is continuously differentiable and

$$\langle \Phi'(u), v\rangle = \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v \mathrm{d}x - \int_\Omega f(x,u)v\mathrm{d}x$$

for every $u, v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Therefore, the weak solutions of problem (1) correspond to the critical points of Φ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [21, 26]).

Next, we recall some useful concepts about Morse Theory (see [10, 22]) and some abstract critical point theorems (see [17, 24]). Consider $\Phi\colon X \to \mathbb{R}$, a C^1-functional defined in a Banach space $(X, \|\cdot\|)$. Let $\mathcal{K} = \big\{u \in X\colon \Phi'(u) = 0\big\}$ be the critical set of Φ, and $\Phi^c = \{u \in X\colon \Phi(u) \le c\}$ denotes the sublevel sets of Φ. The k-th critical group of Φ at an isolated critical point $u \in \mathcal{K}$ is described by

$$C_k(\Phi, u) = H_k(\Phi^c \cap \mathcal{U}, \Phi^c \cap \mathcal{U} \setminus \{u\}), \quad k \in \mathbb{Z},$$

where $\mathcal{U}$ is a neighborhood of u containing no other critical points, and $H_k(A, B)$ denotes the k-th singular relative homology group of the topological pair (A, B) with the coefficients in a field $\mathbb{F}$. For any $a < \Phi(\mathcal{K})$, the k-th critical group of Φ at infinity is defined by

$$C_k(\Phi, \infty) = H_k(X, \Phi^a), \quad k \in \mathbb{Z}.$$

Definition 2.1. *[18]. Let X be a real Banach space such that $X = V \oplus W$, where V and W are closed subspaces of X. Let $\Phi : X \to \mathbb{R}$ be a C^1-functional. We say that Φ has a local linking near the origin 0 (with respect to the decomposition $X = V \oplus W$), if there exists $\rho > 0$ such that*

$$\begin{aligned} &u \in V : \|u\| \le \rho \Longrightarrow \Phi(u) \le 0, \\ &u \in W : 0 < \|u\| \le \rho \Longrightarrow \Phi(u) > 0. \end{aligned}$$

Theorem 2.2. *[19, Theorem 2.1]. Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ be bounded from below and satisfying the (PS) condition. Assume that Φ has a critical point u which is homologically nontrivial, that is, $C_k(\Phi, u) \ncong \{0\}$ for some k, and it is not a minimizer for Φ. Then Φ admits at least three critical points.*

Definition 2.3. *[24, Definition 2.1]. Let Q be a submanifold of a Banach space X with relative boundary ∂Q, S be a closed subset of a Banach space Y and G be a subset of $C^0(\partial Q, Y \setminus S)$. We say S and ∂Q are G-link if for any map $h \in C^0(Q, Y \setminus S)$ such that $h|_{\partial Q} \in G$ there holds $h(Q) \cap S \neq \emptyset$.*

Theorem 2.4. *[24, G-link Theorem]. Suppose that X, Y are Banach spaces. Consider a closed subset $S \subset Y$ and a submanifold $Q \subset X$ with relative boundary ∂Q, G is a subset of $C^0(\partial Q, Y \setminus S)$. Set $\Gamma = \{h \in C^0(Q, Y) : h|_{\partial Q} \in G\}$. Suppose S and ∂Q are G-link and $\Phi \in C^1(Y, \mathbb{R})$ satisfies*

(a) There exists $h_0 \in \Gamma$ and a positive constant r such that

$$\sup_{x \in Q} \Phi(h_0(x)) < r;$$

(b) There exist $\beta > \alpha$ such that

$$\inf_{y \in S} \Phi(y) \ge \beta \quad \textit{and} \quad \sup_{x \in \partial Q} \Phi(h(x)) \le \alpha, \quad \forall h \in \Gamma;$$

(c) Φ satisfies the (PS) condition. Then, the number

$$c := \inf_{h \in \Gamma} \sup_{x \in Q} \Phi(h(x))$$

defines a critical values $c \ge \beta$ of Φ.

Remark 2.5. *If the (PS) condition is replaced by the weaker (C) condition (see [3]), the deformation lemma (see [22]) can be proved. Hence, it is still sufficient to ensure that the above abstract critical point theorems hold.*

Lemma 2.6. *Assume that* (I_1) *and* (I_5') *hold, then* Φ *is coercive on* $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$*, that is,* $\Phi(u)\to+\infty$ *as* $\|u\|\to\infty$.

Proof. By contradiction, let $\{u_n\}\subset W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ be such that

$$\begin{aligned} &\|u_n\|\to\infty \quad \text{as } n\to\infty, \quad \text{and}\\ &\Phi(u_n)\le M_1, \end{aligned} \tag{4}$$

for some $M_1\in\mathbb{R}$. Let $v_n:=\frac{u_n}{\|u_n\|}$, then we can assume that there exists $v\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, up to a subsequence still denoted by $\{v_n\}$, such that

$$\begin{aligned} &v_n\rightharpoonup v \text{ weakly in } W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega),\\ &v_n\to v \text{ strongly in } L^{\gamma}(\Omega), \quad \gamma\in\big[1,p_2^*\big). \end{aligned} \tag{5}$$

From (4), it follows that there is $M_2>0$ such that

$$\begin{aligned} \frac{M_1}{\|u_n\|^p} &\ge \frac{\Phi(u_n)}{\|u_n\|^p}\\ &=\frac{1}{p}\int_\Omega|\Delta v_n|^p\mathrm{d}x-\frac{1}{\|u_n\|^p}\int_\Omega F(x,u_n)\mathrm{d}x\\ &=\frac{1}{p}\big(\|v_n\|^p-\lambda_1\|v_n\|_p^p\big)-\frac{1}{\|u_n\|^p}\int_\Omega\Big(F(x,u_n)-\frac{\lambda_1}{p}|u_n|^p\Big)\mathrm{d}x\\ &\ge\frac{1}{p}\big(\|v_n\|^p-\lambda_1\|v_n\|_p^p\big)-\frac{M_2}{\|u_n\|^p}. \end{aligned} \tag{6}$$

Therefore, the combination of (4), (5) and (6) implies that

$$\limsup_{n\to\infty}\|v_n\|^p\le\lambda_1\|v\|_p^p. \tag{7}$$

From the lower semi-continuity of the norm and the Poincaré inequality, it follows that

$$\begin{aligned} \lambda_1\|v\|_p^p\le\|v\|^p&\le\liminf_{n\to\infty}\|v_n\|^p\\ &\le\limsup_{n\to\infty}\|v_n\|^p. \end{aligned} \tag{8}$$

Combining (7) and (8), we have $\|v\|^p=\lambda_1\|v\|_p^p$ and $v_n\to v$ in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ with $\|v\|=1$. Thus, we have $v=\pm\phi_1$, and we take $v=\phi_1$. It follows from the definition of v_n and (I_5') that

$$u_n(x)\to+\infty \quad \text{uniformly for } x\in\Omega,$$

which implies that

$$M_1\ge-\int_\Omega\Big(F(x,u_n)-\frac{\lambda_1}{p}|u_n|^p\Big)\mathrm{d}x\to+\infty \quad \text{as } n\to\infty,$$

a contradiction with (4). So Φ is coercive on $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. The proof of Lemma 2.6 is completed. □

Lemma 2.7. *Suppose that* (I_1)*,* (I_5) *and* (I_6) *(or* (I_7) *and* (I_8) *) hold, then the functional* Φ *satisfies the (C) condition in* $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.

Proof. Here, we only prove this lemma under conditions (I_5) and (I_6), because the proof under conditions (I_7) and (I_8) is similar.

Firstly, we prove that Φ satisfies the (C) condition, that is, for any $c \in \mathbb{R}$ and every sequence $\{u_n\} \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\Phi(u_n) \to c \text{ and } \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{as } n \to \infty \tag{9}$$

has a convergent subsequence. If $\{u_n\}$ is unbounded, without loss of generality, we can suppose that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from (9) that there exists a constant M_3 such that

$$\begin{aligned} M_3 &\geq \liminf_{n\to\infty} \big(p\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \big) \\ &= \liminf_{n\to\infty} \int_\Omega \big(f(x, u_n)u_n - pF(x, u_n)\big)\mathrm{d}x. \end{aligned} \tag{10}$$

For any $\varepsilon_1 > 0$, from (I_6), there is a positive constant $M_4 = M_4(\varepsilon_1)$ such that

$$\big|F(x, t)\big| \leq \frac{\lambda_{k+1} + \varepsilon_1}{p}|t|^p + M_4 \tag{11}$$

for all $(x, t) \in \Omega \times \mathbb{R}$. Combining (9) and (11), for n large enough, we have

$$\begin{aligned} c + 1 &\geq \Phi(u_n) \\ &= \frac{1}{p}\int_\Omega |\Delta u_n|^p \mathrm{d}x - \int_\Omega F(x, u_n)\mathrm{d}x \\ &\geq \frac{1}{p}\int_\Omega |\Delta u_n|^p \mathrm{d}x - \Big(\frac{\lambda_{k+1} + \varepsilon_1}{p}\Big)\int_\Omega |u_n|^p \mathrm{d}x - M_4|\Omega| \\ &\geq \frac{1}{p}\|u_n\|^p - \Big(\frac{\lambda_{k+1} + \varepsilon_1}{p}\Big)\|u_n\|_p^p - M_4|\Omega|, \end{aligned} \tag{12}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Now, define $z_n := \frac{u_n}{\|u_n\|}$, then we can also assume, up to a subsequence, that there exists $z \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $z_n \rightharpoonup z$ weakly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $z_n \to z$ strongly in $L^\gamma(\Omega)$, $\gamma \in \big[1, p_2^*\big)$. Particularly, dividing by $\|u_n\|^p$ and letting $n \to \infty$ in (12), we obtain

$$1 \leq \big(\lambda_{k+1} + \varepsilon_1\big)\|z_n\|_p^p,$$

which limplies that there exists $\widetilde{\Omega} \subset \Omega$ with $meas(\widetilde{\Omega}) > 0$ satisfying $z(x) \neq 0$ for $x \in \widetilde{\Omega}$. Therefore, for $x \in \widetilde{\Omega}$, we have

$$|u(x)| \to \infty \quad \text{as} \quad n \to \infty.$$

In addition, by Fatou's lemma and (I_5), it follows that there is a constant M_5 such that

$$\begin{aligned} \liminf_{n\to\infty} \int_\Omega \big(f(x, u_n)u_n - pF(x, u_n)\big)\mathrm{d}x &= \liminf_{n\to\infty} \int_{\widetilde{\Omega}} \big(f(x, u_n)u_n - pF(x, u_n)\big)\mathrm{d}x \\ &\quad + \liminf_{n\to\infty} \int_{\Omega\setminus\widetilde{\Omega}} \big(f(x, u_n)u_n - pF(x, u_n)\big)\mathrm{d}x \\ &\geq \int_{\widetilde{\Omega}} \liminf_{n\to\infty} \big(f(x, u_n)u_n - pF(x, u_n)\big)\mathrm{d}x \\ &\quad + M_5|\Omega \setminus \widetilde{\Omega}| \\ &\to +\infty, \end{aligned}$$

we obtain a contradiction with (11). Hence, we conclude that $\{u_n\}$ is bounded in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, i.e., there exists a constant $M_6 > 0$ such that

$$\|u_n\| \le M_6 \tag{13}$$

for all $n \in \mathbb{N}$. Therefore, there is a $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),\\ &u_n \to u \text{ strongly in } L^{\gamma}(\Omega), \quad \gamma \in \big[1, p_2^*\big). \end{aligned} \tag{14}$$

It follows from (I_1), Hölder's inequality and (2) that

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n-u)\mathrm{d}x\right| \le &\int_\Omega \big|f(x,u_n)\big|\big|u_n-u\big|\mathrm{d}x\\ \le &C\int_\Omega \big(1+|u_n|^{q-1}\big)\big|u_n-u\big|\mathrm{d}x\\ \le &C\bigg(\int_\Omega |u_n|^{(q-1)\frac{q}{q-1}}\mathrm{d}x\bigg)^{\frac{q-1}{q}}\bigg(\int_\Omega |u_n-u|^q\mathrm{d}x\bigg)^{\frac{1}{q}}\\ &+C\|u_n-u\|_1\\ = &C\|u_n\|_q^{q-1}\|u_n-u\|_q + C\|u_n-u\|_1\\ \le &CC_0^{q-1}\|u_n\|^{q-1}\|u_n-u\|_q + C\|u_n-u\|_1. \end{aligned}$$

By (13) and (14), we have

$$\lim_{n\to\infty}\left|\int_\Omega f(x,u_n)(u_n-u)\mathrm{d}x\right| = 0. \tag{15}$$

In addition, from (9), it holds that

$$\begin{aligned} \Big|\big\langle \Phi'(u_n), u_n-u\big\rangle\Big| &\le \|\Phi'(u_n)\|\|u_n-u\|\\ &\le \|\Phi'(u_n)\|\big(M_6+\|u\|\big)\\ &\to 0 \end{aligned} \tag{16}$$

as $n\to\infty$. By (15) and (16), one has

$$\begin{aligned} \big\langle \Phi'(u_n), u_n-u\big\rangle - \int_\Omega f(x,u_n)(u_n-u)\mathrm{d}x &= \int_\Omega |\Delta u_n|^{p-2}\Delta u_n\Delta\big(u_n-u\big)\mathrm{d}x\\ &\to 0 \end{aligned} \tag{17}$$

as $n\to\infty$. From (15), it follows that

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta\big(u_n-u\big)\mathrm{d}x \to 0 \tag{18}$$

as $n\to\infty$. Combining (17) and (18), we obtain that

$$\int_\Omega \Big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\Big)\Delta\big(u_n-u\big)\mathrm{d}x \to 0$$

as $n \to \infty$. Based on Clarkson's inequality [8], that is, there exists a positive constant L_p such that

$$|x-y|^p \leq \begin{cases} L_p\big(|x|^{p-2}x-|y|^{p-2}y\big)\big(x-y\big), & p \geq 2, \\ L_p\Big(\big(|x|^{p-2}x-|y|^{p-2}y\big)\big(x-y\big)\Big)^{\frac{p}{2}}\big(|x|+|y|\big)^{\frac{(2-p)p}{2}}, & 1<p<2, \end{cases}$$

for all $x, y \in \mathbb{R}$. So, we have

$$\int_\Omega \big|\Delta u_n - \Delta u\big|^p \mathrm{d}x \to 0$$

as $n \to \infty$, which implies that $u_n \to u$ strongly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and the proof is completed. □

Lemma 2.8. *Assume that F satisfies assumptions (I_1), (I_5) and (I_6) , then the functional Φ satisfies the conditions (a) and (b) in Theorem 2.4.*

Proof. Define a symmetric closed subset of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ by

$$\mathcal{C}_{k+1} = \Big\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\colon \int_\Omega |\Delta u|^p \mathrm{d}x \geq \lambda_{k+1} \int_\Omega |u|^p \mathrm{d}x\Big\}.$$

For any $\varepsilon_2 > 0$, it follows from (I_1) and (I_6) that there is a positive constant $M_7 = M_7(\varepsilon_2)$ such that

$$F(x,t) \geq \frac{\lambda_k + 2\varepsilon_2}{p}|t|^p - M_7 \tag{19}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. By the definition of λ_k, there is $\Lambda \in \Sigma_k$ such that

$$\sup_{u \in \Lambda} \int_\Omega |\Delta u|^p \mathrm{d}x \leq \lambda_k + \varepsilon_2. \tag{20}$$

For any $u \in \Lambda$ and $s > 0$, from (19) and (20), it holds that

$$\begin{aligned} \Phi(su) &\leq \frac{1}{p}\|su\|^p - \frac{\lambda_k + 2\varepsilon_2}{p}\|su\|_p^p + M_7|\Omega| \\ &\leq -\frac{\varepsilon_2}{p}s^p + M_7|\Omega|. \end{aligned} \tag{21}$$

Moreover, let $H(x,t) = F(x,t) - \frac{\lambda_{k+1}}{p}|t|^p$, we have

$$H'(x,t)t - pH(x,t) = f(x,t)t - pF(x,t).$$

It follows from (I_5) that for any $L > 0$, there exists a constant $M_8 = M_8(L) > 0$ such that

$$H'(x,t)t - pH(x,t) \geq L \tag{22}$$

for $x \in \Omega$ and $|t| > M_8$. Hence, as $t > M_8$, from (22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{H(x,t)}{t^p}\bigg) = \frac{H'(x,t)t - pH(x,t)}{t^{p+1}} \geq \frac{L}{t^{p+1}} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg(-\frac{L}{pt^p}\bigg). \tag{23}$$

Integrating (23) over the interval $[t,s] \subset [M_8, \infty)$, we obtain

$$\frac{H(x,t)}{t^p} \leq \frac{H(x,s)}{s^p} + \frac{L}{p}\bigg(\frac{1}{s^p} - \frac{1}{t^p}\bigg). \tag{24}$$

By (I_6), we have

$$\limsup_{s\to+\infty}\frac{H(x,s)}{s^p}\le 0.$$

Letting $s\to+\infty$ in (24), we get

$$H(x,t)\le -\frac{L}{p}$$

for $x\in\Omega$ and $t\ge M_8$. Similarly, we have $H(x,t)\le -\frac{L}{p}$ for $x\in\Omega$ and $t\le -M_8$. So, by the arbitrariness of L, we deduce that

$$\lim_{|t|\to\infty}H(x,t)=-\infty\quad \text{uniformly for } x\in\Omega. \tag{25}$$

Hence, it follows from (I_1) that there exists $M_9>0$ such that

$$H(x,t)<M_9 \tag{26}$$

for all $(x,t)\in\Omega\times\mathbb{R}$. Therefore, for any $u\in\mathcal{C}_{k+1}$, by (26), one has

$$\begin{aligned}\Phi(u)&=\frac{1}{p}\|u\|^p-\int_\Omega F(x,u)\mathrm{d}x\\&=\frac{1}{p}\big(\|u\|^p-\lambda_{k+1}\|u\|_p^p\big)-\int_\Omega H(x,u)\mathrm{d}x\\&\ge-\int_\Omega H(x,u)\mathrm{d}x\\&\ge -M_9|\Omega|.\end{aligned}\tag{27}$$

Therefore, for fixed $\Lambda\in\Sigma_k$, the combination of (21) and (27) implies that there exists $R>0$ such that

$$\alpha:=\max_{u\in\Lambda,s\ge R}\Phi(su)<-M_9|\Omega|=:\beta. \tag{28}$$

Moreover, we note $R\Lambda=\{su\colon u\in\Lambda, s\ge R\}$ and $G=\{h\in C(S^{k-1},W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))\colon h$ is odd and $h(S^{k-1})\subset R\Lambda\}$, where $S^{k-1}=\partial B^k$ denotes the boundary of the closed unit ball B^k in $\mathbb{R}^k$. From (28), it follows that for any $h\in G$, we obtain

$$h\big(S^{k-1}\big)\cap\mathcal{C}_{k+1}=\emptyset,$$

which limplies that $G\subset C(S^{k-1},W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\mathcal{C}_{k+1})$.

Now, we claim that let

$$\Gamma=\{h\in C(B^k,W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))\colon\ h|_{S^{k-1}}\in G\},$$

then Γ is nonempty, and

$$h\big(B^k\big)\cap\mathcal{C}_{k+1}\neq\emptyset$$

for all $h\in\Gamma$. If this claim is true, it means that $\mathcal{C}_{k+1}$ and S^{k-1} are G-link. On the other hand, from (28) and the compactness of B^k, it follows that (a) and (b) of the Theorem 2.4 is satisfied.

According to the definition of Σ_k, there exists a continuous odd surjection $h\colon S^{k-1} \to \Lambda$. Now, defining $T\colon B^k \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ as

$$T(su) = sRh(u) \qquad \text{for all } (u,s) \in \big(S^{k-1} \times [0,1]\big).$$

Therefore, $T \in \Gamma$, which implies that Γ is nonempty. Moreover, taking $h \in \Gamma$, if there is a $u \in h(B^k)$ with $\|u\|_p = 0$, we have $h\big(B^k\big) \cap \mathcal{C}_{k+1} \neq \emptyset$. Otherwise we define $\eta\colon S^k \to \Sigma$ as

$$\eta(x_1, x_2, ..., x_{k+1}) = \begin{cases} \pi \circ h(x_1, x_2, ..., x_k) & \text{if } \ x_{k+1} \geq 0, \\ -\pi \circ h(-x_1, -x_2, ..., -x_k) & \text{if } \ x_{k+1} < 0, \end{cases}$$

where $\pi(u) = \frac{u}{\|u\|_p^p}$. Using the definition of η, it is easy to see that $\eta(S^k) \in \Sigma_{k+1}$. Thus, there is a $u_0 \in \eta(S^k)$ such that $\|u_0\|^p \geq \lambda_{k+1}$, that is, $u_0 \in \mathcal{C}_{k+1}$. We can conclude that $\pi \circ h(x) \in \mathcal{C}_{k+1}$, which means that $h(x) \in \mathcal{C}_{k+1}$, i.e., $h\big(B^k\big) \cap \mathcal{C}_{k+1} \neq \emptyset$. Now the proof of Lemma 2.8 is completed. □

Proof of Theorem 1.1. By Lemma 2.6, it is clear that Φ is coercive on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and satisfies (PS) condition. As Lemma 2.2 in [21], we can get that Φ has a local linking near the origin 0 (with respect to the decomposition $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = V \oplus W$) under (I_1) and (I_3). By Theorem 2.1 in [17], we get that 0 is a critical point of Φ and $C_1(\Phi, 0) \not\cong \{0\}$. Associating Theorem 2.2, it is sufficient to prove that the functional Φ has two nontrivial critical points in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

Proof of Theorem 1.3. By Lemma 2.7, it is clear that Φ satisfies (C) condition in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Associating Lemma 2.8 and Theorem 2.4, then functional Φ has a critical point in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

Proof of Theorem 1.4. By Lemma 2.7, it is clear that Φ satisfies (C) condition in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Using a method similar to that in Lemma 2.8, we can obtain the same result under conditions (I_7) and (I_8). Therefore, Φ satisfies all assumptions in Theorem 2.4. Then Φ has a critical point in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

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