Closed-Form Solutions of Leonardo-Type Sequences: Pell-Padovan, Jacobsthal-Padovan, and Narayana Families as Homogeneous Counterparts

Abstract. In this study, we derive closed-form expressions for third-order nonhomogeneous linear recurrence relations, termed generalized Leonardo-type sequences, in which the input function is taken to be a polynomial. We explore several notable cases of generalized Tribonacci numbers, which emerge as the homogeneous analogues of the original nonhomogeneous relations. Our analysis includes classical examples such as the adjusted Pell-Padovan sequence, third order Lucas-Pell sequence, third order Fibonacci-Pell sequence, Pell-Perrin sequence, Pell-Padovan sequence, adjusted Jacobsthal-Padovan sequence, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) sequence, Jacobsthal-Padovan sequence, Narayana sequence and Narayana-Lucas sequence.

2020 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Pell-Perrin numbers, Pell-Padovan numbers, Jacobsthal-Perrin numbers, Jacobsthal-Padovan numbers, Narayana numbers, Narayana-Lucas numbers, Leonardo numbers, nonhomogeneous recurrence relations, homogeneous recurrence relations, closed-form solutions, particular solutions.

1. Introduction

Recurrence-based sequences have long been a cornerstone of mathematics, radiating into fields as varied as physics, engineering, architecture, biology, computer science, and even artistic studies. Their definitions may look simple at first glance, yet they conceal profound depth, capturing models of growth, oscillation, and symbolic form. Within the classical second-order families, the Fibonacci, Lucas, Pell, and Jacobsthal sequences stand as enduring exemplars of this tradition.

The horizon, however, stretches far beyond second-order constructions. Higher-order recurrence sequences enrich both theoretical inquiry and practical application, extending the classical framework while unveiling subtle algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences embody this expansion, each shaped by characteristic polynomials whose root arrangements determine closed-form solutions. Homogeneous recurrences emphasize the relationship between characteristic polynomials and root multiplicities, whereas non-homogeneous cases introduce symbolic terms that interact with root structures to produce resonance effects. Collectively, these families weave a unified framework that connects classical recurrence identities with the advancing discipline of symbolic recurrence theory.

In this study, we extend the established line of inquiry by deriving closed-form expressions for third-order nonhomogeneous linear recurrence relations, formulated as generalized Leonardo-type sequences with polynomial inputs. Our analysis further reveals notable instances of generalized Tribonacci numbers, which arise as the homogeneous counterparts of the original nonhomogeneous relations. Particular attention is devoted to classical families, including the adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin and Pell-Padovan sequence; adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (the latter also referred to as Jacobsthal-Perrin-Lucas) and Jacobsthal-Padovan sequences; Narayana and Narayana-Lucas sequences.

Let the third order nonhomogeneous linear recurrence relation, referred to as generalized Leonardo-type sequences, be given by

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + p(n) \tag{1.1}$$

with initial conditions $W_0 = k_0, W_1 = k_1, W_2 = k_2$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i,$$

and the recurrence coefficients a_1, a_2, a_3 are complex scalars or polynomials in $\mathbb{C}[x]$. For more information on generalized Leonardo-type sequences, see Soykan [5] and [6].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

Suppose that θ_1, θ_2 and θ_3 are the roots of the characteristic equation

$$z^3 - a_1z^2 - a_2z - a_3 = 0 \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^3 - a_1 z^2 - a_2 z - a_3 = (z-1)^3 = z^3 - 3z^2 + 3z - 1 = 0$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1$ and (1.2) reduces to

$$V_n = 3V_{n-1} - 3V_{n-2} + V_{n-3}.$$

In earlier work, particular solutions to third-order nonhomogeneous linear recurrence relations (1.1) with polynomial inputs were obtained for $s = 0, 1, 2, 3$; see Soykan [7]. Building on those results, the present paper derives unified closed-form solutions by applying Theorem 1.1. Each recurrence is decomposed into homogeneous and particular components, the latter determined through an iterative coefficient scheme.

The analysis is organized by two parameters: the multiplicity r of 1 as a root of the characteristic equation (1.2), and the degree s of the polynomial $p(n)$. Explicit formulas are obtained for all cases $r = 0, 1, 2, 3$ and $s = 0, 1, 2, 3$, clarifying how multiplicity and polynomial degree jointly shape the solution. While the general framework covers all multiplicities, in this paper we restrict to the non-resonant case $r = 0$, relevant to the examples of generalized Pell-Padovan, Jacobsthal-Padovan, and Narayana numbers. Accordingly, the theorem is stated in full detail only for $r = 0$, with the cases $r = 1, 2, 3$ obtainable analogously but not required here.

THEOREM 1.1.

(a): *[5, Theorem 7.7. (a)] The case $r = 0$, i.e., all three roots of the characteristic equation of (1.2) is distinct from* 1.

The solution of (1.1) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

is the particular solution of (1.1). For each $0 \le i \le s$, A_i can be calculated with the iteration

$$A_s = -\frac{c_s}{a_1 + a_2 + a_3 - 1},\ \text{for } n = s$$

and

$$A_n = -\frac{1}{a_1 + a_2 + a_3 - 1}\left(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+1} \binom{k}{n} (a_1 + 2^{k-n} a_2 + 3^{k-n} a_3) A_k\right),\ \text{for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \\ W_2^{(p)} &= \sum_{i=0}^{s} 2^i A_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i, \sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) W_2^{(p)} + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) W_1^{(p)} + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) W_0^{(p)}$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) W_2^{(p)} + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) W_1^{(p)} + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) W_0^{(p)}$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) W_2^{(p)} + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) W_1^{(p)} + a_3 (a_3 W_1^2 - a_3 W_0 W_2) W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

i.e.,

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) \sum_{i=0}^{s} 2^i A_i + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) \sum_{i=0}^{s} A_i + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) \sum_{i=0}^{s} 2^i A_i + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) \sum_{i=0}^{s} A_i + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) \sum_{i=0}^{s} 2^i A_i + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) \sum_{i=0}^{s} A_i + a_3 (a_3 W_1^2 - a_3 W_0 W_2) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + \sum_{i=0}^{s} A_i n^i$$

- **(b):** *The case $r = 1$, i.e., 1 is a simple root of the characteristic equation of (1.2). Its solution is obtained directly in analogy with case (a).*
- **(c):** *The case $r = 2$, i.e., 1 is a double root of the characteristic equation of (1.2). The construction follows the same method as case (a), with multiplicity corrections applied.*
- **(d):** *The case $r = 3$, i.e., 1 is a triple root of the characteristic equation of (1.2). The solution is again derived analogously to case (a), adjusted for higher multiplicity.*

Proof. The result follows by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104-105), and Theorem 3.1 (pp. 88-89) for the case $m = 3$, as established in Soykan [5]. In general, the theorem applies to all multiplicities $r = 0, 1, 2, 3$: the case $r = 0$ corresponds to the non-resonant situation where all three roots are distinct from 1, while $r = 1, 2, 3$ represent simple, double, or triple roots of 1 in the characteristic equation (1.2). Since this paper requires only the non-resonant case $r = 0$, where all three roots are distinct from 1, the explicit formulations for $r = 1, 2, 3$ are omitted. They can be constructed analogously to case (a) with multiplicity corrections, but are not required here. ☐

2. Examples: Closed-Form Solutions of Third Order nonhomogeneous Linear Recurrence Relations

In the following subsection, we consider special cases of Theorem 1.1 (a) for the special third order nonhomogeneous linear recurrence relations.

2.1. Generalized Pell-Padovan Numbers.

In this subsection, we consider the case $a_1 = 0$, $a_2 = 2$ and $a_3 = 1$. A generalized Pell-Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0,V_1,V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-2} + V_{n-3} \tag{2.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -2V_{-(n-1)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n. For more details on the generalized Pell-Padovan numbers, see Soykan [2].

Binet formula of generalized padovan numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2.2}$$

Here, α, β and γ are the roots of the cubic equation $z^3 - 2z - 1 = 0$. Moreover

$$\begin{aligned} \alpha &= \frac{1+\sqrt{5}}{2}, \\ \beta &= \frac{1-\sqrt{5}}{2}, \\ \gamma &= -1. \end{aligned}$$

Now we define five special cases of the sequence $\{V_n\}$. Adjusted Pell-Padovan sequence $\{M_n\}_{n\geq 0}$, third order Lucas-Pell sequence $\{B_n\}_{n\geq 0}$ (OEIS: A099925, [1]), third order Fibonacci-Pell sequence $\{G_n\}_{n\geq 0}$ (OEIS: A008346, [1]), Pell-Perrin sequence $\{C_n\}_{n\geq 0}$, Pell-Padovan sequence$\{R_n\}_{n\geq 0}$ (OEIS: A066983, [1]), are defined, respectively, by the third-order recurrence relations

$$M_{n+3} = 2M_{n+1} + M_n, \quad M_0 = 0, M_1 = 1, M_2 = 0, \tag{2.3}$$

$$B_{n+3} = 2B_{n+1} + B_n, \quad B_0 = 3, B_1 = 0, B_2 = 4 \tag{2.4}$$

$$G_{n+3} = 2G_{n+1} + G_n, \quad G_0 = 1, G_1 = 0, G_2 = 2, \tag{2.5}$$

$$C_{n+3} = 2C_{n+1} + C_n, \quad C_0 = 3, C_1 = 0, C_2 = 2, \tag{2.6}$$

$$R_{n+3} = 2R_{n+1} + R_n, \quad R_0 = 1, R_1 = 1, R_2 = 1. \tag{2.7}$$

The sequences $\{M_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, $\{G_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= -2M_{-(n-1)} + M_{-(n-3)}, \\ B_{-n} &= -2B_{-(n-1)} + B_{-(n-3)}, \\ G_{-n} &= -2G_{-(n-1)} + G_{-(n-3)}, \\ C_{-n} &= -2C_{-(n-1)} + C_{-(n-3)}, \\ R_{-n} &= -2R_{-(n-1)} + R_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.7) hold for all integer n.

Note that for all integers n, adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers can be expressed using Binet's formulas as

$$\begin{aligned}
M_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1} \\
&= (\frac{1}{2}-\frac{1}{10}\sqrt{5})\alpha^n+(\frac{1}{2}+\frac{1}{10}\sqrt{5})\beta^n-\gamma^n, \\
B_n &= \alpha^n+\beta^n+\gamma^n, \\
G_n &= \frac{1}{\sqrt{5}}\alpha^n-\frac{1}{\sqrt{5}}\beta^n+\gamma^n, \\
C_n &= (2-\frac{3}{\sqrt{5}})\alpha^n+(2+\frac{3}{\sqrt{5}})\beta^n-\gamma^n, \\
R_n &= (1-\frac{1}{\sqrt{5}})\alpha^n+(1+\frac{1}{\sqrt{5}})\beta^n-\gamma^n,
\end{aligned}$$

respectively, see Soykan [2] for more details.

B_n is the sequence A099925 in [1] associated with the relation

$$B_n = L_n + (-1)^n$$

where L_n is Lucas sequence which is given as

$$L_n = L_{n-1} + L_{n-2} \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

G_n is the sequence A008346 in [1] associated with the relation

$$G_n = F_n + (-1)^n$$

where F_n is Fibonacci sequence which is given as

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

C_n is not indexed in [1].

R_n is the sequence A066983 in [1] associated with the relation

$$R_{n+2} = R_{n+1} + R_n + (-1)^n, \quad \text{with} \quad R_1 = R_2 = 1.$$

Since

$$F_{-n} = (-1)^{n+1}F_n \text{ and } L_{-n} = (-1)^n L_n$$

we get

$$G_{-n} = (-1)^{n+1}G_n + 1 + (-1)^n = (-1)^n(1 - F_n)$$

and

$$B_{-n} = (-1)^n B_n - 1 + (-1)^n = (-1)^n(L_n + 1).$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2 = 2,\ a_3 = 1, \\ W_0 &= 0,\ W_1 = 1,\ W_2 = 0, \end{aligned}$$

so that we have the case $V_n = M_n$ (adjusted Pell-Padovan numbers).

EXAMPLE 2.1. *In this example, we consider the case* $a_1 = 0,\ a_2 = 2,\ a_3 = 1, W_0 = 0,\ W_1 = 1,\ W_2 = 0$ *in Theorem 1.1 (a). So* $V_n(0,1,0) = V_n$ *represents nth adjusted Pell-Padovan number, i.e.,* $V_n = M_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_0$$

is given by

$$W_n(0,1,0) = \frac{1}{2}\left(c_0 + 2\right) M_n + \frac{1}{2} c_0 M_{n-1} + \frac{1}{2} c_0 M_{n-2} - \frac{1}{2} c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(0,1,0) = \frac{1}{4}(2c_0 + 9c_1 + 4)M_n + \frac{1}{4}(2c_0 + 11c_1)M_{n-1} + \frac{1}{4}(2c_0 + 7c_1)M_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1,0) = \frac{1}{4}\left(2c_0 + 9c_1 + 48c_2 + 4\right) M_n + \frac{1}{4}\left(2c_0 + 11c_1 + 68c_2\right) M_{n-1} + \frac{1}{4}\left(2c_0 + 7c_1 + 32c_2\right) M_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1 + 32c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,0) \;=\; \frac{1}{8}\left(4c_0 + 18c_1 + 96c_2 + 639c_3 + 8\right) M_n \;+\; \frac{1}{8}\left(4c_0 + 22c_1 + 136c_2 + 985c_3\right) M_{n-1} \;+\; \frac{1}{8}\left(4c_0 + 14c_1 + 64c_2 + 401c_3\right) M_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 21c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2 + 48c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 14c_1 + 64c_2 + 401c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2 = 2,\ a_3 = 1, \\ W_0 &= 3,\ W_1 = 0,\ W_2 = 4, \end{aligned}$$

so that we have the case $V_n = B_n$ (third order Lucas-Pell numbers).

EXAMPLE 2.2. *In this example, we consider the case $a_1 = 0,\ a_2 = 2,\ a_3 = 1, W_0 = 3,\ W_1 = 50,\ W_2 = 4$ in Theorem 1.1 (a). So $V_n(3,0,4) = V_n$ represents nth third order Lucas-Pell number, i.e., $V_n = B_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_0$$

is given by

$$W_n(3,0,4) = -\frac{1}{10}\left(7c_0 - 10\right) B_n + \frac{9}{10}c_0B_{n-1} + \frac{11}{10}c_0B_{n-2} - \frac{1}{2}c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_1 n + c_0$$

is given by

$W_n(3,0,4) = -\frac{1}{20}(14c_0+29c_1-20)B_n + \frac{1}{20}(18c_0+53c_1)B_{n-1} + \frac{1}{20}(22c_0+57c_1)B_{n-2} + A_1 n + A_0$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0+7c_1). \end{aligned}$$

(c): *The case $s=2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,0,4) = -\frac{1}{20}\left(14c_0+29c_1+32c_2-20\right)B_n + \frac{1}{20}\left(18c_0+53c_1+184c_2\right)B_{n-1} + \frac{1}{20}\left(22c_0+57c_1+156c_2\right)B_{n-2}$
$A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+7c_2), \\ A_0 &= -\frac{1}{4}(2c_0+7c_1+32c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,0,4) = -\frac{1}{40}(28c_0+58c_1+64c_2-437c_3-40)B_n + \frac{1}{40}(36c_0+106c_1+368c_2+1771c_3)B_{n-1} + \frac{1}{40}(44c_0+114c_1+312c_2+1059c_3)B_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2+21c_3), \\ A_1 &= -\frac{1}{2}(c_1+7c_2+48c_3), \\ A_0 &= -\frac{1}{8}(4c_0+14c_1+64c_2+401c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2 = 2,\ a_3 = 1, \\ W_0 &= 1,\ W_1 = 0,\ W_2 = 2, \end{aligned}$$

so that we have the case $V_n = G_n$ (third order Fibonacci-Pell numbers).

Example 2.3. *In this example, we consider the case* $a_1 = 0,\ a_2 = 2,\ a_3 = 1, W_0 = 1,\ W_1 = 0,\ W_2 = 2$ *in Theorem 1.1 (a). So* $V_n(1,0,2) = V_n$ *represents nth third order Fibonacci-Pell number, i.e.,* $V_n = G_n$*.*

(a): *The case* $s = 0$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_0$$

is given by

$$W_n(1,0,2) = \frac{1}{2}(c_0+2)G_n + \frac{1}{2}c_0G_{n-1} - \frac{1}{2}c_0G_{n-2} - \frac{1}{2}c_0.$$

(b): *The case* $s = 1$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_1n + c_0$$

is given by

$$W_n(1,0,2) = \frac{1}{4}(2c_0+7c_1+4)G_n + \frac{1}{4}(2c_0+9c_1)G_{n-1} - \frac{1}{4}(2c_0+3c_1)G_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0+7c_1). \end{aligned}$$

(c): *The case* $s = 2$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(1,0,2) = \frac{1}{4}(2c_0+7c_1+32c_2+4)G_n + \frac{1}{4}(2c_0+9c_1+48c_2)G_{n-1} - \frac{1}{4}(2c_0+3c_1-4c_2)G_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+7c_2), \\ A_0 &= -\frac{1}{4}(2c_0+7c_1+32c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(1,0,2) = \frac{1}{8}(4c_0 + 14c_1 + 64c_2 + 401c_3 + 8)G_n + \frac{1}{8}(4c_0 + 18c_1 + 96c_2 + 639c_3)G_{n-1} - \frac{1}{8}(4c_0 + 6c_1 - 8c_2 - 183c_3)G_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 21c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2 + 48c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 14c_1 + 64c_2 + 401c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2 = 2,\ a_3 = 1, \\ W_0 &= 3,\ W_1 = 0,\ W_2 = 2, \end{aligned}$$

so that we have the case $V_n = C_n$ (Pell-Perrin numbers).

Example 2.4. *In this example, we consider the case $a_1 = 0$, $a_2 = 2$, $a_3 = 1$, $W_0 = 3$, $W_1 = 0$, $W_2 = 2$ in Theorem 1.1 (a). So $V_n(3,0,2) = V_n$ represents nth Pell-Perrin number, i.e., $V_n = C_n$.*

(a): *The case $s=0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_0$$

is given by

$$W_n(3,0,2) = \frac{1}{22}(13c_0 + 22)C_n - \frac{3}{22}c_0C_{n-1} - \frac{5}{22}c_0C_{n-2} - \frac{1}{2}c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,0,2) = \frac{1}{44}(26c_0 + 83c_1 + 44)C_n - \frac{1}{44}(6c_0 - 13c_1)C_{n-1} - \frac{5}{44}(2c_0 + 3c_1)C_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,0,2) = \frac{1}{44}(26c_0 + 83c_1 + 320c_2 + 44)C_n - \frac{1}{44}(6c_0 - 13c_1 - 224c_2)C_{n-1} - \frac{1}{44}(10c_0 + 15c_1 - 36c_2)C_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1 + 32c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2_2W_{n-2} + W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,0,2) = \frac{1}{88}(52c_0 + 166c_1 + 640c_2 + 3397c_3 + 88)C_n - \frac{1}{88}(12c_0 - 26c_1 - 448c_2 - 4139c_3)C_{n-1} - \frac{1}{88}(20c_0 + 30c_1 - 72c_2 - 1347c_3)C_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 21c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2 + 48c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 14c_1 + 64c_2 + 401c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2 = 2,\ a_3 = 1, \\ W_0 &= 1,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = R_n$ (Pell-Padovan numbers).

Example 2.5. *In this example, we consider the case* $a_1 = 0,\ a_2 = 2,\ a_3 = 1, W_0 = 1,\ W_1 = 1,\ W_2 = 1$ *in Theorem 1.1 (a). So* $V_n(1,1,1) = V_n$ *represents nth Pell-Padovan number, i.e.,* $V_n = R_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_0$$

is given by

$$W_n(1,1,1) = \frac{1}{2}(c_0+2)R_n - \frac{1}{2}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(1,1,1) = \frac{1}{4}\left(2c_0+7c_1+4\right)R_n + \frac{3}{4}c_1R_{n-1} + \frac{1}{4}c_1R_{n-2} - \frac{1}{2}c_1 n - \frac{1}{4}(2c_0+7c_1).$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_2n^2 + c_1 n + c_0$$

is given by

$$W_n(1,1,1) = \tfrac{1}{4}(2c_0+7c_1+30c_2+4)R_n + \tfrac{1}{4}(3c_1+28c_2)R_{n-1} + \tfrac{1}{4}(c_1+10c_2)R_{n-2} + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+7c_2), \\ A_0 &= -\frac{1}{4}(2c_0+7c_1+32c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-2} + W_{n-3} + c_3n^3 + c_2n^2 + c_1 n + c_0$$

is given by

$$W_n(1,1,1) = \tfrac{1}{8}\left(4c_0+14c_1+60c_2+347c_3+8\right)R_n + \tfrac{1}{8}\left(6c_1+56c_2+465c_3\right)R_{n-1} + \tfrac{1}{8}\left(2c_1+20c_2+173c_3\right)R_{n-2}+$$
$$A_3n^3 + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2+21c_3), \\ A_1 &= -\frac{1}{2}(c_1+7c_2+48c_3), \\ A_0 &= -\frac{1}{8}(4c_0+14c_1+64c_2+401c_3). \end{aligned}$$

2.2. Generalized Jacobsthal-Padovan Numbers. In this subsection, we consider the case $a_1 = 0$, $a_2 = 1$ and $a_3 = 2$. A generalized Jacobsthal-Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-2} + 2V_{n-3} \tag{2.8}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.8) holds for all integer n. For more information on Jacobsthal-Padovan sequence, see Soykan [3].

Binet formula of generalized Jacobsthal-Padovan numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$\begin{aligned} b_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, & (2.9) \\ b_2 &= V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, & (2.10) \\ b_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. & (2.11) \end{aligned}$$

Here, α, β and γ are the roots of the cubic equation

$$z^3 - z - 2 = 0.$$

Moreover

$$\begin{aligned} \alpha &= \sqrt[3]{1+\frac{\sqrt{78}}{9}} + \sqrt[3]{1-\frac{\sqrt{78}}{9}} \simeq 1.521379706804568, \\ \beta &= \omega\sqrt[3]{1+\frac{\sqrt{78}}{9}} + \omega^2\sqrt[3]{1-\frac{\sqrt{78}}{9}}, \\ \gamma &= \omega^2\sqrt[3]{1+\frac{\sqrt{78}}{9}} + \omega\sqrt[3]{1-\frac{\sqrt{78}}{9}}, \end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n\geq 0}$ (OEIS: A159287, [1]), Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) sequence $\{L_n\}_{n\geq 0}$ (OEIS: A072328, [1]), Jacobsthal-Padovan sequence $\{Q_n\}_{n\geq 0}$ (OEIS: A159284, [1]) are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} K_{n+3} &= K_{n+1} + 2K_n, \qquad K_0 = 0, K_1 = 1, K_2 = 0, && (2.12) \\ L_{n+3} &= L_{n+1} + 2L_n, \qquad L_0 = 3, L_1 = 0, L_2 = 2, && (2.13) \\ Q_{n+3} &= Q_{n+1} + 2Q_n, \qquad Q_0 = 1, Q_1 = 1, Q_2 = 1. && (2.14) \end{aligned}$$

The sequences $\{K_n\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$, and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} K_{-n} &= -\frac{1}{2}K_{-(n-1)} + \frac{1}{2}K_{-(n-3)}, \\ L_{-n} &= -\frac{1}{2}L_{-(n-1)} + \frac{1}{2}L_{-(n-3)}, \\ Q_{-n} &= -\frac{1}{2}Q_{-(n-1)} + \frac{1}{2}Q_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.12)-(2.14) hold for all integer n.

Note that for all integers n, adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan numbers can be expressed using Binet's formulas as

$$\begin{aligned} K_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \\ L_n &= \alpha^n + \beta^n + \gamma^n, \\ Q_n &= \frac{(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{(\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \end{aligned}$$

respectively, see Soykan [3] for more details.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2 = 1,\ a_3 = 2, \\ W_0 &= 0,\ W_1 = 1,\ W_2 = 0, \end{aligned}$$

so that we have the case $V_n = K_n$ (adjusted Jacobsthal-Padovan numbers).

Example 2.6. *In this example, we consider the case* $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $W_0 = 0$, $W_1 = 1$, $W_2 = 0$ *in Theorem 1.1 (a). So* $V_n(0,1,0) = V_n$ *represents nth adjusted Jacobsthal-Padovan number, i.e.,* $V_n = K_n$.

- **(a):** *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(0,1,0) = \frac{1}{2}\left(c_0+2\right)K_n + \frac{1}{2}c_0K_{n-1} + c_0K_{n-2} - \frac{1}{2}c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(0,1,0) = \frac{1}{2}\left(c_0+5c_1+2\right)K_n + \frac{1}{2}\left(c_0+6c_1\right)K_{n-1} + (c_0+4c_1)K_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{2}(c_0+4c_1). \end{aligned}$$

(c): *The case $s=2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,0) = \frac{1}{2}\left(c_0+5c_1+30c_2+2\right)K_n + \frac{1}{2}\left(c_0+6c_1+41c_2\right)K_{n-1} + (c_0+4c_1+21c_2)K_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+8c_2), \\ A_0 &= -\frac{1}{2}(c_0+4c_1+21c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,0) = \frac{1}{2}\left(c_0+5c_1+30c_2+227c_3+2\right)K_n + \frac{1}{2}\left(c_0+6c_1+41c_2+333c_3\right)K_{n-1} + (c_0+4c_1+21c_2+151c_3)K_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned}
A_3 &= -\frac{1}{2}c_3, \\
A_2 &= -\frac{1}{2}(c_2+12c_3), \\
A_1 &= -\frac{1}{2}(c_1+8c_2+63c_3), \\
A_0 &= -\frac{1}{2}(c_0+4c_1+21c_2+151c_3).
\end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned}
a_1 &= 0,\ a_2=1,\ a_3=2, \\
W_0 &= 3,\ W_1=0,\ W_2=2,
\end{aligned}$$

so that we have the case $V_n = L_n$ (Jacobsthal-Padovan numbers).

EXAMPLE 2.7. *In this example, we consider the case* $a_1 = 0,\ a_2 = 1,\ a_3 = 2, W_0 = 3,\ W_1 = 0,\ W_2 = 2$ *in Theorem 1.1 (a). So* $V_n(3,0,2) = V_n$ *represents nth Jacobsthal-Padovan number, i.e.,* $V_n = L_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(3,0,2) = \frac{1}{26}(5c_0+26)L_n + \frac{9}{52}c_0L_{n-1} + \frac{1}{26}c_0L_{n-2} - \frac{1}{2}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,0,2) = \frac{1}{104}(20c_0+81c_1+104)L_n + \frac{1}{104}(18c_0+95c_1)L_{n-1} + \frac{1}{52}(2c_0+25c_1)L_{n-2} + A_1n + A_0$$

where

$$\begin{aligned}
A_1 &= -\frac{1}{2}c_1, \\
A_0 &= -\frac{1}{2}(c_0+4c_1).
\end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,0,2) = \frac{1}{104}(20c_0+81c_1+427c_2+104)L_n + \frac{1}{104}(18c_0+95c_1+591c_2)L_{n-1} + \frac{1}{52}(2c_0 + 25c_1+213c_2)L_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+8c_2), \\ A_0 &= -\frac{1}{2}(c_0+4c_1+21c_2). \end{aligned}$$

(d): *The case* $s=3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,0,2) = \frac{1}{104}\left(20c_0+81c_1+427c_2+3066c_3+104\right)L_n + \frac{1}{104}\left(18c_0+95c_1+591c_2+4556c_3\right)L_{n-1} + \frac{1}{52}\left(2c_0+25c_1+213c_2+1864c_3\right)L_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{2}(c_2+12c_3), \\ A_1 &= -\frac{1}{2}(c_1+8c_2+63c_3), \\ A_0 &= -\frac{1}{2}(c_0+4c_1+21c_2+151c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 0,\ a_2=1,\ a_3=2, \\ W_0 &= 1,\ W_1=1,\ W_2=1, \end{aligned}$$

so that we have the case $V_n = Q_n$ (Jacobsthal-Padovan numbers).

EXAMPLE 2.8. *In this example, we consider the case* $a_1=0,\ a_2=1,\ a_3=2, W_0=1,\ W_1=1,\ W_2=1$ *in Theorem 1.1 (a). So* $V_n(1,1,1)=V_n$ *represents nth Jacobsthal-Padovan number, i.e.,* $V_n=Q_n$.

(a): *The case* $s=0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + a_3W_{n-3} + c_0$$

is given by

$$W_n(1,1,1) = \frac{1}{2}(c_0+2)Q_n - \frac{1}{2}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(1,1,1) = \frac{1}{4}\left(2c_0 + 7c_1 + 4\right)Q_n + \frac{3}{4}c_1 Q_{n-1} + \frac{1}{2}c_1 Q_{n-2} - \frac{1}{2}c_1 n - \frac{1}{2}(c_0 + 4c_1).$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(1,1,1) = \tfrac{1}{4}(2c_0+7c_1+31c_2+4)Q_n+\tfrac{1}{4}(3c_1+29c_2)Q_{n-1}+\tfrac{1}{2}(c_1+11c_2)Q_{n-2}+A_2n^2+A_1n+A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 8c_2), \\ A_0 &= -\frac{1}{2}(c_0 + 4c_1 + 21c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-2} + 2W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(1,1,1) = \frac{1}{4}(2c_0 + 7c_1 + 31c_2 + 196c_3 + 4)Q_n + \frac{1}{4}(3c_1 + 29c_2 + 258c_3)Q_{n-1} + \frac{1}{2}(c_1 + 11c_2 + 106c_3)Q_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{2}(c_2 + 12c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 8c_2 + 63c_3), \\ A_0 &= -\frac{1}{2}(c_0 + 4c_1 + 21c_2 + 151c_3). \end{aligned}$$

2.3. Generalized Narayana Numbers. In this subsection, we consider the case $a_1 = 1$, $a_2 = 0$ and $a_3 = 1$. A generalized Narayana sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-3} \tag{2.15}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.15) holds for all integer n. For more information on Narayana sequence, see Soykan [4].

Binet formula of generalized Narayana numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2.16}$$

Here, α, β and γ are the roots of the cubic equation $z^3 - z^2 - 1 = 0$.

Moreover

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Narayana sequence$\{N_n\}_{n\geq 0}$ (OEIS: A000930, [1]) and Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ (OEIS: A001609, [1]) are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}
N_{n+3} &= N_{n+2} + N_n, \qquad N_0 = 0, N_1 = 1, N_2 = 1, && (2.17) \\
U_{n+3} &= U_{n+2} + U_n, \qquad U_0 = 3, U_1 = 1, U_2 = 1. && (2.18)
\end{aligned}$$

The sequences $\{N_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
N_{-n} &= -N_{-(n-2)} + N_{-(n-3)}, \\
U_{-n} &= -U_{-(n-2)} + U_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.17)- (2.18) hold for all integer n. For more information on generalized Narayana numbers, see Soykan [4].

Binet's formulas of Narayana and Narayana-Lucas numbers, respectively, are

$$\begin{aligned} N_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ U_n &= \alpha^n + \beta^n + \gamma^n. \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 0,\ a_3 = 1, \\ W_0 &= 0,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = N_n$ (Narayana numbers).

Example 2.9. *In this example, we consider the case* $a_1 = 1,\ a_2 = 0,\ a_3 = 1, 0,\ W_1 = 1,\ W_2 = 1$ *in Theorem 1.1 (a). So* $V_n(0, 1, 1) = V_n$ *represents nth Narayana number, i.e.,* $V_n = N_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_0$$

is given by

$$W_n(0, 1, 1) = (c_0 + 1)N_n + c_0 N_{n-2} - c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(0, 1, 1) = (c_0 + 5c_1 + 1)N_n + c_1 N_{n-1} + (c_0 + 4c_1)N_{n-2} - c_1 n - (c_0 + 4c_1).$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(0, 1, 1) = (c_0 + 5c_1 + 31c_2 + 1)N_n + (c_1 + 11c_2)N_{n-1} + (c_0 + 4c_1 + 22c_2)N_{n-2} + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_2 &= -c_2, \\ A_1 &= -(c_1 + 8c_2), \\ A_0 &= -(c_0 + 4c_1 + 22c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,1) = (c_0 + 5c_1 + 31c_2 + 251c_3 + 1)N_n + (c_1 + 11c_2 + 109c_3)N_{n-1} + (c_0 + 4c_1 + 22c_2 + 172c_3)N_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -c_3, \\ A_2 &= -(c_2 + 12c_3), \\ A_1 &= -(c_1 + 8c_2 + 66c_3), \\ A_0 &= -(c_0 + 4c_1 + 22c_2 + 172c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 0,\ a_3 = 1, \\ W_0 &= 3,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = U_n$ (Narayana-Lucas numbers).

EXAMPLE 2.10. *In this example, we consider the case* $a_1 = 1,\ a_2 = 0,\ a_3 = 1, W_0 = 3,\ W_1 = 1,\ W_2 = 1$ *in Theorem 1.1 (a). So* $V_n(3,1,1) = V_n$ *represents nth Narayana-Lucas number, i.e.,* $V_n = U_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{31}\left(13c_0 + 31\right)U_n + \frac{6}{31}c_0U_{n-1} + \frac{4}{31}c_0U_{n-2} - c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{31}\left(13c_0 + 62c_1 + 31\right)U_n + \frac{1}{31}\left(6c_0 + 31c_1\right)U_{n-1} + \frac{1}{31}\left(4c_0 + 31c_1\right)U_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -c_1, \\ A_0 &= -(c_0 + 4c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,1,1) = \frac{1}{31}(13c_0 + 62c_1 + 388c_2 + 31)U_n + \frac{1}{31}(6c_0 + 31c_1 + 191c_2)U_{n-1} + \frac{1}{31}(4c_0 + 31c_1 + 241c_2)U_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -c_2, \\ A_1 &= -(c_1 + 8c_2), \\ A_0 &= -(c_0 + 4c_1 + 22c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,1,1) = \frac{1}{31}\left(13c_0 + 62c_1 + 388c_2 + 3206c_3 + 31\right)U_n + \frac{1}{31}\left(6c_0 + 31c_1 + 191c_2 + 1525c_3\right)U_{n-1} + \frac{1}{31}\left(4c_0 + 31c_1 + 241c_2 + 2143c_3\right)U_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_3 &= -c_3, \\ A_2 &= -(c_2 + 12c_3), \\ A_1 &= -(c_1 + 8c_2 + 66c_3), \\ A_0 &= -(c_0 + 4c_1 + 22c_2 + 172c_3). \end{aligned}$$

References

[1] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[2] Soykan, Y., Generalized Pell-Padovan Numbers, Asian Journal of Advanced Research and Reports, 11(2), 8-28, 2020. DOI: 10.9734/AJARR/2020/v11i230259

[3] Soykan, Y., A Study on Generalized Jacobsthal-Padovan Numbers, Earthline Journal of Mathematical Sciences, 4(2), 227-251, 2020. https://doi.org/10.34198/ejms.4220.227251

[4] Soykan Y., On Generalized Narayana Numbers, Int. J. Adv. Appl. Math. and Mech. 7(3), 43-56, 2020.

[5] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026. ISSN: 2347-2529

[6] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

[7] Soykan, Y., Explicit Formulas for Polynomial-type Particular Solutions of Generalized Leonardo-type Sequences, Asian Journal of Advanced Research and Reports, 20(5), 164-208, 2026. https://doi.org/10.9734/ajarr/2026/v20i51361