Closed-Form Solutions of Second-Order Leonardo-Type Sequences: Homogeneous Counterparts in Jacobsthal and Mersenne Numbers

Abstract. We present in this work explicit closed-form solutions for second-order nonhomogeneous linear recurrence relations, identified as generalized Leonardo-type sequences, where the input term is a polynomial. Our study focuses on several distinguished cases of generalized Fibonacci numbers, which arise as the homogeneous analogues of the original nonhomogeneous relations. In particular, we consider classical sequences such as the Jacobsthal sequence, Jacobsthal–Lucas sequence, Mersen sequence and Mersen-Lucas sequence.

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1. Introduction

Recurrence relations have long formed a cornerstone of mathematics, extending their reach into physics, engineering, architecture, biology, computer science, and even artistic inquiry. Though their definitions often appear elementary, they conceal remarkable depth, modeling processes of growth, oscillation, and symbolic construction. Classical second-order families—most prominently the Fibonacci, Lucas, Pell, and Jacobsthal sequences—remain enduring exemplars of this tradition.

The framework, however, extends far beyond second-order cases. Higher-order recurrence sequences enrich both theory and practice, broadening the classical foundation while uncovering intricate algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences illustrate this expansion, each governed by characteristic polynomials whose root configurations determine closed-form formulas. Homogeneous recurrences emphasize the interplay between characteristic polynomials and root multiplicities, while nonhomogeneous forms introduce symbolic terms that interact with root structures to produce resonance phenomena. Collectively, these families establish a coherent framework that unites classical recurrence identities with the evolving discipline of symbolic recurrence theory.

This work seeks to enrich the mathematical narrative by unveiling closed-form solutions for second-order nonhomogeneous linear recurrence relations, articulated as generalized Leonardo-type sequences with polynomial inputs. Within this architecture, we discern the emergence of generalized Fibonacci numbers, which stand as the homogeneous reflections of their nonhomogeneous counterparts. Our exploration traverses several classical lineages—most prominently the Jacobsthal and Jacobsthal–Lucas sequences, together with the Mersen sequence and Mersen-Lucas families-each forming a distinct thread in the broader tapestry of recurrence theory.

Let the second order nonhomogeneous linear recurrence relation be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + p(n) \tag{1.1}$$

with initial conditions $W_0 = u_0, W_1 = u_1$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i$$

and the recurrence coefficients a_1, a_2 are complex scalars or polynomials in $\mathbb{C}[x]$.

A sequence $\{W_n\}_{n\geq 0}$ defined by (1.1) is called a generalized Leonardo-type sequence with input function $p(n)$**.** For more information on generalized Leonardo-type sequences, see Soykan [3] and [2].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1.$$

Suppose that θ_1 and θ_2 are the roots of the characteristic equation of (1.2)

$$z^2 - a_1 z - a_2 = 0, \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^2 - a_1 z - a_2 = (z-1)^2 = z^2 - 2z + 1 = 0$$

so that $a_1 = 2,\ a_2 = -1$ and (1.2) reduces to

$$V_n = 2V_{n-1} - V_{n-2}. \tag{1.4}$$

In earlier work, particular solutions of second-order nonhomogeneous recurrences (1.1) with polynomial inputs were obtained for degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$; see Soykan [4]. Here, we extend those results to unified closed-form solutions by applying Theorem 1.1. Each recurrence is expressed as the sum of its homogeneous and particular components, with the particular solution determined through iterative coefficient relations. The framework is organized by two parameters: the multiplicity r of 1 as a root of the characteristic equation (1.3), and the polynomial degree s. Explicit formulas are obtained for all cases $r = 0, 1, 2$ and $s = 0, 1, 2, 3, 4, 5, 6, 7$, showing how resonance and multiplicity corrections arise. Since our examples—generalized Jacobsthal numbers ($r = 0$) and Mersenne numbers ($r = 1$)—fall into these categories, we present the theorem in detail only for $r = 0$ and $r = 1$, noting that the double root case $r = 2$ can be derived analogously.

THEOREM 1.1.

(a): *[2, Theorem 7.6. (a)] The case $r = 0$, i.e., all two roots of the characteristic equation of (1.3) is distinct from 1. The solution of (1.1) is given by*

$$\begin{aligned} W_n(W_0, W_1) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

is the particular solution of (1.1). For each $0 \leq i \leq s$, A_i can be calculated with the iteration

$$A_s = -\frac{c_s}{a_1 + a_2 - 1}, \text{ for } n = s$$

and

$$A_n = -\frac{1}{a_1 + a_2 - 1}(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+1} \binom{k}{n} (a_1 + 2^{k-n} a_2) A_k), \text{ for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i) \\ &= \frac{(W_1 - a_1 W_0) \sum_{i=0}^{s} A_i - a_2 W_0 A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_n(W_0, W_1) + \frac{-a_2(W_0 \sum_{i=0}^{s} A_i - A_0 W_1)}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n-1}(W_0, W_1) \\ &= -\frac{W_0 \sum_{i=0}^{s} A_i - A_0 W_1}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n+1}(W_0, W_1) + \frac{W_1 \sum_{i=0}^{s} A_i - (W_0 a_2 + W_1 a_1) A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_n(W_0, W_1). \end{aligned}$$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1)=\left(-\frac{(W_1-a_1W_0)\sum_{i=0}^{s}A_i-a_2W_0A_0}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+\frac{a_2(W_0\sum_{i=0}^{s}A_i-A_0W_1)}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n-1}(W_0,W_1)+\sum_{i=0}^{s}A_in^i$$

i.e.,

$$W_n(W_0,W_1)=\frac{W_0\sum_{i=0}^{s}A_i-A_0W_1}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n+1}(W_0,W_1)+\left(-\frac{W_1\sum_{i=0}^{s}A_i-(W_0a_2+W_1a_1)A_0}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+\sum_{i=0}^{s}A_in^i$$

(b): *The case $r=1$, i.e., 1 is a simple root of the characteristic equation of (1.3). The solution of (1.1) is given by*

$$\begin{aligned}W_n(W_0,W_1) &= W_n^{(h)}+W_n^{(p)}\\ &= V_n(W_0,W_1)-V_n(W_0^{(p)},W_1^{(p)})+W_n^{(p)}\end{aligned}$$

where $W_n^{(h)}=V_n(W_0,W_1)-V_n(W_0^{(p)},W_1^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)}=n\sum_{i=0}^{s}A_in^i=n(A_0+\sum_{i=1}^{s}A_in^i)$$

is the particular solution of (1.1). For each $0\leq i\leq s$, A_i can be calculated with the iteration

$$A_s=(-1)^{1+1}\frac{c_s}{(\sum_{j=1}^{2}ja_j)\binom{s+1}{1}}=\frac{c_s}{(a_1+2a_2)(s+1)},\quad \textit{for } n=s$$

and

$$A_n=\frac{1}{(a_1+2a_2)(n+1)}(c_n-\sum_{k=n+1}^{s}(-1)^{k-n+2}\binom{k+1}{n}(a_1+2^{k-n+1}a_2)A_k),\ \textit{for } n=s-1,s-2,...,2,1,0.$$

Here

$$\begin{aligned}W_0^{(p)} &= 0,\\ W_1^{(p)} &= \sum_{i=0}^{s}A_i,\end{aligned}$$

and

$$\begin{aligned}V_n(W_0^{(p)},W_1^{(p)}) &= V_n(0,\sum_{i=0}^{s}A_i)\\ &= \frac{(W_1-a_1W_0)\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_n(W_0,W_1)+\frac{-a_2W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n-1}(W_0,W_1)\\ &= -\frac{W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n+1}(W_0,W_1)+\frac{W_1\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_n(W_0,W_1).\end{aligned}$$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1)=\left(-\frac{(W_1-a_1W_0)\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+\frac{a_2W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n-1}(W_0,W_1)+n\sum_{i=0}^{s}A_in^i$$

i.e.,

$$W_n(W_0,W_1)=\frac{W_0\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n+1}(W_0,W_1)+\left(-\frac{W_1\sum_{i=0}^{s}A_i}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+n\sum_{i=0}^{s}A_in^i$$

(c): *The case* $r = 2$, *i.e.,* 1 *is a double root of the characteristic equation of (1.3). The construction follows the same method as in case (a) and (b), with appropriate multiplicity corrections.*

Proof. The solution is obtained by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104-105), and Theorem 3.1 (pp. 88-89) for $m = 2$, as presented in Soykan [2]. In general, the theorem applies to all multiplicities $r = 0, 1, 2$: the case $r = 0$ corresponds to the non-resonant situation where both roots are distinct from 1, while $r = 1$ and $r = 2$ represent simple and double roots of 1 in the characteristic equation (1.3). Since our applications involve only the non-resonant case $r = 0$ and the simple root case $r = 1$, the double root case $r = 2$ is omitted. It can be obtained in the same manner with multiplicity adjustments, but is not needed for the examples considered here.

2. Examples: Closed-Form Solutions of Second Order nonhomogeneous Linear Recurrence Relations

In the following subsection, we consider special cases of Theorem 1.1 for the special second order nonhomogeneous linear recurrence relations.

2.1. Generalized Jacobsthal Numbers: The case $a_1 = 1$ **and** $a_2 = 2$**.** Now, we consider the case $a_1 = 1, a_2 = 2$. A generalized Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$V_n = V_{n-1} + 2V_{n-2}, \tag{2.1}$$

with the initial values $V_0 = v_0, V_1 = v_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

The Binet formula of generalized Jacobsthal numbers can be written as

$$\begin{aligned} V_n &= \frac{V_1 - \beta V_0}{\alpha - \beta}\alpha^n - \frac{V_1 - \alpha V_0}{\alpha - \beta}\beta^n \\ &= \frac{p_1\alpha^n - p_2\beta^n}{\alpha - \beta} \end{aligned} \tag{2.2}$$

where α and β are the roots of the quadratic equation $z^2 - z - 2 = 0$ and

$$\begin{aligned} p_1 &= V_1 - \beta V_0 \\ p_2 &= V_1 - \alpha V_0. \end{aligned}$$

Moreover

$$\begin{aligned} \alpha &= 2 \\ \beta &= -1 \end{aligned}$$

So

$$V_n = \frac{(V_1 - \beta V_0) \times 2^n - (V_1 - \alpha V_0) \times (-1)^n}{3}.$$

Now we define two special cases of the sequence $\{V_n\}$. Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$J_n = J_{n-1} + 2J_{n-2}, \qquad J_0 = 0, J_1 = 1, \tag{2.3}$$

$$j_n = j_{n-1} + 2j_{n-2}, \qquad j_0 = 2, j_1 = 1, \tag{2.4}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} + \frac{1}{2}J_{-(n-2)} \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} + \frac{1}{2}j_{-(n-2)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.4) hold for all integer n.

For all integers n, Jacobsthal and Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} J_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{3}, \\ j_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 2, \\ W_0 &= 0, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = J_n$ (Jacobsthal numbers).

Example 2.1. *In this example, we consider the case $a_1 = 1, a_2 = 2, W_0 = 0, W_1 = 1$ in Theorem 1.1 (a). So $V_n(0,1) = V_n$ represents nth Jacobshal number i.e., $V_n = J_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + a_2 W_{n-2} + c_0$$

is given by

$$W_n(0,1) = \frac{1}{2}(c_0 + 2)J_n + c_0 J_{n-1} - \frac{1}{2}c_0$$

i.e.,

$$W_n(0,1) = \frac{1}{2}c_0 J_{n+1} + J_n - \frac{1}{2}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(0,1) = \frac{1}{4}(2c_0 + 7c_1 + 4)J_n + \frac{1}{2}(2c_0 + 5c_1)J_{n-1} + A_1 n + A_0$$

i.e.,

$$W_n(0,1) = \frac{1}{4}(2c_0 + 5c_1)J_{n+1} + \frac{1}{2}(c_1 + 2)J_n + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(0,1) = \frac{1}{4}(2c_0 + 7c_1 + 28c_2 + 4)J_n + \frac{1}{2}(2c_0 + 5c_1 + 16c_2)J_{n-1} + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(0,1) = \frac{1}{4}(2c_0 + 5c_1 + 16c_2)J_{n+1} + \frac{1}{2}(c_1 + 6c_2 + 2)J_n + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1 + 16c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3)J_{n-1} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3)J_{n+1} + \frac{1}{4}\left(2c_1 + 12c_2 + 65c_3 + 4\right)J_n + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3). \end{aligned}$$

(e): *The case $s = 4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 1592c_4 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4)J_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4)J_{n+1} + \frac{1}{4}(2c_1 + 12c_2 + 65c_3 + 396c_4 + 4)J_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_4 &= -\frac{1}{2}c_4, \\ A_3 &= -\frac{1}{2}(c_3 + 10c_4), \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4). \end{aligned}$$

(f): *The case $s = 5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 1592c_4 + 11\,534c_5 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5)J_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5)J_{n+1} + \frac{1}{4}(2c_1 + 12c_2 + 65c_3 + 396c_4 + 2882c_5 + 4)J_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_5 &= -\frac{1}{2}c_5, \\ A_4 &= -\frac{1}{4}(2c_4 + 25c_5), \\ A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5), \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5). \end{aligned}$$

(g): *The case $s = 6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{8}(4c_0 + 14c_1 + 56c_2 + 269c_3 + 1592c_4 + 11\,534c_5 + 99\,896c_6 + 8)J_n + \frac{1}{4}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49\,952c_6)J_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49\,952c_6)J_{n+1} + \frac{1}{4}(2c_1 + 12c_2 + 65c_3 + 396c_4 + 2882c_5 + 24\,972c_6 + 4)J_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_6 &= -\frac{1}{2}c_6, \\ A_5 &= -\frac{1}{2}(c_5 + 15c_6), \\ A_4 &= -\frac{1}{4}(2c_4 + 25c_5 + 240c_6), \\ A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5 + 695c_6), \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5 + 6000c_6), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5 + 8655c_6), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49952c_6). \end{aligned}$$

(h): *The case $s = 7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{16}(8c_0 + 28c_1 + 112c_2 + 538c_3 + 3184c_4 + 23068c_5 + 199792c_6 + 2017813c_7 + 16)J_n + \frac{1}{8}(8c_0 + 20c_1 + 64c_2 + 278c_3 + 1600c_4 + 11\,540c_5 + 99904c_6 + 1008923c_7)J_{n-1} + A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{16}(8c_0 + 20c_1 + 64c_2 + 278c_3 + 1600c_4 + 11\,540c_5 + 99904c_6 + 1008923c_7)J_{n+1} + \frac{1}{8}(4c_1 + 24c_2 + 130c_3 + 792c_4 + 5764c_5 + 49944c_6 + 504445c_7 + 8)J_n + A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned}
A_7 &= -\frac{1}{2}c_7, \\
A_6 &= -\frac{1}{4}(2c_6 + 35c_7), \\
A_5 &= -\frac{1}{2}(c_5 + 15c_6 + 168c_7), \\
A_4 &= -\frac{1}{8}(4c_4 + 50c_5 + 480c_6 + 4865c_7), \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5 + 695c_6 + 7000c_7), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5 + 6000c_6 + 60585c_7), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5 + 8655c_6 + 87416c_7), \\
A_0 &= -\frac{1}{16}(8c_0 + 20c_1 + 64c_2 + 278c_3 + 1600c_4 + 11540c_5 + 99904c_6 + 1008923c_7).
\end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned}
a_1 &= 1, \\
a_2 &= 2, \\
W_0 &= 2, \\
W_1 &= 1,
\end{aligned}$$

so that we have the case $V_n = j_n$ (Jacobsthal-Lucas numbers).

EXAMPLE 2.2. *In this example, we consider the case $a_1 = 1, a_2 = 2, W_0 = 2, W_1 = 1$ in Theorem 1.1 (a). So $V_n(2,1) = V_n$ represents nth Jacobsthal-Lucas number, i.e., $V_n = j_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_0$$

is given by

$$W_n(2,1) = \frac{1}{18}(5c_0 + 18)j_n + \frac{1}{9}c_0 j_{n-1} - \frac{1}{2}c_0$$

i.e.,

$$W_n(2,1) = \frac{1}{18}c_0 j_{n+1} + \frac{1}{9}(2c_0 + 9)j_n - \frac{1}{2}c_0$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(2,1) = \frac{1}{36}(10c_0 + 27c_1 + 36)j_n + \frac{1}{18}(2c_0 + 9c_1)j_{n-1} + A_1 n + A_0$$

i.e.,

$$W_n(2,1) = \frac{1}{36}(2c_0 + 9c_1)j_{n+1} + \frac{1}{18}(4c_0 + 9c_1 + 18)j_n + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(2,1) = \frac{1}{36}(10c_0 + 27c_1 + 92c_2 + 36)j_n + \frac{1}{18}(2c_0 + 9c_1 + 40c_2)j_{n-1} + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(2,1) = \frac{1}{36}(2c_0 + 9c_1 + 40c_2)j_{n+1} + \frac{1}{18}(4c_0 + 9c_1 + 26c_2 + 18)j_n + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 5c_1 + 16c_2), \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(2,1) = \frac{1}{72}(20c_0+54c_1+184c_2+825c_3+72)j_n+\frac{1}{36}(4c_0+18c_1+80c_2+399c_3)j_{n-1}+A_3n^3+A_2n^2+A_1n+A_0$$

i.e.,

$$W_n(2,1) = \frac{1}{72}(4c_0+18c_1+80c_2+399c_3)j_{n+1}+\frac{1}{36}(8c_0+18c_1+52c_2+213c_3+36)j_n+A_3n^3+A_2n^2+A_1n+A_0$$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3). \end{aligned}$$

(e): *The case* $s = 4$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1) = \frac{1}{72}(20c_0 + 54c_1 + 184c_2 + 825c_3 + 4792c_4 + 72)j_n + \frac{1}{36}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4)j_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(2,1) = \frac{1}{72}(4c_0+18c_1+80c_2+399c_3+2384c_4)j_{n+1}+\frac{1}{36}(8c_0+18c_1+52c_2+213c_3+1204c_4+36)j_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_4 &= -\frac{1}{2}c_4, \\ A_3 &= -\frac{1}{2}(c_3 + 10c_4), \\ A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4), \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4), \\ A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4). \end{aligned}$$

(f): *The case $s = 5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1) = \frac{1}{72}(20c_0 + 54c_1 + 184c_2 + 825c_3 + 4792c_4 + 34614c_5 + 72)j_n + \frac{1}{36}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5)j_{n-1} + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

i.e.,

$W_n(2,1) = \frac{1}{72}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5)j_{n+1} + \frac{1}{36}(8c_0 + 18c_1 + 52c_2 + 213c_3 + 1204c_4 + 8658c_5 + 36)j_n + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned}
A_5 &= -\frac{1}{2}c_5, \\
A_4 &= -\frac{1}{4}(2c_4 + 25c_5), \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5), \\
A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5).
\end{aligned}$$

(g): *The case $s = 6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-2} + c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1) = \frac{1}{72}(20c_0 + 54c_1 + 184c_2 + 825c_3 + 4792c_4 + 34614c_5 + 299704c_6 + 72)j_n + \frac{1}{36}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5 + 149840c_6)j_{n-1} + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

i.e.,

$W_n(2,1) = \frac{1}{72}(4c_0 + 18c_1 + 80c_2 + 399c_3 + 2384c_4 + 17298c_5 + 149840c_6)j_{n+1} + \frac{1}{36}(8c_0 + 18c_1 + 52c_2 + 213c_3 + 1204c_4 + 8658c_5 + 74932c_6 + 36)j_n + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$
\begin{aligned}
A_6 &= -\frac{1}{2}c_6, \\
A_5 &= -\frac{1}{2}(c_5 + 15c_6), \\
A_4 &= -\frac{1}{4}(2c_4 + 25c_5 + 240c_6), \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5 + 695c_6), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5 + 6000c_6), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5 + 8655c_6), \\
A_0 &= -\frac{1}{8}(4c_0 + 10c_1 + 32c_2 + 139c_3 + 800c_4 + 5770c_5 + 49952c_6).
\end{aligned}
$$

(h): *The case $s = 7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = -W_{n-1} + 2W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,1) = \frac{1}{144}(40c_0 + 108c_1 + 368c_2 + 1650c_3 + 9584c_4 + 69228c_5 + 599408c_6 + 6053505c_7 + 144)j_n + \frac{1}{72}(8c_0 + 36c_1 + 160c_2 + 798c_3 + 4768c_4 + 34596c_5 + 299680c_6 + 3026703c_7)j_{n-1} + A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(2,1) = \frac{1}{144}(8c_0 + 36c_1 + 160c_2 + 798c_3 + 4768c_4 + 34596c_5 + 299680c_6 + 3026703c_7)j_{n+1} + \frac{1}{72}(16c_0 + 36c_1 + 104c_2 + 426c_3 + 2408c_4 + 17\,316c_5 + 149864c_6 + 1513401c_7 + 72)j_n + A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$
\begin{aligned}
A_7 &= -\frac{1}{2}c_7, \\
A_6 &= -\frac{1}{4}(2c_6 + 35c_7), \\
A_5 &= -\frac{1}{2}(c_5 + 15c_6 + 168c_7), \\
A_4 &= -\frac{1}{8}(4c_4 + 50c_5 + 480c_6 + 4865c_7), \\
A_3 &= -\frac{1}{2}(c_3 + 10c_4 + 80c_5 + 695c_6 + 7000c_7), \\
A_2 &= -\frac{1}{4}(2c_2 + 15c_3 + 96c_4 + 695c_5 + 6000c_6 + 60585c_7), \\
A_1 &= -\frac{1}{2}(c_1 + 5c_2 + 24c_3 + 139c_4 + 1000c_5 + 8655c_6 + 87416c_7), \\
A_0 &= -\frac{1}{16}(8c_0 + 20c_1 + 64c_2 + 278c_3 + 1600c_4 + 11540c_5 + 99904c_6 + 1008923c_7).
\end{aligned}
$$

2.2. Generalized Mersenne Numbers: The case $a_1 = 3$ and $a_2 = -2$. In this subsection, we consider the case $a_1 = 3, a_2 = -2$. A generalized Mersenne sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$V_n = 3V_{n-1} - 2V_{n-2} \tag{2.5}$$

with the initial values $V_0 = c_0, V_1 = c_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = \frac{3}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.5) holds for all integer n. For more information on generalized Mersenne numbers, see Soykan [1].

The Binet formula of generalized Mersenne numbers can be written as

$$V_n = \frac{V_1 - \beta V_0}{\alpha - \beta}\alpha^n - \frac{V_1 - \alpha V_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation

$$z^2 - 3z + 2 = 0. \tag{2.6}$$

Moreover

$$\begin{aligned} \alpha &= 2 \\ \beta &= 1 \end{aligned}$$

So

$$V_n = (V_1 - V_0)2^n - (V_1 - 2V_0). \tag{2.7}$$

Now, we define two special cases of the sequence $\{V_n\}$. Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$M_n = 3M_{n-1} - 2M_{n-2}, \qquad M_0 = 0, M_1 = 1, \tag{2.8}$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \qquad H_0 = 2, H_1 = 3, \tag{2.9}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.8)-(2.9) hold for all integer n.

For all integers n, Mersenne and Mersenne-Lucas can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (b) for

$$\begin{aligned} a_1 &= 3, \ a_2 = -2, \\ W_0 &= 0, \ W_1 = 1, \end{aligned}$$

so that we have the case $V_n = M_n$ (Mersenne numbers).

EXAMPLE 2.3. *In this example, we consider the case* $a_1 = 3$, $a_2 = -2, W_0 = 0$, $W_1 = 1$ *in Theorem 1.1 (b). So* $V_n(0,1) = V_n$ *represents nth Mersenne number, i.e.,* $V_n = M_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_0$$

is given by

$$W_n(0,1) = (c_0 + 1)M_n - c_0 n.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(0,1) = (c_0 + 3c_1 + 1)M_n + A_1 n^2 + A_0 n$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{2}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 1)M_n + A_2 n^3 + A_1 n^2 + A_0 n$$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 1)M_n + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$$

where

$$\begin{aligned}
A_3 &= -\frac{1}{4}c_3, \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3), \\
A_1 &= -\frac{1}{4}(2c_1 + 10c_2 + 49c_3), \\
A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2 + 216c_3).
\end{aligned}$$

(e): *The case $s = 4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 1)M_n + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$$

where

$$\begin{aligned}
A_4 &= -\frac{1}{5}c_4, \\
A_3 &= -\frac{1}{4}(c_3 + 10c_4), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4), \\
A_1 &= -\frac{1}{4}(2c_1 + 10c_2 + 49c_3 + 288c_4), \\
A_0 &= -\frac{1}{30}(30c_0 + 75c_1 + 245c_2 + 1080c_3 + 6239c_4).
\end{aligned}$$

(f): *The case $s = 5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 1)M_n + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$$

where

$$\begin{aligned}
A_5 &= -\frac{1}{6}c_5, \\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5), \\
A_3 &= -\frac{1}{12}(3c_3 + 30c_4 + 245c_5), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5), \\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5), \\
A_0 &= -\frac{1}{30}(30c_0 + 75c_1 + 245c_2 + 1080c_3 + 6239c_4 + 45000c_5).
\end{aligned}$$

(g): *The case $s = 6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 1)M_n + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$\begin{aligned}
A_6 &= -\frac{1}{7}c_6, \\
A_5 &= -\frac{1}{6}(c_5 + 15c_6), \\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5 + 245c_6), \\
A_3 &= -\frac{1}{12}(3c_3 + 30c_4 + 245c_5 + 2160c_6), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5 + 6239c_6), \\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5 + 54000c_6), \\
A_0 &= -\frac{1}{210}(210c_0 + 525c_1 + 1715c_2 + 7560c_3 + 43673c_4 + 315000c_5 + 2726645c_6).
\end{aligned}$$

(h): *The case $s = 7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = (c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 189171c_7 + 1)M_n + A_7n^8 + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$
\begin{aligned}
A_7 &= -\frac{1}{8}c_7, \\
A_6 &= -\frac{1}{14}(2c_6 + 35c_7), \\
A_5 &= -\frac{1}{12}(2c_5 + 30c_6 + 343c_7), \\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5 + 245c_6 + 2520c_7), \\
A_3 &= -\frac{1}{24}(6c_3 + 60c_4 + 490c_5 + 4320c_6 + 43673c_7), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5 + 6239c_6 + 63000c_7), \\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5 + 54000c_6 + 545329c_7), \\
A_0 &= -\frac{1}{210}(210c_0 + 525c_1 + 1715c_2 + 7560c_3 + 43673c_4 + 315000c_5 + 2726645c_6 + 27536040c_7).
\end{aligned}
$$

In the following Example, we consider Theorem 1.1 (b) for

$$
\begin{aligned}
a_1 &= 3,\ a_2 = -2, \\
W_0 &= 2,\ W_1 = 3,
\end{aligned}
$$

so that we have the case $V_n = H_n$ (Mersenne-Lucas numbers).

EXAMPLE 2.4. *In this example, we consider the case $a_1 = 3$, $a_2 = -2, W_0 = 2$, $W_1 = 3$ in Theorem 1.1 (b). So $V_n(2,3) = V_n$ represents nth Mersenne-Lucas number, i.e., $V_n = H_n$).*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_0$$

is given by

$$W_n(2,3) = (3c_0 + 1)H_n - 4c_0H_{n-1} - c_0n,$$

i.e.,

$$W_n(2,3) = 2c_0H_{n+1} - (3c_0 - 1)\,H_n - c_0n.$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_1n + c_0$$

is given by

$$W_n(2,3) = (3c_0 + 9c_1 + 1)H_n - 4(c_0 + 3c_1)H_{n-1} + A_1n^2 + A_0n$$

i.e.,

$$W_n(2,3) = 2(c_0 + 3c_1)H_{n+1} - (3c_0 + 9c_1 - 1)H_n + A_1 n^2 + A_0 n$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{2}(2c_0 + 5c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 1)H_n - 4(c_0 + 3c_1 + 11c_2)H_{n-1} + A_2 n^3 + A_1 n^2 + A_0 n$$

i.e.,

$$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2)H_{n+1} - (3c_0 + 9c_1 + 33c_2 - 1)H_n + A_2 n^3 + A_1 n^2 + A_0 n$$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 5c_2), \\ A_0 &= -\frac{1}{6}(6c_0 + 15c_1 + 49c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3)H_{n-1} + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 - 1)H_n + A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n$

where

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{6}(2c_2+15c_3), \\ A_1 &= -\frac{1}{4}(2c_1+10c_2+49c_3), \\ A_0 &= -\frac{1}{6}(6c_0+15c_1+49c_2+216c_3). \end{aligned}$$

(e): *The case $s=4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,3) = (3c_0+9c_1+33c_2+153c_3+897c_4+1)H_n - 4(c_0+3c_1+11c_2+51c_3+299c_4)H_{n-1} + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

i.e.,

$W_n(2,3) = 2(c_0+3c_1+11c_2+51c_3+299c_4)H_{n+1} - (3c_0+9c_1+33c_2+153c_3+897c_4-1)H_n + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$\begin{aligned} A_4 &= -\frac{1}{5}c_4, \\ A_3 &= -\frac{1}{4}(c_3+10c_4), \\ A_2 &= -\frac{1}{6}(2c_2+15c_3+98c_4), \\ A_1 &= -\frac{1}{4}(2c_1+10c_2+49c_3+288c_4), \\ A_0 &= -\frac{1}{30}(30c_0+75c_1+245c_2+1080c_3+6239c_4). \end{aligned}$$

(f): *The case $s=5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,3) = (3c_0+9c_1+33c_2+153c_3+897c_4+6489c_5+1)H_n - 4(c_0+3c_1+11c_2+51c_3+299c_4+2163c_5)H_{n-1} + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

i.e.,

$W_n(2,3) = 2(c_0+3c_1+11c_2+51c_3+299c_4+2163c_5)H_{n+1} - (3c_0+9c_1+33c_2+153c_3+897c_4+6489c_5-1)H_n + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$\begin{aligned}
A_5 &= -\frac{1}{6}c_5,\\
A_4 &= -\frac{1}{10}(2c_4+25c_5),\\
A_3 &= -\frac{1}{12}(3c_3+30c_4+245c_5),\\
A_2 &= -\frac{1}{6}(2c_2+15c_3+98c_4+720c_5),\\
A_1 &= -\frac{1}{12}(6c_1+30c_2+147c_3+864c_4+6239c_5),\\
A_0 &= -\frac{1}{30}(30c_0+75c_1+245c_2+1080c_3+6239c_4+45000c_5).
\end{aligned}$$

(g): *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,3) = (3c_0+9c_1+33c_2+153c_3+897c_4+6489c_5+56\,193c_6+1)H_n - 4(c_0+3c_1+11c_2+51c_3+299c_4+2163c_5+18\,731c_6)H_{n-1} + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

i.e.,

$W_n(2,3) = 2(c_0+3c_1+11c_2+51c_3+299c_4+2163c_5+18\,731c_6)H_{n+1} - (3c_0+9c_1+33c_2+153c_3+897c_4+6489c_5+56\,193c_6-1)H_n + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$\begin{aligned}
A_6 &= -\frac{1}{7}c_6,\\
A_5 &= -\frac{1}{6}(c_5+15c_6),\\
A_4 &= -\frac{1}{10}(2c_4+25c_5+245c_6),\\
A_3 &= -\frac{1}{12}(3c_3+30c_4+245c_5+2160c_6),\\
A_2 &= -\frac{1}{6}(2c_2+15c_3+98c_4+720c_5+6239c_6),\\
A_1 &= -\frac{1}{12}(6c_1+30c_2+147c_3+864c_4+6239c_5+54000c_6),\\
A_0 &= -\frac{1}{210}(210c_0+525c_1+1715c_2+7560c_3+43673c_4+315000c_5+2726645c_6).
\end{aligned}$$

(h): *The case $s=7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 2W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,3) = (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 56193c_6 + 567513c_7 + 1)H_n - 4(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 189171c_7)H_{n-1} + A_7n^8 + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

i.e.,

$W_n(2,3) = 2(c_0 + 3c_1 + 11c_2 + 51c_3 + 299c_4 + 2163c_5 + 18731c_6 + 189171c_7)H_{n+1} - (3c_0 + 9c_1 + 33c_2 + 153c_3 + 897c_4 + 6489c_5 + 56193c_6 + 567513c_7 - 1)H_n + A_7n^8 + A_6n^7 + A_5n^6 + A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n$

where

$$\begin{aligned}
A_7 &= -\frac{1}{8}c_7, \\
A_6 &= -\frac{1}{14}(2c_6 + 35c_7), \\
A_5 &= -\frac{1}{12}(2c_5 + 30c_6 + 343c_7), \\
A_4 &= -\frac{1}{10}(2c_4 + 25c_5 + 245c_6 + 2520c_7), \\
A_3 &= -\frac{1}{24}(6c_3 + 60c_4 + 490c_5 + 4320c_6 + 43673c_7), \\
A_2 &= -\frac{1}{6}(2c_2 + 15c_3 + 98c_4 + 720c_5 + 6239c_6 + 63000c_7), \\
A_1 &= -\frac{1}{12}(6c_1 + 30c_2 + 147c_3 + 864c_4 + 6239c_5 + 54000c_6 + 545329c_7), \\
A_0 &= -\frac{1}{210}(210c_0 + 525c_1 + 1715c_2 + 7560c_3 + 43673c_4 + 315000c_5 + 2726645c_6 + 27536040c_7).
\end{aligned}$$

Conclusions

This study presents explicit closed-form solutions for second-order nonhomogeneous linear recurrence relations with polynomial input terms. By using the corresponding homogeneous recurrence relation, the solutions are written as combinations of homogeneous sequence terms and particular polynomial components. The manuscript gives detailed formulas for cases in which the polynomial input has degree from 0 to 7. The examples focus on generalized Jacobsthal, Jacobsthal-Lucas, Mersenne, and Mersenne-Lucas type sequences. These examples show that the proposed method can be applied systematically to obtain closed-form representations for several related recurrence families. The results also demonstrate the role of the characteristic equation in determining the form of the particular solution, especially in cases where 1 is a root. Overall, the paper provides a collection of explicit formulas that may be useful for further work on nonhomogeneous recurrence relations and their connections with classical integer sequences.

Limitations

The present work is limited to second-order linear recurrence relations with polynomial input terms of degree up to 7. Although the method is described in a general framework, the detailed examples are restricted to Jacobsthal-type and Mersenne-type homogeneous counterparts. The double-root case is only discussed briefly and is not developed through detailed examples. The manuscript also focuses on deriving explicit formulas rather than investigating applications, asymptotic behavior, computational efficiency, or combinatorial interpretations. Further work may extend the same approach to higher-degree polynomial inputs, higher-order recurrence relations, and additional families of classical sequences. Independent verification of the longer formulas may also be useful because of their algebraic complexity.

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