A Study on Explicit Particular Solutions for Second and Third Order Generalized Leonardo-Type Recurrences with Polynomial-Exponential Forcing

Abstract. Recurrence relations offer a versatile framework for analyzing numerical sequences, with applications across both classical and modern branches of mathematics. In earlier work, explicit iterative formulas were established for polynomial–exponential type particular solutions of generalized Leonardo-type sequences. The present article builds on that foundation by presenting concrete examples for orders

$$m = 1, 2,$$

where the input function takes the form

$$C(n) = p(n)d^n,$$

with $p(n) = \sum_{i=0}^{s} c_i n^i$ a polynomial in n. For such sequences, we construct particular solutions of the form

$$W_n^{(C)} = n^r \left(\sum_{i=0}^{s} A_i n^i \right) d^n,$$

and illustrate the computation of the coefficients A_i using the established iterative scheme. These examples show how the multiplicity r of the root d in the characteristic equation shapes the structure of the particular solution, and they highlight resonance phenomena in non-homogeneous cases. By working through explicit instances, the paper provides a clear and accessible demonstration of the general theory, strengthening the link between abstract recurrence relations and concrete symbolic computation.

We present two representative examples that demonstrate how resonance and root multiplicities influence the construction of particular solutions in polynomial–exponential-driven recurrence relations. In the case of the generalized Fibonacci sequence, the input polynomial-exponential is non-resonant, allowing the particular solution to be obtained directly without the need for correction. By contrast, the generalized Mersenne sequence highlights the resonant situation, where the root 2 of the characteristic equation necessitates a multiplicity-aware adjustment in the solution process.

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1. Introduction

Recurrence sequences defined by recurrence relations have long occupied a central position in mathematics, with applications extending to physics, engineering, architecture, biology, computer science, and even artistic domains. Beneath their seemingly simple construction lies remarkable depth: they embody growth processes, oscillatory phenomena, and symbolic identities. Among the most distinguished second-order recurrences are the Fibonacci, Lucas, Pell, and Jacobsthal sequences. The Fibonacci numbers, in particular, gained lasting fame through Leonardo de Pisa's celebrated rabbit problem in his 1202 treatise Liber Abaci. Both Fibonacci and Lucas sequences continue to provide abundant sources of elegant identities, combinatorial interpretations, and analytic connections.

Beyond second-order families, higher-order recurrence sequences play an equally significant role in both theoretical and applied contexts. These generalizations expand the classical framework and reveal deeper algebraic and analytic structures. Third-order examples such as the Tribonacci sequence, fourth-order examples such as the Tetranacci sequence, and fifth-order examples such as the Pentanacci sequence extend the paradigm, each governed by characteristic polynomials whose root configurations determine closed-form solutions. Homogeneous recurrences emphasize the role of characteristic polynomials and root multiplicities, while non-homogeneous recurrences incorporate symbolic input terms whose interaction with the root structure produces resonance phenomena. Collectively, these families form a unified framework that bridges classical recurrence identities with modern symbolic recurrence theory.

The classical Leonardo sequence is introduced by the non-homogeneous recurrence

$$L_n = L_{n-1} + L_{n-2} + 1, \quad n \geq 2,$$

with initial conditions $L_0 = 1$ and $L_1 = 1$. While the defining relation is straightforward, the historical background of the sequence is somewhat opaque. Its recognition developed gradually, with generalizations and related extensions appearing in the literature well before the designation "Leonardo numbers' became standard. Modern interest has been rekindled largely through the study of explicit examples and their diverse applications.

Over time, the sequence has been revisited both for its inherent mathematical elegance and for its adaptability in modeling systems that combine homogeneous recurrence dynamics with external non-homogeneous terms. This dual character has made it a rich subject for symbolic analysis, linking classical recurrence theory to contemporary computational approaches. Recent contributions emphasize its structural depth, its ability to capture intricate interactions, and its significance in both theoretical investigations and applied contexts.

From an instructional perspective, the simplicity of its recurrence and the availability of concrete examples make the Leonardo sequence especially suitable for teaching. It offers students a clear demonstration of how non-homogeneous recurrences function, while also serving as a gateway to advanced symbolic techniques and resonance phenomena. Thus, the sequence continues to be both a focus of scholarly research and a valuable pedagogical resource (see, for instance, [1, 5, 14, 16, 19, 25, 26, 36, 37, 38, 28, 33, 34, 35, 30, 32]).

To our knowledge, the first systematic generalization of the Leonardo sequence was carried out by J. A. Jeske in a trilogy of papers published in *The Fibonacci Quarterly* during 1963–1964 (see [20, 21, 22]). For a concise survey of this literature–particularly contributions in *The Fibonacci Quarterly*–as well as selected works outside the journal that advance Leonardo-type recurrences, see Soykan [39, Section 5].

We first recall the definition of m-order homogeneous linear recurrence relations.

Definition 1.1. *A sequence $\{V_n\}_{n\geq 0}$ is called a homogeneous (linear) recurrence relation order $m\in\mathbb{N}$ if it satisfies*

$$V_n=\sum_{k=1}^{m}a_kV_{n-k}=a_1V_{n-1}+a_2V_{n-2}...+a_mV_{n-m} \tag{1.1}$$

for

$$m\geq 1$$

with the initial conditions $V_0,V_1,...,V_{m-1}$

and

$$V_n=a_0, \tag{1.2}$$

for

$$m=0.$$

The recurrence coefficients $a_1,a_2,\ldots,a_m$ and the initial conditions $V_0,V_1,...,V_{m-1}$ are complex scalars. We allow each coefficient a_i, for $1\leq i\leq m$, to be identically zero.

The integer m is called the order of the linear recurrence.

The characteristic polynomial of the sequence $(V_n)_{n\geq 0}$ is given by

$$A(z)=z^m-\sum_{k=1}^{m}a_kz^{m-k}=z^m-a_1z^{m-1}-a_2z^{m-2}-...-a_{m-1}z-a_m=(z-\theta_1)^{u_1}(z-\theta_2)^{u_2}...(z-\theta_v)^{u_v}$$

with distinct $\theta_1,\theta_2,...,\theta_v$ and $u_1+u_2+...+u_v=m$. $\theta_1,\theta_2,...,\theta_v$ are called the (characteristic) root of characteristic equation

$$A(z)=z^m-a_1z^{m-1}-a_2z^{m-2}-...-a_{m-1}z-a_m=(z-\theta_1)^{u_1}(z-\theta_2)^{u_2}...(z-\theta_v)^{u_v}=0. \tag{1.3}$$

For $m\geq 1$, consider the sequence (W_n) defined by the recurrence relation (a **generalized Leonardo-type sequence**)

$$W_n=\sum_{k=1}^{m}a_kW_{n-k}+p(n)bd^n=\sum_{k=1}^{m}a_kW_{n-k}+C(n) \tag{1.4}$$

with initial conditions $W_0,W_1,\ldots,W_{m-1}$ and the recurrence coefficients $a_1,a_2,\ldots,a_m$ are complex scalars or polynomials in $\mathbb{C}[x]$ and with the input function

$$C(n)=p(n)bd^n$$

where

$$p(n):=p(n,x)=\sum_{i=0}^{s}c_in^i$$

denotes a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b\in\mathbb{C}[x]$ or $\mathbb{C}$, and $d\in\mathbb{C}$ or $\mathbb{R}$. For more information on generalized Leonardo-type sequences, see Soykan [40] and [39].

We consider the homogeneous recurrence relation (1.1) and its characteristic equation (1.3), corresponding to the sequence (W_n) defined by (1.4).

The particular solution $W_n^{(C)}$ of (1.4) is of the form

$$W_n^{(C)} = n^r(\sum_{i=0}^{s} A_i n^i)d^n = n^r(A_0 + \sum_{i=1}^{s} A_i n^i)d^n, \tag{1.5}$$

where the coefficients $A_i \in \mathbb{C}[x]$ or $\mathbb{C}$ and r is the multiplicity of d as a root of the characteristic equation (1.3), (if d is not a root of characteristic equation (1.3) then $r = 0$).

We now state a theorem that furnishes explicit iterative formulas for determining the coefficients of the particular solution $W_n^{(C)}$ of (1.4). The construction hinges on whether the parameter d coincides with a characteristic root of (1.3); in cases where d is a root of multiplicity r, this multiplicity dictates the precise modifications required in the iterative scheme.

Theorem 1.2. *[40, p.100, Theorem 5.1] For each $0 \le i \le s$, A_i given in (1.5) can be calculated with the iteration as follows:*

- *If $r = 0$, i.e., none of the roots of the characteristic equation (1.3) equals d, then*

$$A_s = -\frac{c_s b d^m}{a_1 d^{m-1} + a_2 d^{m-2} + a_2 d^{m-3} + \ldots + a_{m-2} d^2 + a_{m-1} d + a_m - d^m} = -\frac{c_s b d^m}{-d^m + \sum_{j=1}^{m} a_j d^{m-j}}, \text{ for } n = s$$

  and

$$A_n = -\frac{1}{-d^m + \sum_{j=1}^{m} a_j d^{m-j}}(c_n b d^m - \sum_{k=n+1}^{s} (-1)^{k-n+1}\binom{k}{n}(\sum_{j=1}^{m} j^{k-n} a_j d^{m-j})A_k)$$

  for $n = s-1, s-2, \ldots, 2, 1, 0$.

- *If $r > 0$ then*

$$A_s = (-1)^{r+1}\frac{c_s b d^m}{(\sum_{j=1}^{m} j^r a_j \times d^{m-j})\binom{s+r}{r}}, \text{ for } n = s$$

  and

$$A_n = (-1)^{r+1}\frac{1}{(\sum_{j=1}^{m} j^r a_j \times d^{m-j})\binom{n+r}{r}}(c_n b d^m - \sum_{k=n+1}^{s} (-1)^{k+r-n+1}\binom{k+r}{n}(\sum_{j=1}^{m} j^{k+r-n} a_j \times d^{m-j})A_k)$$

  for $n = s-1, s-2, \ldots, 2, 1, 0$.

In the following sections, we present explicit particular solutions to (1.4) for $m = 1, 2, 3, 4$, where

$$C(n) = p(n)bd^n, \quad p(n) \text{ is a polynomial in } n.$$

We seek solutions of the form

$$W_n^{(C)} = P(n)d^n,$$

where $P(n)$ is itself a polynomial in n.

2. Special Cases

2.1. The Case $m = 1$. Consider the homogeneous relation

$$V_n = a_1 V_{n-1} \tag{2.1}$$

with the same initial condition as V_0. Suppose that θ_1 is the the roots of characteristic equation

$$z - a_1 = 0 \tag{2.2}$$

associated with (2.1). Note that if the root of (2.2) equals to d then

$$z - a_1 = z - d = 0$$

so that $a_1 = d$ and (2.1) reduces to

$$V_n = dV_{n-1}.$$

We now turn to an example that illustrates the results derived above.

Example 2.1. *Consider the sequence (W_n) defined by the recurrence relation*

$$W_n = a_1 W_{n-1} + p(n) b d^n$$

where $p(n) := p(n, x)$ denotes a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. We seek a particular solution

$$W_n^C = P(n) d^n$$

for the cases $s = 0, 1, 2, 3, 4, 5, 6, 7$ where $P(n)$ is itself a polynomial in n. The order (degree) and coffecients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.2) and W_n^C satisfy

$$W_n^C = a_1 W_{n-1}^C + p(n) b d^n$$

i.e.,

$$P(n) d^n = a_1 P(n-1) d^{n-1} + p(n) b d^n.$$

In each case of s, we consider the homogeneous relation (2.1) and its characteristic equation (2.2), corresponding to the sequence (W_n) with the same initial conditions as W_n, i.e.,

$$V_0 = W_0.$$

We investigate all cases of multiplicity r of of d as a root of the characteristic equation (2.2):

- **(a):** $m = 1$, $s = 0$. *Consider the sequence (W_n) defined by*

  $$W_n = a_1 W_{n-1} + c_0 b d^n.$$

  - **(i):** *Case $r = 0$, i.e., the root of the characteristic equation does not equal d:*

    $$W_n^C = A_0 d^n, \quad A_0 = -\frac{c_0 b d}{a_1 - d}.$$

  - **(ii):** *Case $r = 1$, i.e., the root of the characteristic equation equals d:*

    $$W_n^C = n A_0 d^n, \quad A_0 = b c_0.$$

(b): *$m=1$, $s=1$. Consider the sequence (W_n) defined by*

$$W_n = a_1 W_{n-1} + (c_1 n + c_0) b d^n.$$

(i): *Case $r=0$ (the root does not equal to d):*

$$W_n^C = (A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1 b d}{a_1 - d}, \\ A_0 &= -\frac{1}{a_1 - d}(c_0 b d - a_1 A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1 b d}{a_1 - d}, \\ A_0 &= -\frac{bd}{(a_1 - d)^2}(-c_0 d + a_1 (c_0 + c_1)). \end{aligned}$$

(ii): *Case $r=1$ (the root equals to d):*

$$W_n^C = n(A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= \frac{1}{2} b c_1, \\ A_0 &= \frac{1}{2} b(2c_0 + c_1). \end{aligned}$$

(c): *$m=1$, $s=2$. Consider the sequence (W_n) defined by*

$$W_n = a_1 W_{n-1} + (c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): *Case $r=0$ (the root does not equal to d):*

$$W_n^C = (A_2 n^2 + A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_2 &= -\frac{c_2 b d}{a_1 - d}, \\ A_1 &= -\frac{1}{a_1 - d}(c_1 b d - 2 a_1 A_2), \\ A_0 &= -\frac{1}{a_1 - d}(c_0 b d - a_1 A_1 + a_1 A_2), \end{aligned}$$

i.e.,

$$\begin{aligned} A_2 &= -\frac{c_2 b d}{a_1 - d}, \\ A_1 &= -\frac{bd}{(a_1 - d)^2}(-c_1 d + a_1 (c_1 + 2c_2)), \\ A_0 &= -\frac{bd}{(a_1 - d)^3}(c_0 d^2 + a_1^2 (c_0 + c_1 + c_2) - a_1 (2c_0 + c_1 - c_2) d). \end{aligned}$$

(ii): *Case* $r = 1$ *(the root equals to* d*):*

$$W_n^C = n(A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_2 &= \frac{1}{3}bc_2, \\ A_1 &= \frac{1}{2}b(c_1 + c_2), \\ A_0 &= \frac{1}{6}b(6c_0 + 3c_1 + c_2). \end{aligned}$$

(d): $m = 1$, $s = 3$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1} + (c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): *Case* $r = 0$ *(the root does not equal to* d*):*

$$W_n^C = (A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= -\frac{c_3bd}{a_1 - d}, \\ A_2 &= -\frac{1}{a_1 - d}(c_2bd - 3a_1A_3), \\ A_1 &= -\frac{1}{a_1 - d}(c_1bd - 2a_1A_2 + 3a_1A_3), \\ A_0 &= -\frac{1}{a_1 - d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3), \end{aligned}$$

i.e.,

$$\begin{aligned} A_3 &= -\frac{c_3bd}{a_1 - d}, \\ A_2 &= -\frac{bd}{(a_1 - d)^2}(-c_2d + a_1(c_2 + 3c_3)), \\ A_1 &= -\frac{bd}{(a_1 - d)^3}(c_1d^2 + a_1^2(c_1 + 2c_2 + 3c_3) - a_1(2c_1 + 2c_2 - 3c_3)d), \\ A_0 &= -\frac{bd}{(a_1 - d)^4}(-c_0d^3 + a_1^3(c_0 + c_1 + c_2 + c_3) - a_1^2(3c_0 + 2c_1 - 4c_3)d + a_1(3c_0 + c_1 - c_2 + c_3)d^2). \end{aligned}$$

(ii): *Case* $r = 1$ *(the root equals to* d*):*

$$W_n^C = n(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{1}{4}bc_3, \\ A_2 &= \frac{1}{6}b(2c_2 + 3c_3), \\ A_1 &= \frac{1}{4}b(2c_1 + 2c_2 + c_3), \\ A_0 &= \frac{1}{6}b(6c_0 + 3c_1 + c_2). \end{aligned}$$

(e): $m=1,\ s=4$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + (c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0) b d^n .$$

(i): *Case* $r=0$ *(the root does not equal to* d*):*

$$W_n^C = (A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_4 &= -\frac{c_4 bd}{a_1-d}, \\ A_3 &= -\frac{1}{a_1-d}\left(c_3 bd - 4a_1 A_4\right), \\ A_2 &= -\frac{1}{a_1-d}\left(c_2 bd - 3a_1 A_3 + 6a_1 A_4\right), \\ A_1 &= -\frac{1}{a_1-d}\left(c_1 bd - 2a_1 A_2 + 3a_1 A_3 - 4a_1 A_4\right), \\ A_0 &= -\frac{1}{a_1-d}(c_0 bd - a_1 A_1 + a_1 A_2 - a_1 A_3 + a_1 A_4), \end{aligned}$$

i.e.,

$A_4 = -\dfrac{c_4 bd}{a_1-d}$,

$A_3 = -\dfrac{bd}{(a_1-d)^2}(-c_3 d + a_1(4c_4 + c_3))$,

$A_2 = -\dfrac{bd}{(a_1-d)^3}(c_2 d^2 + a_1^2(6c_4 + 3c_3 + c_2) + a_1(6c_4 - 3c_3 - 2c_2)d)$,

$A_1 = -\dfrac{bd}{(a_1-d)^4}(-c_1 d^3 + a_1^3(4c_4 + 3c_3 + 2c_2 + c_1) + a_1^2(16c_4 - 4c_2 - 3c_1)d + a_1(4c_4 - 3c_3 + 2c_2 + 3c_1)d^2$
$)$,

$A_0 = -\dfrac{bd}{(a_1-d)^5}(c_0 d^4 + a_1^4(c_4 + c_3 + c_2 + c_1 + c_0) + a_1^3(11c_4 + 3c_3 - c_2 - 3c_1 - 4c_0)d + a_1^2(11c_4 -$
$3c_3 - c_2 + 3c_1 + 6c_0)d^2 + a_1(c_4 - c_3 + c_2 - c_1 - 4c_0)d^3)$.

(ii): *Case* $r=1$ *(the root equals to* d*):*

$$W_n^C = n(A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_4 &= \frac{1}{5} b c_4, \\ A_3 &= \frac{1}{4} b(c_3 + 2c_4), \\ A_2 &= \frac{1}{6} b(2c_2 + 3c_3 + 2c_4), \\ A_1 &= \frac{1}{4} b(2c_1 + 2c_2 + c_3), \\ A_0 &= \frac{1}{30} b(30c_0 + 15c_1 + 5c_2 - c_4). \end{aligned}$$

(f): $m=1,\ s=5$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + (c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0) b d^n .$$

(i): *Case $r=0$ (the root does not equal to d):*

$$W_n^C = (A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_5 &= -\frac{c_5bd}{a_1-d}, \\
A_4 &= -\frac{1}{a_1-d}(c_4bd - 5a_1A_5), \\
A_3 &= -\frac{1}{a_1-d}(c_3bd - 4a_1A_4 + 10a_1A_5), \\
A_2 &= -\frac{1}{a_1-d}(c_2bd - 3a_1A_3 + 6a_1A_4 - 10a_1A_5), \\
A_1 &= -\frac{1}{a_1-d}(c_1bd - 2a_1A_2 + 3a_1A_3 - 4a_1A_4 + 5a_1A_5), \\
A_0 &= -\frac{1}{a_1-d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3 + a_1A_4 - a_1A_5),
\end{aligned}$$

i.e.,

$A_5 = -\frac{c_5bd}{a_1-d}$,

$A_4 = -\frac{bd}{(a_1-d)^2}(-c_4d + a_1(c_4+5c_5))$,

$A_3 = -\frac{bd}{(a_1-d)^3}(c_3d^2 + a_1^2(c_3+4c_4+10c_5) - 2a_1(c_3+2c_4-5c_5)d)$,

$A_2 = -\frac{bd}{(a_1-d)^4}(-c_2d^3 + a_1^3(c_2+3c_3+6c_4+10c_5) - a_1^2(3c_2+6c_3-40c_5)d + a_1(3c_2+3c_3-6c_4+10c_5)d^2)$,

$A_1 = -\frac{bd}{(a_1-d)^5}(c_1d^4 + a_1^4(c_1+2c_2+3c_3+4c_4+5c_5) - a_1^3(4c_1+6c_2+3c_3-12c_4-55c_5)d + a_1^2(6c_1+6c_2-3c_3-12c_4+55c_5)d^2 - a_1(4c_1+2c_2-3c_3+4c_4-5c_5)d^3)$,

$A_0 = -\frac{bd}{(a_1-d)^6}(-c_0d^5 + a_1(5c_0+c_1-c_2+c_3-c_4+c_5)d^4 - 2a_1^2(5c_0+2c_1-c_2-c_3+5c_4-13c_5)d^3 + 2a_1^3(5c_0+3c_1-3c_3+33c_5)d^2 - a_1^4(5c_0+4c_1+2c_2-2c_3-10c_4-26c_5)d + a_1^5(c_0+c_1+c_2+c_3+c_4+c_5))$.

(ii): *Case $r=1$ (the root equals to d):*

$$W_n^C = n(A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_5 &= \frac{1}{6}bc_5, \\
A_4 &= \frac{1}{10}b(2c_4 + 5c_5), \\
A_3 &= \frac{1}{12}b(3c_3 + 6c_4 + 5c_5), \\
A_2 &= \frac{1}{6}b(2c_2 + 3c_3 + 2c_4), \\
A_1 &= \frac{1}{12}b(6c_1 + 6c_2 + 3c_3 - c_5), \\
A_0 &= \frac{1}{30}b(30c_0 + 15c_1 + 5c_2 - c_4).
\end{aligned}$$

(g): $m=1$, $s=6$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + (c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): *Case $r=0$ (the root does not equal to d):*

$$W_n^C=(A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_6 &= -\frac{c_6bd}{a_1-d},\\
A_5 &= -\frac{1}{a_1-d}(c_5bd-6a_1A_6),\\
A_4 &= -\frac{1}{a_1-d}(c_4bd-5a_1A_5+15a_1A_6),\\
A_3 &= -\frac{1}{a_1-d}(c_3bd-4a_1A_4+10a_1A_5-20a_1A_6),\\
A_2 &= -\frac{1}{a_1-d}(c_2bd-3a_1A_3+6a_1A_4-10a_1A_5+15a_1A_6),\\
A_1 &= -\frac{1}{a_1-d}(c_1bd-2a_1A_2+3a_1A_3-4a_1A_4+5a_1A_5-6a_1A_6),\\
A_0 &= -\frac{1}{a_1-d}(c_0bd-a_1A_1+a_1A_2-a_1A_3+a_1A_4-a_1A_5+a_1A_6),
\end{aligned}$$

i.e.,

$A_6=-\dfrac{c_6bd}{a_1-d}$,

$A_5=-\dfrac{bd}{(a_1-d)^2}(-c_5d+a_1(c_5+6c_6))$,

$A_4=-\dfrac{bd}{(a_1-d)^3}(c_4d^2-a_1(2c_4+5c_5-15c_6)d+a_1^2(c_4+5c_5+15c_6))$,

$A_3=-\dfrac{bd}{(a_1-d)^4}(-c_3d^3+a_1(3c_3+4c_4-10c_5+20c_6)d^2-a_1^2(3c_3+8c_4-80c_6)d+a_1^3(c_3+4c_4+10c_5+20c_6))$,

$A_2=-\dfrac{bd}{(a_1-d)^5}(c_2d^4-a_1(4c_2+3c_3-6c_4+10c_5-15c_6)d^3+3a_1^2(2c_2+3c_3-2c_4-10c_5+55c_6)d^2-a_1^3(4c_2+9c_3+6c_4-30c_5-165c_6)d+a_1^4(c_2+3c_3+6c_4+10c_5+15c_6))$,

$A_1=-\dfrac{bd}{(a_1-d)^6}(-c_1d^5+a_1(5c_1+2c_2-3c_3+4c_4-5c_5+6c_6)d^4-2a_1^2(5c_1+4c_2-3c_3-4c_4+25c_5-78c_6)d^3+2a_1^3(5c_1+6c_2-12c_4+198c_6)d^2-a_1^4(5c_1+8c_2+6c_3-8c_4-50c_5-156c_6)d+a_1^5(c_1+2c_2+3c_3+4c_4+5c_5+6c_6))$,

$A_0=-\dfrac{bd}{(a_1-d)^7}(c_0d^6-a_1(6c_0+c_1-c_2+c_3-c_4+c_5-c_6)d^5+a_1^2(15c_0+5c_1-3c_2-c_3+9c_4-25c_5+57c_6)d^4-2a_1^3(10c_0+5c_1-c_2-4c_3+5c_4+20c_5-151c_6)d^3+a_1^4(15c_0+10c_1+2c_2-8c_3-10c_4+40c_5+302c_6)d^2+a_1^5(-6c_0-5c_1-3c_2+c_3+9c_4+25c_5+57c_6)d+a_1^6(c_0+c_1+c_2+c_3+c_4+c_5+c_6))$.

(ii): *Case $r=1$ (the root equals to d):*

$$W_n^C=n(A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_6 &= \frac{1}{7}bc_6,\\
A_5 &= \frac{1}{6}b(c_5+3c_6),\\
A_4 &= \frac{1}{10}b(2c_4+5c_5+5c_6),\\
A_3 &= \frac{1}{12}b(3c_3+6c_4+5c_5),\\
A_2 &= \frac{1}{6}b(2c_2+3c_3+2c_4-c_6),\\
A_1 &= \frac{1}{12}b(6c_1+6c_2+3c_3-c_5),\\
A_0 &= \frac{1}{210}b(210c_0+105c_1+35c_2-7c_4+5c_6).
\end{aligned}$$

(h): $m=1$, $s=7$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1} + (c_7n^7+c_6n^6+c_5n^5+c_4n^4+c_3n^3+c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case* $r=0$ *(the root does not equal to d):*

$$W_n^C = (A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_7 &= -\frac{c_7bd}{a_1-d},\\
A_6 &= -\frac{1}{a_1-d}(c_6bd-7a_1A_7),\\
A_5 &= -\frac{1}{a_1-d}(c_5bd-6a_1A_6+21a_1A_7),\\
A_4 &= -\frac{1}{a_1-d}(c_4bd-5a_1A_5+15a_1A_6-35a_1A_7),\\
A_3 &= -\frac{1}{a_1-d}(c_3bd-4a_1A_4+10a_1A_5-20a_1A_6+35a_1A_7),\\
A_2 &= -\frac{1}{a_1-d}(c_2bd-3a_1A_3+6a_1A_4-10a_1A_5+15a_1A_6-21a_1A_7),\\
A_1 &= -\frac{1}{a_1-d}(c_1bd-2a_1A_2+3a_1A_3-4a_1A_4+5a_1A_5-6a_1A_6+7a_1A_7),\\
A_0 &= -\frac{1}{a_1-d}(c_0bd-a_1A_1+a_1A_2-a_1A_3+a_1A_4-a_1A_5+a_1A_6-a_1A_7).
\end{aligned}$$

i.e.,

$A_7 = -\dfrac{c_7bd}{a_1-d}$,

$A_6 = -\dfrac{bd}{(a_1-d)^2}(-c_6d+a_1(c_6+7c_7))$,

$A_5 = -\dfrac{bd}{(a_1-d)^3}(c_5d^2-a_1(2c_5+6c_6-21c_7)d+a_1^2(c_5+6c_6+21c_7))$,

$A_4 = -\dfrac{bd}{(a_1-d)^4}(-c_4d^3+a_1(3c_4+5c_5-15c_6+35c_7)d^2-a_1^2(3c_4+10c_5-140c_7)d+a_1^3(c_4+5c_5+ 15c_6+35c_7))$,

$A_3 = -\frac{bd}{(a_1-d)^5}(c_3d^4 - a_1(4c_3+4c_4-10c_5+20c_6-35c_7)d^3 + a_1^2(6c_3+12c_4-10c_5-60c_6+385c_7)d^2$
$- a_1^3(4c_3+12c_4+10c_5-60c_6-385c_7)d + a_1^4(c_3+4c_4+10c_5+20c_6+35c_7)),$
$A_2 = -\frac{bd}{(a_1-d)^6}(-c_2d^5 + a_1(5c_2+3c_3-6c_4+10c_5-15c_6+21c_7)d^4 - 2a_1^2(5c_2+6c_3-6c_4-10c_5+$
$75c_6-273c_7)d^3 + 2a_1^3(5c_2+9c_3-30c_5+693c_7)d^2 - a_1^4(5c_2+12c_3+12c_4-20c_5-150c_6-546c_7)$
$d + a_1^5(c_2+3c_3+6c_4+10c_5+15c_6+21c_7)),$
$A_1 = -\frac{bd}{(a_1-d)^7}(c_1d^6 - a_1(6c_1+2c_2-3c_3+4c_4-5c_5+6c_6-7c_7)d^5 + a_1^2(15c_1+10c_2-9c_3-$
$4c_4+45c_5-150c_6+399c_7)d^4 - 2a_1^3(10c_1+10c_2-3c_3-16c_4+25c_5+120c_6-1057c_7)d^3 + a_1^4(15c_1+$
$20c_2+6c_3-32c_4-50c_5+240c_6+2114c_7)d^2 - a_1^5(6c_1+10c_2+9c_3-4c_4-45c_5-150c_6-399c_7)$
$d + a_1^6(c_1+2c_2+3c_3+4c_4+5c_5+6c_6+7c_7)),$
$A_0 = -\frac{bd}{(a_1-d)^8}(-c_0d^7 + a_1(7c_0+c_1-c_2+c_3-c_4+c_5-c_6+c_7)d^6 + a_1^2(-21c_0-6c_1+4c_2-$
$8c_4+24c_5-56c_6+120c_7)d^5 + a_1^3(35c_0+15c_1-5c_2-9c_3+19c_4+15c_5-245c_6+1191c_7)d^4 +$
$a_1^4(-35c_0-20c_1+16c_3-80c_5+2416c_7)d^3 + a_1^5(21c_0+15c_1+5c_2-9c_3-19c_4+15c_5+245c_6+1191c_7)$
$d^2 - a_1^6(7c_0+6c_1+4c_2-8c_4-24c_5-56c_6-120c_7)d + a_1^7(c_0+c_1+c_2+c_3+c_4+c_5+c_6+c_7)).$

(ii): *Case* $r = 1$ *(the root equals to* d*):*

$$W_n^C = n(A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_7 &= \frac{1}{8}bc_7, \\
A_6 &= \frac{1}{14}b(2c_6+7c_7), \\
A_5 &= \frac{1}{12}b(2c_5+6c_6+7c_7), \\
A_4 &= \frac{1}{10}b(2c_4+5c_5+5c_6), \\
A_3 &= \frac{1}{24}b(6c_3+12c_4+10c_5-7c_7), \\
A_2 &= \frac{1}{6}b(2c_2+3c_3+2c_4-c_6), \\
A_1 &= \frac{1}{12}b(6c_1+6c_2+3c_3-c_5+c_7), \\
A_0 &= \frac{1}{210}b(210c_0+105c_1+35c_2-7c_4+5c_6).
\end{aligned}$$

2.2. The Case $m = 2$.

Consider the homogeneous relation

$$V_n = a_1V_{n-1} + a_2V_{n-2} \tag{2.3}$$

with the initial conditions V_0, V_1.Suppose that θ_1 and θ_2 are roots of the characteristic equation of (2.3):

$$z^2 - a_1z - a_2 = 0. \tag{2.4}$$

Note that if all the roots of (2.4) are equal to d then

$$z^2 - a_1z - a_2 = (z-d)(z-d) = z^2 - 2dz + d^2 = 0$$

so that $a_1 = 2d$, $a_2 = -d^2$ and (2.3) reduces to

$$V_n = 2dV_{n-1} - d^2V_{n-2}.$$

We now turn to an example that illustrates the results derived above.

Example 2.2. *Consider the sequence (W_n) defined by the recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + p(n)bd^n$$

where $p(n) := p(n,x)$ denotes a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. We seek a particular solution

$$W_n^C = P(n)d^n$$

for the cases $s = 0, 1, 2, 3, 4, 5, 6, 7$ where $P(n)$ is itself a polynomial in n. The order (degree) and coffecients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.4) and W_n^C satisfy

$$W_n^C = a_1W_{n-1}^C + a_2W_{n-2}^C + p(n)bd^n$$

i.e.,

$$P(n)d^n = a_1P(n-1)d^{n-1} + a_2P(n-2)d^{n-2} + p(n)bd^n.$$

In each case of s, we consider the homogeneous relation (2.3) and its characteristic equation (2.4), corresponding to the sequence (W_n) with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1.$$

We investigate all cases of multiplicity r of of d as a root of the characteristic equation (2.4):

- **(a):** $m = 2$, $s = 0$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + c_0bd^n.$$

  - **(i):** *Case $r = 0$, i.e., none of the roots of the characteristic equation equals d:*

$$W_n^C = A_0d^n, \quad A_0 = -\frac{c_0bd^2}{a_1d + a_2 - d^2}.$$

  - **(ii):** *Case $r = 1$, i.e., exactly one root of the characteristic equation equals d:*

$$W_n^C = nA_0d^n, \quad A_0 = \frac{c_0bd^2}{a_1d + 2a_2}.$$

  - **(iii):** *Case $r = 2$, i.e., all two roots of the characteristic equation equal d:*

$$W_n^C = n^2A_0d^n, \quad A_0 = \frac{1}{2}bc_0.$$

- **(b):** $m = 2$, $s = 1$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_1n + c_0)bd^n.$$

(i): *Case $r=0$ (no root equal to d):*

$$W_n^C=(A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1bd^2}{(a_1d+a_2-d^2)}, \\ A_0 &= -\frac{1}{(a_1d+a_2-d^2)}(c_0bd^2-(a_1d+2a_2)A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1bd^2}{(a_1d+a_2-d^2)}, \\ A_0 &= -\frac{bd^2}{(a_1d+a_2-d^2)^2}(-c_0d^2+a_1(c_0+c_1)d+a_2(c_0+2c_1)). \end{aligned}$$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C=n(A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= \frac{c_1bd^2}{2\left(a_1d+2a_2\right)}, \\ A_0 &= \frac{1}{(a_1d+2a_2)}(c_0bd^2+\left(a_1d+4a_2\right)A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= \frac{c_1bd^2}{2\left(a_1d+2a_2\right)}, \\ A_0 &= \frac{bd^2}{2\left(a_1d+2a_2\right)^2}(a_1(2c_0+c_1)d+4a_2(c_0+c_1)). \end{aligned}$$

(iii): *Case $r=2$ (all two roots equal to d):*

$$W_n^C=n^2(A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= \frac{1}{6}bc_1, \\ A_0 &= \frac{1}{2}b(c_0+c_1). \end{aligned}$$

(c): $m=2$, $s=2$. *Consider the sequence (W_n) defined by*

$$W_n=a_1W_{n-1}+a_2W_{n-2}+(c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case $r=0$ (no root equal to d):*

$$W_n^C=(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_2 &= -\frac{c_2bd^2}{(a_1d+a_2-d^2)}, \\
A_1 &= -\frac{1}{(a_1d+a_2-d^2)}(c_1bd^2-2(a_1d+2a_2)A_2), \\
A_0 &= -\frac{1}{(a_1d+a_2-d^2)}(c_0bd^2-(a_1d+2a_2)A_1+(a_1d+4a_2)A_2),
\end{aligned}$$

i.e.,

$$A_2 = -\frac{c_2bd^2}{(a_1d+a_2-d^2)},$$

$$A_1 = -\frac{bd^2}{(a_1d+a_2-d^2)^2}(-c_1d^2+a_1(c_1+2c_2)d+a_2(c_1+4c_2)),$$

$$A_0 = -\frac{bd^2}{(a_1d+a_2-d^2)^3}(c_0d^4+a_1^2(c_0+c_1+c_2)d^2+a_2^2(c_0+2c_1+4c_2)-a_1(2c_0+c_1-c_2)d^3- 2a_2(c_0+c_1-2c_2)d^2+a_1a_2(2c_0+3c_1+3c_2)d).$$

(ii): *Case* $r=1$ *(exactly one root equal to* d*):*

$$W_n^C = n(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_2 &= \frac{c_2bd^2}{3\,(a_1d+2a_2)}, \\
A_1 &= \frac{1}{2\,(a_1d+2a_2)}(c_1bd^2+3\,(a_1d+4a_2)\,A_2), \\
A_0 &= \frac{1}{(a_1d+2a_2)}(c_0bd^2+(a_1d+4a_2)\,A_1-(a_1d+8a_2)\,A_2),
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_2 &= \frac{c_2bd^2}{3\,(a_1d+2a_2)}, \\
A_1 &= \frac{bd^2}{2\,(a_1d+2a_2)^2}(a_1(c_1+c_2)d+2a_2(c_1+2c_2)), \\
A_0 &= \frac{bd^2}{6\,(a_1d+2a_2)^3}(a_1^2(6c_0+3c_1+c_2)d^2+8a_2^2(3c_0+3c_1+2c_2)+2a_1a_2(12c_0+9c_1+2c_2)d).
\end{aligned}$$

(iii): *Case* $r=2$ *(all two roots equal to* d*):*

$$W_n^C = n^2(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_2 &= \frac{1}{12}bc_2, \\
A_1 &= \frac{1}{6}b(c_1+2c_2), \\
A_0 &= \frac{1}{12}b(6c_0+6c_1+5c_2).
\end{aligned}$$

(d): $m=2$, $s=3$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1}+a_2W_{n-2}+(c_3n^3+c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case $r=0$ (no root equal to d):*

$$W_n^C=(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_3 &= -\frac{c_3bd^2}{(a_1d+a_2-d^2)}, \\
A_2 &= -\frac{1}{(a_1d+a_2-d^2)}(c_2bd^2-3(a_1d+2a_2)A_3), \\
A_1 &= -\frac{1}{(a_1d+a_2-d^2)}(c_1bd^2-2(a_1d+2a_2)A_2+3(a_1d+4a_2)A_3), \\
A_0 &= -\frac{1}{(a_1d+a_2-d^2)}(c_0bd^2-(a_1d+2a_2)A_1+(a_1d+4a_2)A_2-(a_1d+8a_2)A_3),
\end{aligned}$$

i.e.,

$A_3=-\dfrac{c_3bd^2}{(a_1d+a_2-d^2)}$,

$A_2=-\dfrac{bd^2}{(a_1d+a_2-d^2)^2}(-c_2d^2+a_1(c_2+3c_3)d+a_2(c_2+6c_3))$,

$A_1=-\dfrac{bd^2}{(a_1d+a_2-d^2)^3}(c_1d^4+a_1^2(c_1+2c_2+3c_3)d^2+a_2^2(c_1+4c_2+12c_3)-a_1(2c_1+2c_2-3c_3)d^3-2a_2(c_1+2c_2-6c_3)d^2+a_1a_2(2c_1+6c_2+9c_3)d)$,

$A_0=-\dfrac{bd^2}{(a_1d+a_2-d^2)^4}(-c_0d^6+a_1^3(c_0+c_1+c_2+c_3)d^3+a_2^3(c_0+2c_1+4c_2+8c_3)-a_1^2(3c_0+2c_1-4c_3)d^4-a_2^2(3c_0+4c_1-32c_3)d^2+a_1(3c_0+c_1-c_2+c_3)d^5+a_2(3c_0+2c_1-4c_2+8c_3)d^4-2a_1a_2(3c_0+3c_1-c_2-9c_3)d^3+a_1a_2^2(3c_0+5c_1+7c_2+5c_3)d+a_1^2a_2(3c_0+4c_1+4c_2+4c_3)d^2)$.

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C=n(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_3 &= \frac{c_3bd^2}{4\left(a_1d+2a_2\right)}, \\
A_2 &= \frac{1}{3\left(a_1d+2a_2\right)}(c_2bd^2+6(a_1d+4a_2)A_3), \\
A_1 &= \frac{1}{2\left(a_1d+2a_2\right)}(c_1bd^2+3(a_1d+4a_2)A_2-4(a_1d+8a_2)A_3), \\
A_0 &= \frac{1}{(a_1d+2a_2)}(c_0bd^2+(a_1d+4a_2)A_1-(a_1d+8a_2)A_2+(a_1d+16a_2)A_3),
\end{aligned}$$

i.e.,

$A_3=\dfrac{c_3bd^2}{4\left(a_1d+2a_2\right)}$,

$A_2=\dfrac{bd^2}{6\left(a_1d+2a_2\right)^2}(a_1(2c_2+3c_3)d+4a_2(c_2+3c_3))$,

$A_1=\dfrac{bd^2}{4\left(a_1d+2a_2\right)^3}(a_1^2(2c_1+2c_2+c_3)d^2+8a_2^2(c_1+2c_2+2c_3)+4a_1a_2(2c_1+3c_2+c_3)d)$,

$A_0=\dfrac{bd^2}{6\left(a_1d+2a_2\right)^4}(a_1^3(6c_0+3c_1+c_2)d^3+16a_2^3(3c_0+3c_1+2c_2)+6a_1a_2^2(12c_0+10c_1+4c_2-3c_3)d+6a_1^2a_2(6c_0+4c_1+c_2)d^2)$.

(iii): *Case $r = 2$ (all two roots equal to d):*

$$W_n^C = n^2(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{1}{20}bc_3, \\ A_2 &= \frac{1}{12}b(c_2 + 3c_3), \\ A_1 &= \frac{1}{12}b(2c_1 + 4c_2 + 5c_3), \\ A_0 &= \frac{1}{12}b(6c_0 + 6c_1 + 5c_2 + 3c_3). \end{aligned}$$

(e): $m = 2$, $s = 4$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): *Case $r = 0$ (no root equal to d):*

$$W_n^C = (A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_4 &= -\frac{c_4bd^2}{(a_1d + a_2 - d^2)}, \\ A_3 &= -\frac{1}{(a_1d + a_2 - d^2)}\left(c_3bd^2 - 4\left(a_1d + 2a_2\right)A_4\right), \\ A_2 &= -\frac{1}{(a_1d + a_2 - d^2)}\left(c_2bd^2 - 3\left(a_1d + 2a_2\right)A_3 + 6\left(a_1d + 4a_2\right)A_4\right), \\ A_1 &= -\frac{1}{(a_1d + a_2 - d^2)}\left(c_1bd^2 - 2\left(a_1d + 2a_2\right)A_2 + 3\left(a_1d + 4a_2\right)A_3 - 4\left(a_1d + 8a_2\right)A_4\right), \\ A_0 &= -\frac{1}{(a_1d + a_2 - d^2)}(c_0bd^2 - (a_1d + 2a_2)A_1 + (a_1d + 4a_2)A_2 - (a_1d + 8a_2)A_3 + (a_1d + 16a_2)A_4), \end{aligned}$$

i.e.,

$A_4 = -\dfrac{c_4bd^2}{(a_1d + a_2 - d^2)}$,

$A_3 = -\dfrac{bd^2}{(a_1d + a_2 - d^2)^2}\left(-c_3d^2 + a_1\left(4c_4 + c_3\right)d + a_2\left(8c_4 + c_3\right)\right)$,

$A_2 = -\dfrac{bd^2}{(a_1d + a_2 - d^2)^3}(c_2d^4 + a_1^2(6c_4 + 3c_3 + c_2)d^2 + a_2^2(24c_4 + 6c_3 + c_2) + a_1(6c_4 - 3c_3 - 2c_2)d^3 + 2a_2(12c_4 - 3c_3 - c_2)d^2 + a_1a_2(18c_4 + 9c_3 + 2c_2)d)$,

$A_1 = -\dfrac{bd^2}{(a_1d + a_2 - d^2)^4}(-c_1d^6 + a_1^3(4c_4 + 3c_3 + 2c_2 + c_1)d^3 + a_2^3(32c_4 + 12c_3 + 4c_2 + c_1) + a_1^2(16c_4 - 4c_2 - 3c_1)d^4 + a_2^2(128c_4 - 8c_2 - 3c_1)d^2 + a_1(4c_4 - 3c_3 + 2c_2 + 3c_1)d^5 + a_2(32c_4 - 12c_3 + 4c_2 + 3c_1)d^4 + 6a_1a_2(12c_4 + c_3 - 2c_2 - c_1)d^3 + a_1a_2^2(20c_4 + 21c_3 + 10c_2 + 3c_1)d + a_1^2a_2(16c_4 + 12c_3 + 8c_2 + 3c_1)d^2)$,

$A_0 = -\dfrac{bd^2}{(a_1d + a_2 - d^2)^5}(c_0d^8 + a_1^4(c_4 + c_3 + c_2 + c_1 + c_0)d^4 + a_2^4(16c_4 + 8c_3 + 4c_2 + 2c_1 + c_0) + a_1^3(11c_4 + 3c_3 - c_2 - 3c_1 - 4c_0)d^5 + 2a_2^3(88c_4 + 12c_3 - 2c_2 - 3c_1 - 2c_0)d^2 + a_1^2(11c_4 - 3c_3 - c_2 + 3c_1 + 6c_0)d^6 + 2a_2^2(88c_4 - 12c_3 - 2c_2 + 3c_1 + 3c_0)d^4 + a_1(c_4 - c_3 + c_2 - c_1 - 4c_0)d^7 + 2a_2(8c_4 - 4c_3 + 2c_2 - c_1 - 2c_0)d^6 +$

$a_1a_2^2(115c_4+45c_3-5c_2-15c_1-12c_0)d^3+2a_1^2a_2(29c_4+9c_3-c_2-6c_1-6c_0)d^4-a_1a_2^3(c_4-13c_3-11c_2-7c_1-4c_0)d+a_1^3a_2(5c_4+5c_3+5c_2+5c_1+4c_0)d^3+a_1^2a_2^2(11c_4+9c_3+11c_2+9c_1+6c_0)d^2+a_1a_2(77c_4-9c_3-7c_2+9c_1+12c_0)d^5)$.

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C = n(A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_4 &= \frac{c_4bd^2}{5(a_1d+2a_2)}, \\
A_3 &= \frac{1}{4(a_1d+2a_2)}(c_3bd^2+10(a_1d+4a_2)A_4), \\
A_2 &= \frac{1}{3(a_1d+2a_2)}(c_2bd^2+6(a_1d+4a_2)A_3-10(a_1d+8a_2)A_4), \\
A_1 &= \frac{1}{2(a_1d+2a_2)}(c_1bd^2+3(a_1d+4a_2)A_2-4(a_1d+8a_2)A_3+5(a_1d+16a_2)A_4), \\
A_0 &= \frac{1}{(a_1d+2a_2)}(c_0bd^2+(a_1d+4a_2)A_1-(a_1d+8a_2)A_2+(a_1d+16a_2)A_3-(a_1d+32a_2)A_4),
\end{aligned}$$

i.e.,

$A_4 = \dfrac{c_4bd^2}{5(a_1d+2a_2)}$,

$A_3 = \dfrac{bd^2}{4\,(a_1d+2a_2)^2}(a_1(c_3+2c_4)d+2a_2(c_3+4c_4))$,

$A_2 = \dfrac{bd^2}{6(a_1d+2a_2)^3}(a_1^2(2c_2+3c_3+2c_4)d^2+8a_2^2(c_2+3c_3+4c_4)+2a_1a_2(4c_2+9c_3+4c_4)d)$,

$A_1 = \dfrac{bd^2}{4\,(a_1d+2a_2)^4}(a_1^3(2c_1+2c_2+c_3)d^3+16a_2^3(c_1+2c_2+2c_3)+8a_1a_2^2(3c_1+5c_2+3c_3-3c_4)d+2a_1^2a_2(6c_1+8c_2+3c_3)d^2)$,

$A_0 = \dfrac{bd^2}{30\,(a_1d+2a_2)^5}(a_1^4(30c_0+15c_1+5c_2-c_4)d^4+32a_2^4(15c_0+15c_1+10c_2-8c_4)+4a_1a_2^3(240c_0+210c_1+100c_2-45c_3-68c_4)d+2a_1^3a_2(120c_0+75c_1+20c_2-4c_4)d^3+6a_1^2a_2^2(120c_0+90c_1+30c_2-15c_3+16c_4)d^2)$.

(iii): *Case $r=2$ (all two roots equal to d):*

$$W_n^C = n^2(A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_4 &= \frac{1}{30}bc_4, \\
A_3 &= \frac{1}{20}b(c_3+4c_4), \\
A_2 &= \frac{1}{12}b(c_2+3c_3+5c_4), \\
A_1 &= \frac{1}{12}b(2c_1+4c_2+5c_3+4c_4), \\
A_0 &= \frac{1}{60}b(30c_0+30c_1+25c_2+15c_3+3c_4).
\end{aligned}$$

(f): *$m=2$, $s=5$. Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): *Case $r=0$ (no root equal to d):*

$$W_n^C = (A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$A_5 = -\frac{c_5bd^2}{a_1d+a_2-d^2},$$

$$A_4 = -\frac{1}{a_1d+a_2-d^2}(c_4bd^2 - 5(a_1d+2a_2)A_5),$$

$$A_3 = -\frac{1}{a_1d+a_2-d^2}(c_3bd^2 - 4(a_1d+2a_2)A_4 + 10(a_1d+4a_2)A_5),$$

$$A_2 = -\frac{1}{a_1d+a_2-d^2}(c_2bd^2 - 3(a_1d+2a_2)A_3 + 6(a_1d+4a_2)A_4 - 10(a_1d+8a_2)A_5),$$

$$A_1 = -\frac{1}{a_1d+a_2-d^2}(c_1bd^2 - 2(a_1d+2a_2)A_2 + 3(a_1d+4a_2)A_3 - 4(a_1d+8a_2)A_4 + 5(a_1d+16a_2)A_5),$$

$$A_0 = -\frac{1}{a_1d+a_2-d^2}(c_0bd^2 - (a_1d+2a_2)A_1 + (a_1d+4a_2)A_2 - (a_1d+8a_2)A_3 + (a_1d+16a_2)A_4 - (a_1d+32a_2)A_5),$$

i.e.,

$$A_5 = -\frac{c_5bd^2}{a_1d+a_2-d^2},$$

$$A_4 = -\frac{bd^2}{(a_1d+a_2-d^2)^2}(-c_4d^2 + a_1(c_4+5c_5)d + a_2(c_4+10c_5)),$$

$$A_3 = -\frac{bd^2}{(a_1d+a_2-d^2)^3}(c_3d^4 + a_1^2(c_3+4c_4+10c_5)d^2 + a_2^2(c_3+8c_4+40c_5) - 2a_1(c_3+2c_4-5c_5)d^3 - 2a_2(c_3+4c_4-20c_5)d^2 + 2a_1a_2(c_3+6c_4+15c_5)d),$$

$$A_2 = -\frac{bd^2}{(a_1d+a_2-d^2)^4}(-c_2d^6 + a_1^3(c_2+3c_3+6c_4+10c_5)d^3 + a_2^3(c_2+6c_3+24c_4+80c_5) - a_1^2(3c_2+6c_3-40c_5)d^4 - a_2^2(3c_2+12c_3-320c_5)d^2 + a_1(3c_2+3c_3-6c_4+10c_5)d^5 + a_2(3c_2+6c_3-24c_4+80c_5)d^4 - 6a_1a_2(c_2+3c_3-2c_4-30c_5)d^3 + a_1a_2^2(3c_2+15c_3+42c_4+50c_5)d + a_1^2a_2(3c_2+12c_3+24c_4+40c_5)d^2),$$

$$A_1 = -\frac{bd^2}{(a_1d+a_2-d^2)^5}(c_1d^8 + a_1^4(c_1+2c_2+3c_3+4c_4+5c_5)d^4 + a_2^4(c_1+4c_2+12c_3+32c_4+80c_5) - a_1^3(4c_1+6c_2+3c_3-12c_4-55c_5)d^5 - 4a_2^3(c_1+3c_2+3c_3-24c_4-220c_5)d^2 + a_1^2(6c_1+6c_2-3c_3-12c_4+55c_5)d^6 + 2a_2^2(3c_1+6c_2-6c_3-48c_4+440c_5)d^4 - a_1(4c_1+2c_2-3c_3+4c_4-5c_5)d^7 - 4a_2(c_1+c_2-3c_3+8c_4-20c_5)d^6 - a_1a_2^2(12c_1+30c_2+15c_3-180c_4-575c_5)d^3 - 2a_1^2a_2(6c_1+12c_2+3c_3-36c_4-145c_5)d^4 + a_1a_2^3(4c_1+14c_2+33c_3+52c_4-5c_5)d + a_1^3a_2(4c_1+10c_2+15c_3+20c_4+25c_5)d^3 + a_1^2a_2^2(6c_1+18c_2+33c_3+36c_4+55c_5)d^2 + a_1a_2(12c_1+18c_2-21c_3-36c_4+385c_5)d^5),$$

$$A_0 = -\frac{bd^2}{(a_1d+a_2-d^2)^6}(-c_0d^{10} + a_1(5c_0+c_1-c_2+c_3-c_4+c_5)d^9 - 2a_1^2(5c_0+2c_1-c_2-c_3+5c_4-13c_5)d^8 + 2a_1^3(5c_0+3c_1-3c_3+33c_5)d^7 - a_1^4(5c_0+4c_1+2c_2-2c_3-10c_4-26c_5)d^6 + a_1^5(c_0+c_1+c_2+c_3+c_4+c_5)d^5 + a_2(5c_0+2c_1-4c_2+8c_3-16c_4+32c_5)d^8 - 2a_2^2(5c_0+4c_1-4c_2-8c_3+80c_4-416c_5)d^6 + 2a_2^3(5c_0+6c_1-24c_3+1056c_5)d^4 - a_2^4(5c_0+8c_1+8c_2-16c_3-160c_4-832c_5)d^2 + a_2^5(c_0+2c_1+4c_2+8c_3+16c_4+32c_5) + 6a_1a_2^2(5c_0+5c_1-c_2-13c_3+23c_4+215c_5)d^5 + 6a_1^2a_2(5c_0+4c_1-c_2-5c_3+5c_4+79c_5)d^6 - 4a_1a_2^3(5c_0+7c_1+5c_2-14c_3-73c_4-98c_5)d^3 - 4a_1^3a_2(5c_0+5c_1+2c_2-4c_3-16c_4-40c_5)d^5 + a_1a_2^4(5c_0+9c_1+15c_2+21c_3+15c_4-51c_5)d + a_1^4a_2(5c_0+6c_1+6c_2+6c_3+6c_4+6c_5)d^4 -$$

$6a_1^2a_2^2(5c_0+6c_1+3c_2-9c_3-27c_4-69c_5)d^4+2a_1^2a_2^3(5c_0+8c_1+11c_2+11c_3+5c_4+23c_5)d^2+$
$2a_1^3a_2^2(5c_0+7c_1+8c_2+7c_3+8c_4+7c_5)d^3-4a_1a_2(5c_0+3c_1-3c_2+15c_4-72c_5)d^7),$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C=n(A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_5=\dfrac{c_5bd^2}{6(a_1d+2a_2)},$

$A_4=\dfrac{1}{5(a_1d+2a_2)}(c_4bd^2+15(a_1d+4a_2)A_5),$

$A_3=\dfrac{1}{4(a_1d+2a_2)}(c_3bd^2+10(a_1d+4a_2)A_4-20(a_1d+8a_2)A_5),$

$A_2=\dfrac{1}{3(a_1d+2a_2)}(c_2bd^2+6(a_1d+4a_2)A_3-10(a_1d+8a_2)A_4+15(a_1d+16a_2)A_5),$

$A_1=\dfrac{1}{2(a_1d+2a_2)}(c_1bd^2+3\left(a_1d+4a_2\right)A_2-4\left(a_1d+8a_2\right)A_3+5\left(a_1d+16a_2\right)A_4-6\left(a_1d+32a_2\right)A_5),$

$A_0=\dfrac{1}{(a_1d+2a_2)}(c_0bd^2+\left(a_1d+4a_2\right)A_1-\left(a_1d+8a_2\right)A_2+\left(a_1d+16a_2\right)A_3-\left(a_1d+32a_2\right)A_4+$
$\left(a_1d+64a_2\right)A_5),$

i.e.,

$A_5=\dfrac{c_5bd^2}{6(a_1d+2a_2)},$

$A_4=\dfrac{bd^2}{10\left(a_1d+2a_2\right)^2}(a_1(2c_4+5c_5)d+4a_2(c_4+5c_5)),$

$A_3=\dfrac{bd^2}{12\left(a_1d+2a_2\right)^3}(a_1^2(3c_3+6c_4+5c_5)d^2+4a_2^2(3c_3+12c_4+20c_5)+4a_1a_2(3c_3+9c_4+5c_5)d),$

$A_2=\dfrac{bd^2}{6\left(a_1d+2a_2\right)^4}(a_1^3(2c_2+3c_3+2c_4)d^3+16a_2^3(c_2+3c_3+4c_4)+12a_1a_2^2(2c_2+5c_3+4c_4-5c_5)d+$
$12a_1^2a_2(c_2+2c_3+c_4)d^2),$

$A_1=\dfrac{bd^2}{12\left(a_1d+2a_2\right)^5}(a_1^4(6c_1+6c_2+3c_3-c_5)d^4+32a_2^4(3c_1+6c_2+6c_3-8c_5)+16a_1a_2^3(12c_1+21c_2+$
$15c_3-9c_4-17c_5)d+4a_1^3a_2(12c_1+15c_2+6c_3-2c_5)d^3+12a_1^2a_2^2(12c_1+18c_2+9c_3-6c_4+8c_5)d^2),$

$A_0=\dfrac{bd^2}{30\left(a_1d+2a_2\right)^6}(a_1^5(30c_0+15c_1+5c_2-c_4)d^5+64a_2^5(15c_0+15c_1+10c_2-8c_4)+40a_1a_2^4(60c_0+$
$54c_1+28c_2-9c_3-20c_4+30c_5)d+10a_1^4a_2(30c_0+18c_1+5c_2-c_4)d^4+40a_1^2a_2^3(60c_0+48c_1+19c_2-$
$9c_3-2c_4+30c_5)d^2+10a_1^3a_2^2(120c_0+84c_1+26c_2-9c_3+8c_4-15c_5)d^3).$

(iii): *Case $r=2$ (all two roots equal to d):*

$$W_n^C=n^2(A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}A_5 &= \frac{1}{42}bc_5,\\ A_4 &= \frac{1}{30}b(c_4+5c_5),\\ A_3 &= \frac{1}{60}b(3c_3+12c_4+25c_5),\\ A_2 &= \frac{1}{12}b(c_2+3c_3+5c_4+5c_5),\\ A_1 &= \frac{1}{12}b(2c_1+4c_2+5c_3+4c_4+c_5),\\ A_0 &= \frac{1}{60}b(30c_0+30c_1+25c_2+15c_3+3c_4-5c_5).\end{aligned}$$

(g): $m=2$, $s=6$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1}+a_2W_{n-2}+(c_6n^6+c_5n^5+c_4n^4+c_3n^3+c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case* $r=0$ *(no root equal to* d*):*

$$W_n^C = (A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_6 = -\dfrac{c_6bd^2}{a_1d+a_2-d^2},$

$A_5 = -\dfrac{1}{a_1d+a_2-d^2}(c_5bd^2-6(a_1d+2a_2)A_6),$

$A_4 = -\dfrac{1}{a_1d+a_2-d^2}(c_4bd^2-5(a_1d+2a_2)A_5+15(a_1d+4a_2)A_6),$

$A_3 = -\dfrac{1}{a_1d+a_2-d^2}(c_3bd^2-4(a_1d+2a_2)A_4+10(a_1d+4a_2)A_5-20(a_1d+8a_2)A_6),$

$A_2 = -\dfrac{1}{a_1d+a_2-d^2}(c_2bd^2-3(a_1d+2a_2)A_3+6(a_1d+4a_2)A_4-10(a_1d+8a_2)A_5+15(a_1d+16a_2)A_6),$

$A_1 = -\dfrac{1}{a_1d+a_2-d^2}(c_1bd^2-2(a_1d+2a_2)A_2+3(a_1d+4a_2)A_3-4(a_1d+8a_2)A_4+5(a_1d+16a_2)A_5-$
$6A_6(a_1d+32a_2)),$

$A_0 = -\dfrac{1}{a_1d+a_2-d^2}(c_0bd^2-(a_1d+2a_2)A_1+(a_1d+4a_2)A_2-(a_1d+8a_2)A_3+(a_1d+16a_2)A_4-$
$(a_1d+32a_2)A_5+(a_1d+64a_2)A_6),$

i.e.,

$A_6 = -\dfrac{c_6bd^2}{a_1d+a_2-d^2},$

$A_5 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^2}(-c_5d^2+a_1(c_5+6c_6)d+a_2(c_5+12c_6)),$

$A_4 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^3}(c_4d^4-a_1(2c_4+5c_5-15c_6)d^3+a_1^2(c_4+5c_5+15c_6)d^2-2a_2(c_4+5c_5-30c_6)d^2+$
$a_2^2(c_4+10c_5+60c_6)+a_1a_2(2c_4+15c_5+45c_6)d),$

$A_3 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^4}(-c_3d^6+a_1(3c_3+4c_4-10c_5+20c_6)d^5-a_1^2(3c_3+8c_4-80c_6)d^4+a_1^3(c_3+4c_4+$
$10c_5+20c_6)d^3+a_2(3c_3+8c_4-40c_5+160c_6)d^4-a_2^2(3c_3+16c_4-640c_6)d^2+a_2^3(c_3+8c_4+40c_5+160c_6)+$
$a_1a_2^2(3c_3+20c_4+70c_5+100c_6)d+a_1^2a_2(3c_3+16c_4+40c_5+80c_6)d^2-2a_1a_2(3c_3+12c_4-10c_5-180c_6)d^3),$

$A_2 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^5}(c_2d^8-a_1(4c_2+3c_3-6c_4+10c_5-15c_6)d^7+3a_1^2(2c_2+3c_3-2c_4-10c_5+55c_6)d^6$
$-a_1^3(4c_2+9c_3+6c_4-30c_5-165c_6)d^5+a_1^4(c_2+3c_3+6c_4+10c_5+15c_6)d^4-2a_2(2c_2+3c_3-12c_4+$

$40c_5-120c_6)d^6+6a_2^2(c_2+3c_3-4c_4-40c_5+440c_6)d^4-2a_2^3(2c_2+9c_3+12c_4-120c_5-1320c_6)d^2$
$+a_2^4(c_2+6c_3+24c_4+80c_5+240c_6)-3a_1a_2^2(4c_2+15c_3+10c_4-150c_5-575c_6)d^3-6a_1^2a_2(2c_2+6c_3+$
$2c_4-30c_5-145c_6)d^4+a_1a_2^3(4c_2+21c_3+66c_4+130c_5-15c_6)d+a_1^3a_2(4c_2+15c_3+30c_4+50c_5+75c_6)d^3$
$+3a_1^2a_2^2(2c_2+9c_3+22c_4+30c_5+55c_6)d^2+3a_1a_2(4c_2+9c_3-14c_4-30c_5+385c_6)d^5),$

$A_1=-\dfrac{bd^2}{(a_1d+a_2-d^2)^6}(-c_1d^{10}+a_1(5c_1+2c_2-3c_3+4c_4-5c_5+6c_6)d^9-2a_1^2(5c_1+4c_2-3c_3-4c_4+$
$25c_5-78c_6)d^8+2a_1^3(5c_1+6c_2-12c_4+198c_6)d^7-a_1^4(5c_1+8c_2+6c_3-8c_4-50c_5-156c_6)d^6+a_1^5(c_1+$
$2c_2+3c_3+4c_4+5c_5+6c_6)d^5+a_2(5c_1+4c_2-12c_3+32c_4-80c_5+192c_6)d^8-2a_2^2(5c_1+8c_2-12c_3-32c_4+$
$400c_5-2496c_6)d^6+2a_2^3(5c_1+12c_2-96c_4+6336c_6)d^4-a_2^4(5c_1+16c_2+24c_3-64c_4-800c_5-4992c_6)$
$d^2+a_2^5(c_1+4c_2+12c_3+32c_4+80c_5+192c_6)+6a_1a_2^2(5c_1+10c_2-3c_3-52c_4+115c_5+1290c_6)d^5+$
$6a_1^2a_2(5c_1+8c_2-3c_3-20c_4+25c_5+474c_6)d^6-4a_1a_2^3(5c_1+14c_2+15c_3-56c_4-365c_5-588c_6)$
$d^3-4a_1^3a_2(5c_1+10c_2+6c_3-16c_4-80c_5-240c_6)d^5+a_1a_2^4(5c_1+18c_2+45c_3+84c_4+75c_5-306c_6)$
$d+a_1^4a_2(5c_1+12c_2+18c_3+24c_4+30c_5+36c_6)d^4-6a_1^2a_2^2(5c_1+12c_2+9c_3-36c_4-135c_5-414c_6)d^4+$
$2a_1^2a_2^3(5c_1+16c_2+33c_3+44c_4+25c_5+138c_6)d^2+2a_1^3a_2^2(5c_1+14c_2+24c_3+28c_4+40c_5+42c_6)$
$d^3-4a_1a_2(5c_1+6c_2-9c_3+75c_5-432c_6)d^7),$

$A_0=-\dfrac{bd^2}{(a_1d+a_2-d^2)^7}(c_0d^{12}-a_1(6c_0+c_1-c_2+c_3-c_4+c_5-c_6)d^{11}+a_1^2(15c_0+5c_1-3c_2-$
$c_3+9c_4-25c_5+57c_6)d^{10}-2a_1^3(10c_0+5c_1-c_2-4c_3+5c_4+20c_5-151c_6)d^9+a_1^4(15c_0+10c_1+$
$2c_2-8c_3-10c_4+40c_5+302c_6)d^8+a_1^5(-6c_0-5c_1-3c_2+c_3+9c_4+25c_5+57c_6)d^7+a_1^6(c_0+$
$c_1+c_2+c_3+c_4+c_5+c_6)d^6-2a_2(3c_0+c_1-2c_2+4c_3-8c_4+16c_5-32c_6)d^{10}+a_2^2(15c_0+10c_1-$
$12c_2-8c_3+144c_4-800c_5+3648c_6)d^8-4a_2^3(5c_0+5c_1-2c_2-16c_3+40c_4+320c_5-4832c_6)$
$d^6+a_2^4(15c_0+20c_1+8c_2-64c_3-160c_4+1280c_5+19\,328c_6)d^4-2a_2^5(3c_0+5c_1+6c_2-4c_3-$
$72c_4-400c_5-1824c_6)d^2+a_2^6(c_0+2c_1+4c_2+8c_3+16c_4+32c_5+64c_6)-2a_1a_2^2(30c_0+25c_1-$
$13c_2-47c_3+179c_4+85c_5-5353c_6)d^7-4a_1^2a_2(15c_0+10c_1-5c_2-8c_3+25c_4+40c_5-755c_6)d^8+$
$2a_1^3a_2(30c_0+25c_1+c_2-26c_3-17c_4+190c_5+1171c_6)d^7+2a_1a_2^3(30c_0+35c_1+7c_2-91c_3-77c_4+$
$1505c_5+5887c_6)d^5-a_1a_2^4(30c_0+45c_1+43c_2-51c_3-437c_4-1275c_5+163c_6)d^3-2a_1^4a_2(15c_0+$
$15c_1+8c_2-6c_3-34c_4-90c_5-202c_6)d^6+a_1a_2^5(6c_0+11c_1+19c_2+29c_3+31c_4-19c_5-281c_6)d+$
$a_1^5a_2(6c_0+7c_1+7c_2+7c_3+7c_4+7c_5+7c_6)d^5+6a_1^2a_2^2(15c_0+15c_1+c_2-27c_3+c_4+225c_5+1261c_6)$
$d^6-4a_1^2a_2^3(15c_0+20c_1+15c_2-22c_3-111c_4-190c_5-615c_6)d^4-2a_1^3a_2^2(30c_0+35c_1+21c_2-28c_3-$
$105c_4-280c_5-609c_6)d^5+a_1^2a_2^4(15c_0+25c_1+37c_2+43c_3+25c_4-5c_5+337c_6)d^2+a_1^4a_2^2(15c_0+$
$20c_1+22c_2+20c_3+22c_4+20c_5+22c_6)d^4+2a_1^3a_2^3(10c_0+15c_1+19c_2+18c_3+13c_4+30c_5-11c_6)$
$d^3+a_1a_2(30c_0+15c_1-17c_2+9c_3+43c_4-255c_5+1003c_6)d^9).$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C=n(A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_6=\dfrac{c_6bd^2}{7\left(a_1d+2a_2\right)},$

$A_5=\dfrac{1}{6\left(a_1d+2a_2\right)}(c_5bd^2+21(a_1d+4a_2)A_6),$

$A_4=\dfrac{1}{5\left(a_1d+2a_2\right)}(c_4bd^2+15(a_1d+4a_2)A_5-35(a_1d+8a_2)A_6),$

$$A_3 = \frac{1}{4\left(a_1 d + 2a_2\right)}(c_3 bd^2 + 10(a_1 d + 4a_2)A_4 - 20(a_1 d + 8a_2)A_5 + 35(a_1 d + 16a_2)A_6),$$

$$A_2 = \frac{1}{3\left(a_1 d + 2a_2\right)}(c_2 bd^2 + 6(a_1 d + 4a_2)A_3 - 10(a_1 d + 8a_2)A_4 + 15(a_1 d + 16a_2)A_5 - 21(a_1 d + 32a_2)A_6),$$

$$A_1 = \frac{1}{2\left(a_1 d + 2a_2\right)}(c_1 bd^2 + 3(a_1 d + 4a_2)A_2 - 4(a_1 d + 8a_2)A_3 + 5(a_1 d + 16a_2)A_4 - 6(a_1 d + 32a_2)A_5 + 7(a_1 d + 64a_2)A_6),$$

$$A_0 = \frac{1}{\left(a_1 d + 2a_2\right)}(c_0 bd^2 + (a_1 d + 4a_2)A_1 - (a_1 d + 8a_2)A_2 + (a_1 d + 16a_2)A_3 - (a_1 d + 32a_2)A_4 + (a_1 d + 64a_2)A_5 - (a_1 d + 128a_2)A_6),$$

i.e.,

$$A_6 = \frac{c_6 bd^2}{7\left(a_1 d + 2a_2\right)},$$

$$A_5 = \frac{bd^2}{6\left(a_1 d + 2a_2\right)^2}(a_1(c_5 + 3c_6)d + 2a_2(c_5 + 6c_6)),$$

$$A_4 = \frac{bd^2}{10\left(a_1 d + 2a_2\right)^3}(a_1^2(2c_4 + 5c_5 + 5c_6)d^2 + 8a_2^2(c_4 + 5c_5 + 10c_6) + 2a_1 a_2(4c_4 + 15c_5 + 10c_6)d),$$

$$A_3 = \frac{bd^2}{12\left(a_1 d + 2a_2\right)^4}(a_1^3(3c_3 + 6c_4 + 5c_5)d^3 + 8a_2^3(3c_3 + 12c_4 + 20c_5) + 12a_1 a_2^2(3c_3 + 10c_4 + 10c_5 - 15c_6)d + 6a_1^2 a_2(3c_3 + 8c_4 + 5c_5)d^2),$$

$$A_2 = \frac{bd^2}{6\left(a_1 d + 2a_2\right)^5}(a_1^4(2c_2 + 3c_3 + 2c_4 - c_6)d^4 + 32a_2^4(c_2 + 3c_3 + 4c_4 - 8c_6) + 8a_1 a_2^3(8c_2 + 21c_3 + 20c_4 - 15c_5 - 34c_6)d + 2a_1^3 a_2(8c_2 + 15c_3 + 8c_4 - 4c_6)d^3 + 12a_1^2 a_2^2(4c_2 + 9c_3 + 6c_4 - 5c_5 + 8c_6)d^2),$$

$$A_1 = \frac{bd^2}{12\left(a_1 d + 2a_2\right)^6}(a_1^5(6c_1 + 6c_2 + 3c_3 - c_5)d^5 + 64a_2^5(3c_1 + 6c_2 + 6c_3 - 8c_5) + 32a_1 a_2^4(15c_1 + 27c_2 + 21c_3 - 9c_4 - 25c_5 + 45c_6)d + 2a_1^4 a_2(30c_1 + 36c_2 + 15c_3 - 5c_5)d^4 + 8a_1^2 a_2^3(60c_1 + 96c_2 + 57c_3 - 36c_4 - 10c_5 + 180c_6)d^2 + 4a_1^3 a_2^2(60c_1 + 84c_2 + 39c_3 - 18c_4 + 20c_5 - 45c_6)d^3),$$

$$A_0 = \frac{bd^2}{210\left(a_1 d + 2a_2\right)^7}(a_1^6(210c_0 + 105c_1 + 35c_2 - 7c_4 + 5c_6)d^6 + 128a_2^6(105c_0 + 105c_1 + 70c_2 - 56c_4 + 160c_6) + 48a_1 a_2^5(840c_0 + 770c_1 + 420c_2 - 105c_3 - 308c_4 + 350c_5 + 860c_6)d + 6a_1^5 a_2(420c_0 + 245c_1 + 70c_2 - 14c_4 + 10c_6)d^5 + 120a_1^2 a_2^4(420c_0 + 350c_1 + 154c_2 - 63c_3 - 56c_4 + 210c_5 - 53c_6)d^2 + 30a_1^4 a_2^2(420c_0 + 280c_1 + 84c_2 - 21c_3 + 14c_4 - 35c_5 + 52c_6)d^4 + 20a_1^3 a_2^3(1680c_0 + 1260c_1 + 448c_2 - 189c_3 + 28c_4 + 315c_5 - 1346c_6) d^3).$$

(iii): *Case $r = 2$ (all two roots equal to d):*

$$W_n^C = (c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0)bd^n$$

where

$$\begin{aligned}
A_6 &= \frac{1}{56} bc_6, \\
A_5 &= \frac{1}{42} b(c_5 + 6c_6), \\
A_4 &= \frac{1}{60} b(2c_4 + 10c_5 + 25c_6), \\
A_3 &= \frac{1}{60} b(3c_3 + 12c_4 + 25c_5 + 30c_6), \\
A_2 &= \frac{1}{24} b(2c_2 + 6c_3 + 10c_4 + 10c_5 + 3c_6), \\
A_1 &= \frac{1}{12} b(2c_1 + 4c_2 + 5c_3 + 4c_4 + c_5 - 2c_6), \\
A_0 &= \frac{1}{420} b(210c_0 + 210c_1 + 175c_2 + 105c_3 + 21c_4 - 35c_5 - 25c_6).
\end{aligned}$$

(h): $m=2$, $s=7$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): *Case* $r=0$ *(no root equal to* d*):*

$$W_n^C = (A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$A_7 = -\dfrac{c_7bd^2}{a_1d+a_2-d^2},$

$A_6 = -\dfrac{1}{a_1d+a_2-d^2}(c_6bd^2 - 7(a_1d+2a_2)A_7),$

$A_5 = -\dfrac{1}{a_1d+a_2-d^2}(c_5bd^2 - 6(a_1d+2a_2)A_6 + 21(a_1d+4a_2)A_7),$

$A_4 = -\dfrac{1}{a_1d+a_2-d^2}(c_4bd^2 - 5(a_1d+2a_2)A_5 + 15(a_1d+4a_2)A_6 - 35(a_1d+8a_2)A_7),$

$A_3 = -\dfrac{1}{a_1d+a_2-d^2}(c_3bd^2 - 4(a_1d+2a_2)A_4 + 10(a_1d+4a_2)A_5 - 20(a_1d+8a_2)A_6 + 35(a_1d+16a_2)A_7),$

$A_2 = -\dfrac{1}{a_1d+a_2-d^2}(c_2bd^2 - 3(a_1d+2a_2)A_3 + 6(a_1d+4a_2)A_4 - 10(a_1d+8a_2)A_5 + 15(a_1d+16a_2)A_6 - 21(a_1d+32a_2)A_7),$

$A_1 = -\dfrac{1}{a_1d+a_2-d^2}(c_1bd^2 - 2(a_1d+2a_2)A_2 + 3(a_1d+4a_2)A_3 - 4(a_1d+8a_2)A_4 + 5(a_1d+16a_2)A_5 - 6(a_1d+32a_2)A_6 + 7(a_1d+64a_2)A_7),$

$A_0 = -\dfrac{1}{a_1d+a_2-d^2}(c_0bd^2 - (a_1d+2a_2)A_1 + (a_1d+4a_2)A_2 - (a_1d+8a_2)A_3 + (a_1d+16a_2)A_4 - (a_1d+32a_2)A_5 + (a_1d+64a_2)A_6 - (a_1d+128a_2)A_7),$

i.e.,

$A_7 = -\dfrac{c_7bd^2}{a_1d+a_2-d^2},$

$A_6 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^2}(-c_6d^2 + a_1(c_6+7c_7)d + a_2(c_6+14c_7)),$

$A_5 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^3}(c_5d^4 - a_1(2c_5+6c_6-21c_7)d^3 + a_1^2(c_5+6c_6+21c_7)d^2 - 2a_2(c_5+6c_6-42c_7)d^2 + a_2^2(c_5+12c_6+84c_7) + a_1a_2(2c_5+18c_6+63c_7)d),$

$A_4 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^4}(-c_4d^6 + a_1(3c_4+5c_5-15c_6+35c_7)d^5 - a_1^2(3c_4+10c_5-140c_7)d^4 + a_1^3(c_4+5c_5+15c_6+35c_7)d^3 + a_2(3c_4+10c_5-60c_6+280c_7)d^4 - a_2^2(3c_4+20c_5-1120c_7)d^2 + a_2^3(c_4+10c_5+60c_6+280c_7) + a_1a_2^2(3c_4+25c_5+105c_6+175c_7)d + a_1^2a_2(3c_4+20c_5+60c_6+140c_7)d^2 - 6a_1a_2(c_4+5c_5-5c_6-105c_7)d^3),$

$A_3 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^5}(c_3d^8 - a_1(4c_3+4c_4-10c_5+20c_6-35c_7)d^7 + a_1^2(6c_3+12c_4-10c_5-60c_6+385c_7)d^6 - a_1^3(4c_3+12c_4+10c_5-60c_6-385c_7)d^5 + a_1^4(c_3+4c_4+10c_5+20c_6+35c_7)d^4 - 4a_2(c_3+2c_4-10c_5+40c_6-140c_7)d^6 + 2a_2^2(3c_3+12c_4-20c_5-240c_6+3080c_7)d^4 - 4a_2^3(c_3+6c_4+10c_5-120c_6-1540c_7)d^2 + a_2^4(c_3+8c_4+40c_5+160c_6+560c_7) - a_1a_2^2(12c_3+60c_4+50c_5-900c_6-4025c_7)d^3 - 2a_1^2a_2(6c_3+24c_4+10c_5-180c_6-1015c_7)d^4 + a_1a_2^3(4c_3+28c_4+110c_5+260c_6-35c_7)d + a_1^3a_2(4c_3+20c_4+50c_5+100c_6+175c_7)d^3 + a_1^2a_2^2(6c_3+36c_4+110c_5+180c_6+385c_7)d^2 + a_1a_2(12c_3+36c_4-70c_5-180c_6+2695c_7)d^5),$

$A_2 = -\dfrac{bd^2}{(a_1d+a_2-d^2)^6}(-c_2d^{10} + a_1(5c_2+3c_3-6c_4+10c_5-15c_6+21c_7)d^9 - 2a_1^2(5c_2+6c_3-6c_4-10c_5+75c_6-273c_7)d^8 + 2a_1^3(5c_2+9c_3-30c_5+693c_7)d^7 - a_1^4(5c_2+12c_3+12c_4-20c_5-$

$150c_6 - 546c_7)d^6 + a_1^5(c_2 + 3c_3 + 6c_4 + 10c_5 + 15c_6 + 21c_7)d^5 + a_2(5c_2 + 6c_3 - 24c_4 + 80c_5 - 240c_6 + 672c_7)d^8 - 2a_2^2(5c_2 + 12c_3 - 24c_4 - 80c_5 + 1200c_6 - 8736c_7)d^6 + 2a_2^3(5c_2 + 18c_3 - 240c_5 + 22176c_7)d^4 - a_2^4(5c_2 + 24c_3 + 48c_4 - 160c_5 - 2400c_6 - 17472c_7)d^2 + a_2^5(c_2 + 6c_3 + 24c_4 + 80c_5 + 240c_6 + 672c_7) + 6a_1a_2^2(5c_2 + 15c_3 - 6c_4 - 130c_5 + 345c_6 + 4515c_7)d^5 + 6a_1^2a_2(5c_2 + 12c_3 - 6c_4 - 50c_5 + 75c_6 + 1659c_7) d^6 - 4a_1a_2^3(5c_2 + 21c_3 + 30c_4 - 140c_5 - 1095c_6 - 2058c_7)d^3 - 4a_1^3a_2(5c_2 + 15c_3 + 12c_4 - 40c_5 - 240c_6 - 840c_7) d^5 + a_1a_2^4(5c_2 + 27c_3 + 90c_4 + 210c_5 + 225c_6 - 1071c_7)d + a_1^4a_2(5c_2 + 18c_3 + 36c_4 + 60c_5 + 90c_6 + 126c_7)d^4 - 6a_1^2a_2^2(5c_2 + 18c_3 + 18c_4 - 90c_5 - 405c_6 - 1449c_7)d^4 + 2a_1^2a_2^3(5c_2 + 24c_3 + 66c_4 + 110c_5 + 75c_6 + 483c_7) d^2 + 2a_1^3a_2^2(5c_2 + 21c_3 + 48c_4 + 70c_5 + 120c_6 + 147c_7)d^3 - 4a_1a_2(5c_2 + 9c_3 - 18c_4 + 225c_6 - 1512c_7)d^7),$

$A_1 = -\dfrac{bd^2}{(a_1d + a_2 - d^2)^7}(c_1d^{12} - a_1(6c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6 - 7c_7)d^{11} + a_1^2(15c_1 + 10c_2 - 9c_3 - 4c_4 + 45c_5 - 150c_6 + 399c_7)d^{10} - 2a_1^3(10c_1 + 10c_2 - 3c_3 - 16c_4 + 25c_5 + 120c_6 - 1057c_7)d^9 + a_1^4(15c_1 + 20c_2 + 6c_3 - 32c_4 - 50c_5 + 240c_6 + 2114c_7)d^8 - a_1^5(6c_1 + 10c_2 + 9c_3 - 4c_4 - 45c_5 - 150c_6 - 399c_7) d^7 + a_1^6(c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 + 6c_6 + 7c_7)d^6 - 2a_2(3c_1 + 2c_2 - 6c_3 + 16c_4 - 40c_5 + 96c_6 - 224c_7) d^{10} + a_2^2(15c_1 + 20c_2 - 36c_3 - 32c_4 + 720c_5 - 4800c_6 + 25536c_7)d^8 - 4a_2^3(5c_1 + 10c_2 - 6c_3 - 64c_4 + 200c_5 + 1920c_6 - 33824c_7)d^6 + a_2^4(15c_1 + 40c_2 + 24c_3 - 256c_4 - 800c_5 + 7680c_6 + 135296c_7)d^4 - 2a_2^5(3c_1 + 10c_2 + 18c_3 - 16c_4 - 360c_5 - 2400c_6 - 12768c_7)d^2 + a_2^6(c_1 + 4c_2 + 12c_3 + 32c_4 + 80c_5 + 192c_6 + 448c_7) - 2a_1a_2^2(30c_1 + 50c_2 - 39c_3 - 188c_4 + 895c_5 + 510c_6 - 37471c_7)d^7 - 4a_1^2a_2(15c_1 + 20c_2 - 15c_3 - 32c_4 + 125c_5 + 240c_6 - 5285c_7)d^8 + 2a_1a_2^3(30c_1 + 70c_2 + 21c_3 - 364c_4 - 385c_5 + 9030c_6 + 41209c_7)d^5 + 2a_1^3a_2(30c_1 + 50c_2 + 3c_3 - 104c_4 - 85c_5 + 1140c_6 + 8197c_7)d^7 - a_1a_2^4(30c_1 + 90c_2 + 129c_3 - 204c_4 - 2185c_5 - 7650c_6 + 1141c_7)d^3 - 2a_1^4a_2(15c_1 + 30c_2 + 24c_3 - 24c_4 - 170c_5 - 540c_6 - 1414c_7)d^6 + a_1a_2^5(6c_1 + 22c_2 + 57c_3 + 116c_4 + 155c_5 - 114c_6 - 1967c_7)d + a_1^5a_2(6c_1 + 14c_2 + 21c_3 + 28c_4 + 35c_5 + 42c_6 + 49c_7) d^5 + 6a_1^2a_2^2(15c_1 + 30c_2 + 3c_3 - 108c_4 + 5c_5 + 1350c_6 + 8827c_7)d^6 - 4a_1^2a_2^3(15c_1 + 40c_2 + 45c_3 - 88c_4 - 555c_5 - 1140c_6 - 4305c_7)d^4 - 2a_1^3a_2^2(30c_1 + 70c_2 + 63c_3 - 112c_4 - 525c_5 - 1680c_6 - 4263c_7)d^5 + a_1^2a_2^4(15c_1 + 50c_2 + 111c_3 + 172c_4 + 125c_5 - 30c_6 + 2359c_7)d^2 + a_1^4a_2^2(15c_1 + 40c_2 + 66c_3 + 80c_4 + 110c_5 + 120c_6 + 154c_7)d^4 + 2a_1^3a_2^3(10c_1 + 30c_2 + 57c_3 + 72c_4 + 65c_5 + 180c_6 - 77c_7)d^3 + a_1a_2(30c_1 + 30c_2 - 51c_3 + 36c_4 + 215c_5 - 1530c_6 + 7021c_7)d^9),$

$A_0 = -\dfrac{bd^2}{(a_1d + a_2 - d^2)^8}(-c_0d^{14} + a_1(7c_0 + c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + c_7)d^{13} + a_1^2(-21c_0 - 6c_1 + 4c_2 - 8c_4 + 24c_5 - 56c_6 + 120c_7)d^{12} + a_1^3(35c_0 + 15c_1 - 5c_2 - 9c_3 + 19c_4 + 15c_5 - 245c_6 + 1191c_7)d^{11} + a_1^4(-35c_0 - 20c_1 + 16c_3 - 80c_5 + 2416c_7)d^{10} + a_1^5(21c_0 + 15c_1 + 5c_2 - 9c_3 - 19c_4 + 15c_5 + 245c_6 + 1191c_7) d^9 - a_1^6(7c_0 + 6c_1 + 4c_2 - 8c_4 - 24c_5 - 56c_6 - 120c_7)d^8 + a_1^7(c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7)d^7 + a_2(7c_0 + 2c_1 - 4c_2 + 8c_3 - 16c_4 + 32c_5 - 64c_6 + 128c_7)d^{12} - a_2^2(21c_0 + 12c_1 - 16c_2 + 128c_4 - 768c_5 + 3584c_6 - 15360c_7)d^{10} + a_2^3(35c_0 + 30c_1 - 20c_2 - 72c_3 + 304c_4 + 480c_5 - 15\,680c_6 + 152448c_7)d^8 - a_2^4(35c_0 + 40c_1 - 128c_3 + 2560c_5 - 309248c_7)d^6 + a_2^5(21c_0 + 30c_1 + 20c_2 - 72c_3 - 304c_4 + 480c_5 + 15\,680c_6 + 152448c_7)d^4 - a_2^6(7c_0 + 12c_1 + 16c_2 - 128c_4 - 768c_5 - 3584c_6 - 15360c_7)d^2 + a_2^7(c_0 + 2c_1 + 4c_2 + 8c_3 + 16c_4 + 32c_5 + 64c_6 + 128c_7) + a_1a_2^2(105c_0 + 75c_1 - 55c_2 - 93c_3 + 545c_4 - 885c_5 - 6055c_6 + 75507c_7) d^9 + a_1^2a_2(105c_0 + 60c_1 - 40c_2 - 24c_3 + 152c_4 - 120c_5 - 1960c_6 + 16776c_7)d^{10} - 4a_1a_2^3(35c_0 + 35c_1 - 5c_2 - 85c_3 + 91c_4 + 1115c_5 - 4565c_6 - 49525c_7)d^7 - 4a_1^3a_2(35c_0 + 25c_1 - 5c_2 - 23c_3 + 19c_4 + 145c_5 - 245c_6 - 6023c_7)d^9 + a_1a_2^4(105c_0 + 135c_1 + 65c_2 - 297c_3 - 751c_4 + 3015c_5 + 31\,265c_6 + 61623c_7) d^5 + a_1^4a_2(105c_0 + 90c_1 + 20c_2 - 72c_3 - 112c_4 + 240c_5 + 2240c_6 + 10128c_7)d^8 - 2a_1a_2^5(21c_0 + 33c_1 + 37c_2 -$

$15c_3-275c_4-1047c_5-1883c_6+7425c_7)d^3-2a_1^5a_2(21c_0+21c_1+13c_2-3c_3-35c_4-99c_5-227c_6-483c_7)d^7+a_1a_2^6(7c_0+13c_1+23c_2+37c_3+47c_4+13c_5-217c_6-1163c_7)d+a_1^6a_2(7c_0+8c_1+8c_2+8c_3+8c_4+8c_5+8c_6+8c_7)d^6-2a_1^2a_2^2(105c_0+90c_1-20c_2-144c_3+232c_4+840c_5-3080c_6-48504c_7)d^8+2a_1^2a_2^3(105c_0+120c_1+40c_2-216c_3-296c_4+1800c_5+8440c_6+40104c_7)d^6+2a_1^3a_2^2(105c_0+105c_1+25c_2-135c_3-119c_4+585c_5+4345c_6+18585c_7)d^7+a_1^2a_2^4(-105c_0-150c_1-140c_2+96c_3+856c_4+2040c_5+1960c_6+18456c_7)d^4+a_1^4a_2^2(-105c_0-120c_1-80c_2+48c_3+256c_4+720c_5+1600c_6+3408c_7)d^6-4a_1^3a_2^3(35c_0+45c_1+35c_2-27c_3-157c_4-315c_5-925c_6-1467c_7)d^5+a_1^2a_2^5(21c_0+36c_1+56c_2+72c_3+56c_4-24c_5+56c_6+2472c_7)d^2+a_1^5a_2^2(21c_0+27c_1+29c_2+27c_3+29c_4+27c_5+29c_6+27c_7)d^5+a_1^3a_2^4(35c_0+55c_1+75c_2+79c_3+51c_4+55c_5+315c_6-1121c_7)d^3+a_1^4a_2^3(35c_0+50c_1+60c_2+56c_3+48c_4+80c_5+176c_7)d^4-2a_1a_2(21c_0+9c_1-11c_2+9c_3+13c_4-111c_5+469c_6-1671c_7)d^{11}).$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C=n(A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_7=\dfrac{c_7bd^2}{8\left(a_1d+2a_2\right)}$,

$A_6=\dfrac{1}{7\left(a_1d+2a_2\right)}(c_6bd^2+28(a_1d+4a_2)A_7)$,

$A_5=\dfrac{1}{6\left(a_1d+2a_2\right)}(c_5bd^2+21(a_1d+4a_2)A_6-56(a_1d+8a_2)A_7)$,

$A_4=\dfrac{1}{5\left(a_1d+2a_2\right)}(c_4bd^2+15(a_1d+4a_2)A_5-35(a_1d+8a_2)A_6+70(a_1d+16a_2)A_7)$,

$A_3=\dfrac{1}{4\left(a_1d+2a_2\right)}(c_3bd^2+10(a_1d+4a_2)A_4-20(a_1d+8a_2)A_5+35(a_1d+16a_2)A_6-56(a_1d+32a_2)A_7)$,

$A_2=\dfrac{1}{3\left(a_1d+2a_2\right)}(c_2bd^2+6(a_1d+4a_2)A_3-10(a_1d+8a_2)A_4+15(a_1d+16a_2)A_5-21(a_1d+32a_2)A_6+28(a_1d+64a_2)A_7)$,

$A_1=\dfrac{1}{2\left(a_1d+2a_2\right)}(c_1bd^2+3(a_1d+4a_2)A_2-4(a_1d+8a_2)A_3+5(a_1d+16a_2)A_4-6(a_1d+32a_2)A_5+7(a_1d+64a_2)A_6-8(a_1d+128a_2)A_7)$,

$A_0=\dfrac{1}{\left(a_1d+2a_2\right)}(c_0bd^2+(a_1d+4a_2)A_1-(a_1d+8a_2)A_2+(a_1d+16a_2)A_3-(a_1d+32a_2)A_4+(a_1d+64a_2)A_5-(a_1d+128a_2)A_6+(a_1d+256a_2)A_7)$,

i.e.,

$A_7=\dfrac{c_7bd^2}{8\left(a_1d+2a_2\right)}$,

$A_6=\dfrac{bd^2}{14\left(a_1d+2a_2\right)^2}(a_1(2c_6+7c_7)d+4a_2(c_6+7c_7))$,

$A_5=\dfrac{bd^2}{12\left(a_1d+2a_2\right)^3}(a_1^2(2c_5+6c_6+7c_7)d^2+8a_2^2(c_5+6c_6+14c_7)+4a_1a_2(2c_5+9c_6+7c_7)d)$,

$A_4=\dfrac{bd^2}{10\left(a_1d+2a_2\right)^4}(a_1^3(2c_4+5c_5+5c_6)d^3+16a_2^3(c_4+5c_5+10c_6)+2a_1a_2^2(12c_4+50c_5+60c_6-105c_7)d+2a_1^2a_2(6c_4+20c_5+15c_6)d^2)$,

$A_3=\dfrac{bd^2}{24\left(a_1d+2a_2\right)^5}(a_1^4(6c_3+12c_4+10c_5-7c_7)d^4+32a_2^4(3c_3+12c_4+20c_5-56c_7)+16a_1a_2^3(12c_3+42c_4+50c_5-45c_6-119c_7)d+8a_1^3a_2(6c_3+15c_4+10c_5-7c_7)d^3+24a_1^2a_2^2(6c_3+18c_4+15c_5-15c_6+28c_7)d^2)$,

$$A_2 = \frac{bd^2}{6\,(a_1d+2a_2)^6}(a_1^5(2c_2+3c_3+2c_4-c_6)d^5+64a_2^5(c_2+3c_3+4c_4-8c_6)+16a_1a_2^4(10c_2+27c_3+28c_4-15c_5-50c_6+105c_7)d+2a_1^4a_2(10c_2+18c_3+10c_4-5c_6)d^4+16a_1^2a_2^3(10c_2+24c_3+19c_4-15c_5-5c_6+105c_7)d^2+2a_1^3a_2^2(40c_2+84c_3+52c_4-30c_5+40c_6-105c_7)d^3),$$

$$A_1 = \frac{bd^2}{12\,(a_1d+2a_2)^7}(a_1^6(6c_1+6c_2+3c_3-c_5+c_7)d^6+128a_2^6(3c_1+6c_2+6c_3-8c_5+32c_7)+192a_1a_2^5(6c_1+11c_2+9c_3-3c_4-11c_5+15c_6+43c_7)d+12a_1^5a_2(6c_1+7c_2+3c_3-c_5+c_7)d^5+24a_1^2a_2^4(60c_1+100c_2+66c_3-36c_4-40c_5+180c_6-53c_7)d^2+12a_1^4a_2^2(30c_1+40c_2+18c_3-6c_4+5c_5-15c_6+26c_7)d^4+8a_1^3a_2^3(120c_1+180c_2+96c_3-54c_4+10c_5+135c_6-673c_7)d^3),$$

$$A_0 = \frac{bd^2}{210\,(a_1d+2a_2)^8}(a_1^7(210c_0+105c_1+35c_2-7c_4+5c_6)d^7+256a_2^7(105c_0+105c_1+70c_2-56c_4+160c_6)+112a_1a_2^6(840c_0+780c_1+440c_2-90c_3-328c_4+300c_5+920c_6-1785c_7)d+14a_1^6a_2(210c_0+120c_1+35c_2-7c_4+5c_6)d^6+336a_1^2a_2^5(420c_0+360c_1+170c_2-60c_3-84c_4+200c_5+85c_6-1190c_7)d^2+42a_1^5a_2^2(420c_0+270c_1+80c_2-15c_3+6c_4-25c_5+40c_6-35c_7)d^5+280a_1^3a_2^4(420c_0+330c_1+130c_2-54c_3-20c_4+135c_5-215c_6-441c_7)d^3+280a_1^4a_2^3(210c_0+150c_1+50c_2-18c_3+5c_4+15c_5-85c_6+273c_7)d^4).$$

(iii): *Case $r = 2$ (all two roots equal to d):*

$$W_n^C = n^2(A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_7 &= \frac{1}{72}bc_7,\ A_6 = \frac{1}{56}b(c_6+7c_7),\ A_5 = \frac{1}{84}b(2c_5+12c_6+35c_7),\\
A_4 &= \frac{1}{60}b(2c_4+10c_5+25c_6+35c_7),\ A_3 = \frac{1}{120}b(6c_3+24c_4+50c_5+60c_6+21c_7),\\
A_2 &= \frac{1}{24}b(2c_2+6c_3+10c_4+10c_5+3c_6-7c_7),\\
A_1 &= \frac{1}{36}b(6c_1+12c_2+15c_3+12c_4+3c_5-6c_6-5c_7),\\
A_0 &= \frac{1}{420}b(210c_0+210c_1+175c_2+105c_3+21c_4-35c_5-25c_6+35c_7).
\end{aligned}$$

3. Specific Examples

We now move from the general theory to explicit computations, presenting illustrative cases that show how the framework functions in practice. These examples emphasize the difference between non-resonant and resonant inputs, the role of root multiplicities, and the corrective terms that appear when the input polynomial interacts with the characteristic equation. By examining low-order recurrences, the abstract formulas are converted into concrete solutions, thereby validating the theory and enhancing pedagogical clarity.

3.1. Generalized Fibonacci Numbers. We begin with a non-resonant case. Recall the homogeneous recurrence relation defining the generalized Fibonacci sequence:

$$V_n = V_{n-1} + V_{n-2}, \tag{3.1}$$

with the initial conditions V_0, V_1. The characteristic equation is

$$z^2 - z - 1 = 0, \tag{3.2}$$

with roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The closed-form solution (Binet's formula) is

$$V_n = \frac{V_1 - \beta V_0}{\alpha - \beta}\alpha^n - \frac{V_1 - \alpha V_0}{\alpha - \beta}\beta^n. \tag{3.3}$$

We now consider a second-order non-homogeneous recurrence where the input polynomial does not coincide with a root of the characteristic equation, so no resonance adjustment is required:

Example 3.1. *Consider the non-homogeneous recurrence:*

$$W_n = W_{n-1} + W_{n-2} + C(n), \tag{3.4}$$

with the initial conditions W_0, W_1. *The input function is*

$$C(n) = (n^3 + 4n^2 - 6n + 5) \times 3^n. \tag{3.5}$$

Since 3 *is not a root of the characteristic equation (3.2) of the associated homogeneous recurrence relation (3.1), the input* $C(n)$ *is non-resonant. So we will use Example* **??** *(d) (i). The particular solution is*

$$W_n^C = (A_3 n^3 + A_2 n^2 + A_1 n + A_0) \times 3^n,$$

satisfying

$$W_n^{(C)} = W_{n-1}^{(C)} + W_{n-2}^{(C)} + (n^3 + 4n^2 - 6n + 5) \times 3^n.$$

Solving yields

$$W_n^{(C)} = \frac{9}{25}(5n^3 + 5n^2 - 19n + 40) \times 3^n.$$

This confirms the non-resonant case, where no correction is necessary.

3.2. Generalized Mersenne Numbers. We next examine a resonant case. Recall the homogeneous recurrence defining the generalized Mersenne numbers:

$$V_n = 3V_{n-1} - 2V_{n-2} \tag{3.6}$$

with the initial values V_0, V_1. The characteristic equation is

$$z^2 - 3z + 2 = 0 \tag{3.7}$$

with roots $\alpha = 2$, $\beta = 1$. The closed-form ((Binet's formula) solution is

$$V_n = (V_1 - V_0)2^n - (V_1 - 2V_0). \tag{3.8}$$

We now consider a second-order non-homogeneous linear recurrence relation exhibiting resonance, where the input function aligns with a root of the characteristic equation of the associated homogeneous recurrence.

Example 3.2. *Consider the non-homogeneous recurrence:*

$$W_n = 3W_{n-1} - 2W_{n-2} + C(n) \tag{3.9}$$

with the initial conditions W_0, W_1. *The input function is*

$$C(n) = (7n^4 + 3n^3 - n^2 + 9n + 8) \times 2^n$$

The input $C(n)$ is resonant since $d = 2$ is a root of characteristic equation (3.7) of the associated homogeneous recurrence relation (3.6), so resonance correction is required. Thus we will use Example **??** *(e) (ii). The particular solution takes the form*

$$\begin{aligned} W_n^C &= n(A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0) \times 2^n \\ &= (A_4n^5 + A_3n^4 + A_2n^3 + A_1n^2 + A_0n) \times 2^n \end{aligned}$$

satisfying

$$W_n^{(C)} = 3W_{n-1}^{(C)} - 2W_{n-2}^{(C)} + (7n^4 + 3n^3 - n^2 + 9n + 8) \times 2^n.$$

Solving yields

$$\begin{aligned} W_n^C &= \frac{n}{10}(28n^4 - 55n^3 + 570n^2 - 2225n + 6762) \times 2^n \\ &= \frac{1}{10}(28n^5 - 55n^4 + 570n^3 - 2225n^2 + 6762n) \times 2^n. \end{aligned}$$

This example illustrates resonance correction, since the input polynomial coincides with the root 2.

Conclusion

In this work, we derived explicit particular solutions of generalized Leonardo-type recurrence relations with polynomial-exponential inputs. By extending Theorem 1.2, closed-form expressions were obtained for the low-order cases $m = 1, 2$, showing how the general framework specializes into concrete computations. These results highlight the interplay between characteristic polynomials, root multiplicities, and resonance phenomena, unifying classical recurrence identities with modern symbolic approaches.

Recurrence sequences remain central across mathematics and its applications, appearing in physics, engineering, biology, computer science, and even artistic domains. Homogeneous relations capture intrinsic dynamics, while non-homogeneous relations incorporate external influences, making them powerful tools for modeling real-world processes.

The contribution of this study lies in presenting clear methods for constructing particular solutions under polynomial-exponential inputs. Explicit formulas simplify abstract theory, clarify resonance effects, and provide templates for interdisciplinary modeling. The examples serve both as verification of the general theorem and as pedagogical illustrations, bridging theory with application. Non-resonant cases, such as the generalized Fibonacci sequence, demonstrate direct solutions, while resonant cases, such as the generalized Mersenne sequence, reveal the adjustments required when characteristic roots coincide with input terms.

Overall, the results strengthen both theoretical understanding and practical accessibility, offering resonance-aware formulas that are valuable for research and teaching alike.

Recurrence relations (sequences) have many applications. Next, we list applications of second-order recurrence relations.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [2].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [4].

- For the application of Jacobsthal numbers to special matrices, see [47].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [18].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [45].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [46].
- For the applications of generalized Fibonacci numbers to binomial sums, see [44].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [41].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [42].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [13].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [29].
- For the application of higher order Jacobsthal numbers to quaternions, see [31].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [15].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [3].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [23].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [24].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [27].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [43].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [17].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [6].
- For the application of Pell, Pell-Lucas and Modified Pell numbers to matrices, see [7].
- For the application of Jacobsthal numbers to matrices, see [8].
- For the application of Jacobsthal and Jacobsthal–Lucas numbers to matrices, see [9].
- For the application of Pell, Pell-Lucas and Modified Pell numbers to Toeplitz matrices, see [10].
- For the application of Fibonacci and Lucas numbers to quaternions, see [11].
- For the application of Lucas numbers to Hyperbolic quaternions, see [12].

References

[1] Abd-Elhameed, W.M., Alqubori, O.M., Alluhaybi, A.A., Amin, A.K., Novel Expressions for Certain Generalized Leonardo Polynomials and Their Associated Numbers, Axioms, 14, 286, 2025. https://doi.org/10.3390/

[2] Azak, A.Z., Pauli Gaussian Fibonacci and Pauli Gaussian Lucas Quaternions. Mathematics, 2022, 10, 4655. https://doi.org/10.3390/math10244655

[3] Bilgin, N.G., Fibonacci Lacunary Statistical Convergence of Order γ in IFNLS, International Journal of Advances in Applied Mathematics and Mechanics, 8(4), 28-36, 2021.

[4] Büyük, H., Taşkara, N., On The Solutions of Three-Dimensional Difference Equation Systems Via Pell Numbers, European Journal of Science and Technology, Special Issue 34, 433-440, 2022.

[5] Catarino, P., Borges, A., On Leonardo Numbers, Acta Mathematica Universitatis Comenianae, 89(1), 75–86, 2020. Available online at: http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005/650.

[6] Daşdemir, A., On Recursive Hyperbolic Fibonacci Quaternions, Communications in Advanced Mathematical Sciences, 4(4), 198-207, 2021. DOI:10.33434/cams.997824

[7] Dasdemir A., On the Pell, Pell-Lucas and Modified Pell numbers by matrix method, Applied Mathematical Sciences, 5(64), 3173–3181, 2011.

[8] Dasdemir A., On the Jacobsthal numbers by matrix method. SDU Journal of Science, 7(1), 69-76, 2012.

[9] Daşdemir, A., A study on the Jacobsthal and Jacobsthal–Lucas numbers by matrix method, DUFED Journal of Sciences, 3(1), 13–18, 2014.

[10] Daşdemir, A., On the norms of Toeplitz matrices with the Pell, Pell-Lucas and Modified Pell numbers, Journal of Engineering Technology and Applied Sciences, 1(1), 1–12, 2016.

[11] Daşdemir, A., Gelin-Cesáro identities for Fibonacci and Lucas quaternions, Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica, 18, 137–144, 2019.

[12] Daşdemir, A., On hyperbolic Lucas quaternions, Ars Combinatoria, 150, 77–84, 2020.

[13] Deveci, Ö., Shannon, A.G., On Recurrence Results From Matrix Transforms, Notes on Number Theory and Discrete Mathematics, 28(4), 589–592, 2022. DOI: 10.7546/nntdm.2022.28.4.589-592

[14] Dikmen, C.D., Properties of Gaussian Generalized Leonardo Numbers, Karaelmas Science and Engineering Journal, 15(1), 134-145, 2025. DOI: 10.7212/karaelmasfen.1578154

[15] Goy, T.,Shattuck, M., Fibonacci and Lucas Identities from Toeplitz-Hessenberg Matrices, Appl. Appl. Math, 14(2), 699–715, 2019.

[16] Gökbaş, H., k-Leonardo Numbers, Palestine Journal of Mathematics, 13(4), 1427-1435, 2024.

[17] Gökbaş¸ H., Topal, S., Smarandache, F., Neutrosophic Number Sequences: An Introductory Study, International Journal of Neutrosophic Science (IJNS), 20(01), 27-48, 2023. https://doi.org/10.54216/IJNS.200103

[18] Gül, K., Generalized k-Order Fibonacci Hybrid Quaternions, Erzincan University Journal of Science and Technology, 15(2), 670-683, 2022. DOI: 10.18185/erzifbed.1132164

[19] İşbilir, Z., Akyiğit, M., Tosun, M., Pauli–Leonardo Quaternions, Notes on Number Theory and Discrete Mathematics, 29(1), 1-16, 2023. DOI: 10.7546/nntdm.2023.29.1.1-16

[20] Jeske, J.A., Linear Recurrence Relations, Part I, The Fibonacci Quarterly, 1(2), 69-74, 1963.

[21] Jeske, J.A., Linear Recurrence Relations, Part II, The Fibonacci Quarterly, 1(4), 34-39, 1963.

[22] Jeske, J.A., Linear Recurrence Relations, Part III, The Fibonacci Quarterly, 2(3), 197-203, 1964.

[23] Kişi, Ö., Tuzcuoglu, I., Fibonacci Lacunary Statistical Convergence in Intuitionistic Fuzzy Normed Linear Spaces, Journal of Progressive Research in Mathematics 16(3), 3001-3007, 2020.

[24] Kişi, Ö., Debnathb, P., Fibonacci Ideal Convergence on Intuitionistic Fuzzy Normed Linear Spaces, Fuzzy Information and Engineering, 1-13, 2022. https://doi.org/10.1080/16168658.2022.2160226

[25] Kuhapatanakul, K., Chobsorn, J, On the Generalized Leonardo Numbers, Integers 22, 2022, #A48.

[26] Kuhapatanakul, K., Ruankong, P., On Generalized Leonardo p-numbers, Journal of Integer Sequences, 27, Article 24.4.6, 2024.

[27] Kumari, M., Prasad, K., Frontczak, R., On the k-Fibonacci and $k-$Lucas Spinors, Notes on Number Theory and Discrete Mathematics, 29(2), 322-335, 2023. DOI: 10.7546/nntdm.2023.29.2.322-335

[28] Prasad, K., Kumari, M., The Leonardo Polynomials and Their Algebraic Properties. Proceedings of the Indian National Science Academy, 2024. https://doi.org/10.1007/s43538-024-00348-0

[29] Prasad, K., Mahato, H., Cryptography Using Generalized Fibonacci Matrices with Affine-Hill Cipher, Journal of Discrete Mathematical Sciences & Cryptography, 25(8-A), 2341–2352, 2022. DOI : 10.1080/09720529.2020.1838744

[30] Özimamoğlu, H., On Leonardo Sedenions, Afrika Matematika (2023) 34:26, 2023. https://doi.org/10.1007/s13370-023-01065-5

[31] Özkan, E., Uysal, M., On Quaternions with Higher Order Jacobsthal Numbers Components, Gazi University Journal of Science, 36(1), 336-347, 2023. DOI: 10.35378/gujs. 1002454

[32] Özkan, E., Akkuş, H., Generalized Bronze Leonardo Sequence, Notes on Number Theory and Discrete Mathematics, 30(4), 811-824, 2024. DOI: 10.7546/nntdm.2024.30.4.811-824

[33] Shannon, A.G., A Note On Generalized Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 25(3), 97–101, 2019. DOI: 10.7546/nntdm.2019.25.3.97-101

[34] Shannon, A.G., Deveci, Ö., A Note on Generalized and Extended Leonardo Sequences, Notes on Number Theory and Discrete Mathematics, 28(1), 109–114, 2022. DOI: 10.7546/nntdm.2022.28.1.109-114

[35] Shannon, A.G., Shiue, P.J.S., Huang, S.C., Notes on Generalized and Extended Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 29(4), 752–773, 2023. DOI: 10.7546/nntdm.2023.29.4.752-773

[36] Soykan, Y., Generalized Horadam-Leonardo Numbers and Polynomials, Asian Journal of Advanced Research and Reports, 17(8), 128-169, 2023. https://doi.org/10.9734/ajarr/2023/v17i8511

[37] Soykan, Y., Interrelations between Horadam and Generalized Horadam-Leonardo Polynomials via Identities, International Journal of Advances in Applied Mathematics and Mechanics, 11(1), 42-55, 2023. ISSN: 2347-2529

[38] Soykan, Y., Generalized Leonardo Numbers, Journal of Progressive Research in Mathematics, 18(4), 58-84, 2021.

[39] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026.

[40] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

[41] Soykan, Y., Taşdemir, E., A Study On Hyperbolic Numbers With Generalized Jacobsthal Numbers Components, International Journal of Nonlinear Analysis and Applications, 13(2), 1965–1981, 2022. http://dx.doi.org/10.22075/ijnaa.2021.22113.2328

[42] Soykan, Y., On Dual Hyperbolic Generalized Fibonacci Numbers, Indian J Pure Appl Math, 2021. https://doi.org/10.1007/s13226-021-00128-2

[43] Şentürk, G.Y., Gürses, N., Yüce, S., Construction of Dual-Generalized Complex Fibonacci and Lucas Quaternions, Carpathian Math. Publ. 2022, 14 (2), 406-418, 2022. doi:10.15330/cmp.14.2.406-41

[44] Ulutaş, Y.T., Toy, D., Some Equalities and Binomial Sums about the Generalized Fibonacci Number u_n, Notes on Number Theory and Discrete Mathematics, 28(2), 252–260, 2022. DOI: 10.7546/nntdm.2022.28.2.252-260

[45] Yılmaz, N., Split Complex Bi-Periodic Fibonacci and Lucas Numbers, Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat. 71(1), 153–164, 2022. DOI:10.31801/cfsuasmas.704435

[46] Yılmaz, N., The Generalized Bivariate Fibonacci and Lucas Matrix Polynomials, Mathematica Montisnigri, Vol LIII, 33-44, 2022. DOI: 10.20948/mathmontis-2022-53-5

[47] Vasanthi, S., Sivakumar, B., Jacobsthal Matrices and their Properties. Indian Journal of Science and Technology 15(5): 207-215, 2022, https://doi.org/10.17485/IJST/v15i5.1948