# Convergence Analysis of Generalized Non-smooth Equations Using Gauss-Type Proximal Point Method


## Abstract

This work discusses the Gauss-type proximal point algorithm for solving non-smooth generalized equations like $0 \in q(x) + Q(x)$, where a set-valued mapping $Q: X \rightrightarrows 2^Y$ acting between two real or complex Banach spaces $X$ and $Y$ with closed graph and $q: U \subseteq X \to Y$ is a single-valued mapping. In order to ensure the existence as well as convergence of any sequence produced by this algorithm under appropriate circumstances, we develop the convergence criteria of this approach by utilize metric regularity condition and point-based approximation. Lastly, we present a numerical example to validate the semi-local convergence of this algorithm.



## Keywords

Non-smooth Generalized Equation, Set-valued mapping, Metrically regular mapping, Point Based Approximation, Semi-local convergence.


## 1. Introduction

This study deals with non-smooth generalized equations. Non-smooth generalized equations are those that have a set-valued function and a non-smooth single-valued function. Let $X$ as well as $Y$ be two different Banach spaces, $q: U \subseteq X \to Y$ be a continuous non-smooth function, that is, $q$ continuous but does not have a Frechet derivative on $U \subseteq X$ as well as $Q: X \rightrightarrows 2^Y$ be a closed graph set-valued function, where $U$ be an open subset of $X$. To this end, the problem of finding a point $x \in U \subseteq X$ which satisfies the generalized non- smooth equation

$$0 \in q(x) + Q(x). \tag{1}$$

The idea of generalized equations, which was first presented by Robinson [1, 2]; can be used to describe a wide range of diverse problems, such as system of nonlinear equations, system of inequalities, variational inequalities, equilibrium problems, complementary problems, etc, see [1, 3–5]. They are also widely used in mathematical programming, engineering, economics, as well as applied computational sciences. Consult [2, 3, 6, 7] for comprehensive information on this application. Robinson [8] first began to point-based approximation for the non-smooth equations $q(x) = 0$, as well as using an assumption of the Newton-Kantorovich type hypothesis, they demonstrated the convergence of Newton's technique. Argyros [9] generalized the Robinson's method [8] using weaker assumptions in the point-based approximation as well as presented a semi-local convergence for Newton's method considering Holderian instead of Lipchitzian properties.

To solve the non-smooth generalized equation (1), when the single-valued part of the generalized equation is Fréchet differentiable, Newton-type methods, the proximal point approach, as well as other iterative approaches were presented; see, for example, [4, 5, 10-12]. In the setting of $q = 0$ as well as $Y = X$ a Hilbert space, one of the most efficient approaches to tackle the problem (1) is the proximal point algorithm (PPA). Regarding the root of PPA for variational inequalities, we refer to the writings of Martinet [13]. More broadly, this PPA has been generalized and applied to convex programs, variational inequality problems as well as convex-concave saddle point problems, see [3, 5, 14]. For a smooth function, i.e., when $q$ is Frechet differentiable, a Gauss-type proximal point technique for the solution of (1) has been introduced by Alom and Rashid [15].

Numerous publications examine the outcomes of local and semi-local convergence and deal with Newton-type approaches for solving the non-smooth generalized equation (1), see [9, 16, 17]. Semi-local analysis has developed some useful results for specific situations, such as the Newton method for non-smooth equations as well as Newton-type method for nonlinear least square problems, see [18, 19]. Rashid *et al.* [20] introduced the Gauss-Newton type technique for smooth functions, which yields results for semi-local as well as local convergence and may be used to estimate the solution of (1).

We denote the subset of $X$ by $(\xi_k, x)$ by any $x \in X$ as well as for any sequence of positive numbers $\xi_k$. This subset can be defined as:

$$D(\xi_\kappa, x) = \{d \in X : 0 \in A(x, x + d) + Q(x + d)\}. \tag{2}$$

In section 2, we have a point-based approximation of a function $q$ on $U$, which we denote as $A(x, \cdot)$. In the book [3, Ch. 6], Dontchev and Rockafellar provided the following PPA for the non-smooth generalized equation (1):

**Algorithm1**(PPA)

Step-1: Let $\xi > 0$, $x_0 \in X$, $k = 0$.

Step-2: If $0 \in D(\xi_k, x_k)$, then end the program; otherwise, return to Step-3.

Step-3: Set $\{\xi_k\} \subseteq (0, \xi)$ as well as if $0 \notin D(\xi_k, x_k)$, choose $d_k$ such that $d_k \in D(\xi_k, x_k)$.

Step-4: Write $x_{k+1} = x_k + d_k$.

Step-5: Put $k = k + 1$ and return to Step-2.

Specifically, not all of the results generated by algorithm1 are convergent, and those that are sufficiently near to a solution (limit point of convergences) are not unique. A solution to the non-smooth generalized equation (1) is linearly convergent to one of the sequences generated by Algorithm1. Thus, from the standpoint of practical application, such methods would not be appropriate for real-world use. We are confident enough to suggest a novel convergence analysis method for the Gauss-type proximal point algorithm (G-PPA) because of this shortcoming. For the non-smooth issue (1), we then provide the G-PPA (see Algorithm2). The main variation between the Algorithm1 and our suggested Algorithm2 is that the Algorithm1 does not produce more than one converging sequence, but the G-PPA provides these sequences.

**Algorithm2**(G-PPA)

Step-1: Let $\xi > 0, x_0 \in X, \eta \geq 1$, $k = 0$.

Step-2: If $0 \in D(\xi_k, x_k)$, then end the program; otherwise, return to Step-3.

Step-3: Set $\{\xi_k\} \subseteq (0, \xi)$ as well as if $0 \notin D(\xi_k, x_k)$, choose $d_k$ such that $d_k \in D(\xi_k, x_k)$ as well as $\|d_k\| \leq \eta d(0, D(\xi_k, x_k))$.

Step-4: Write $x_{k+1} = x_k + d_k$.

Step-5: Put $k = k + 1$ and return to Step-2.

In Algorithm 2, we see that

(i) if $\eta = 1, D(\xi_k, x_k)$ is singleton, then Algorithm2 is same as Algorithm1,

(ii) Algorithm2 is similar to the general G-PPA for smooth generalized equations, examined in Alom and Rashid [15], if the single-valued function $q$ is Frechet differentiable.

Rashid introduced the Gauss-Newton method for non-smooth generalized equations in [16], where he earned the local and semi-local super linear convergence results. Additionally, Rashid [18] demonstrated the results of local and semi-local convergence and offered an extended Newton-type method to solve the non-smooth generalized equation (1). The non-smooth generalized equation (1) has solved by using a constrained Newton-type approach by Rashid and Yuan [17], who also examine the semi-local and local convergence outcomes. Alom and Rashid [21] presented the generic Gauss-type proximal point method for solving (1) and offered semi-local as well as local convergence results for smooth functions.

In this context, we demonstrated that the sequence produced by the Gauss-type proximal point method, as defined by method2, converges semi-locally to a solution of the generalized equation (1) that is not smooth. In this work for set-valued functions, we have used the property of metric regularity as well as the point-based approximation. This method for semi-local convergence analysis utilizing the Gauss-type proximal point approach specified by Algorithm 2 has not, as far as we are aware, been studied before. We might thus conclude that the contributions put out in this work appear to be innovative.

## 2. Notations as well as Preliminaries

This chapter provides an overview of some basic concepts, mathematical facts, as well as simple notations, which will be frequently alluded in the latter section. Consider two Banach spaces, $X$ and $Y$, which might be real or complex. For example, let $Q$ be a set-valued function $Q: X \rightrightarrows 2^Y$ from the collection of $X$ into the subsets of $Y$. Assume $r > 0$ as well as $x \in X$. We refer to the closed ball with radius $r$ that is centered at $x$ as such $\mathbb{B}_r(x)$. The domain $domQ$, the inverse $Q^{-1}$ as well as the graph *gph Q* are defined by

$$domQ = \{x \in X \text{ such that } Q(x) \neq \emptyset\},$$

$$Q^{-1}(y) = \{x \in X \text{ such that } y \in Q(x)\},$$

as well as $gphQ = \{(x, y) \in X \times Y$ such that $y \in Q(x)\}$.

Every norm is indicated with $\|\cdot\|$. The distance from $x$ to $B \subseteq X$ is defined by

$$d(x, B) = inf\ \{\|x - b\| \text{ such that } b \in B\} \text{ for each } x \in X.$$

The definition of the excess between class $A \subseteq$ X as well as class $B$ is

$$e(A, B) = sup\ \{d(x, B) \text{ such that } x \in A\}.$$

**Definition 1: Normed Linear space:** By a normed linear space (briefly normed space) is meant a real or complex vector space $E$ in which every vector $w$ is associated with a real number $|w|$, called its absolute value or norm, in such a manner that

i. $\|w\| \geq 0$, as well as $\|w\| = 0$ iff $w = 0$,
*ii.* $\|w + z\| \leq \|w\| + \|z\|$,
iii. $\|\alpha w\| = |\alpha|\|w\|$, for all $w, z \in W$ as well as $\alpha \in k$.

**Definition 2: Hilbert space:** If an inner product space is complete as a metric space, it is referred to as a Hilbert space. The norm of the inner product induces the metric.

**Definition 3: Inner product space:** An inner product space is a vector space with an inner product defined on it. The inner product is a function that associates a scalar with every pair of vectors.

**Definition 4: Banach space:** A Banach space $(X, ||\ ||)$ is a normed vector space (over the real or complex numbers) that is complete with respect to the metric $d(x, y) = ||x - y||$.

From [22], we collect the definition of metric regularity for set-valued mapping.

**Definition 5: Metrically Regular Mapping:** Consider $Q: X \rightrightarrows 2^Y$ be a set-valued function with $(x_1, y_1) \in gphQ$. Let $r_{x_1} > 0$, $r_{y_1} > 0$ as well as $k > 0$. It is then considered to be metrically regular at $(x_1, y_1)$ on $\mathbb{B}_{r_{x_1}}(x_1) \times \mathbb{B}_{r_{y_1}}(y_1)$ with constant $k$ if

$$d\big(x, Q^{-1}(y)\big) \leq k\ d(y, Q(x)) \text{ for every } x \in \mathbb{B}_{r_{x_1}}(x_1), y \in \mathbb{B}_{r_{y_1}}(y_1). \qquad (3)$$

Also, in [22], we think back to the ideas of Lipchitz-like as well as pseudo-Lipschitz consistency of set-valued functions.

**Definition 6: Lipschitz Like Continuity:** Suppose that $I: Y \rightrightarrows 2^X$ a set-valued function as well as let $(x_1, y_1) \in gph\, I$. Let $r_{x_1} > 0,\ r_{y_1} > 0$ as well as $c > 0$. Then $I$ is known to be

(a) Lipschitz-like on $\mathbb{B}_{r_{y_1}}(y_1) \times \mathbb{B}_{r_{x_1}}(x_1)$ with constant $c$ if the following inequality holds:

$$e\left(I(y') \cap \mathbb{B}_{r_{x_1}}(x_1)\ ,\ I(y'')\right) \leq c\|y' - y''\| \text{ for any } y', y'' \in \mathbb{B}_{r_{y_1}}(y_1). \tag{4}$$

(b) Pseudo-Lipschitz around $(y_1, x_1)$ if there exist constants $r'_{y_1} > 0,\ r'_{x_1} > 0$ as well as $c' > 0$ such that $I$ is Lipschitz-like on $\mathbb{B}_{r'_{y_1}}(y_1) \times \mathbb{B}_{r'_{x_1}}(x_1)$ with constant $c'$.

The connection among the Lipschitz-like continuity of the inverse $Q^{-1}$ at $(y_1, x_1)$ as well as the metric regularity of a function $Q$ at $(x_1, y_1)$ is given by the following result. It is extracted from [23]. We present the proof of the following lemmas, which are a modification of the evidence in [3] as well as are necessary to prove our major findings. Motivated by previous concepts [11], we have reached this conclusion.

**Lemma 1:** Let $Q: X \rightrightarrows 2^Y$ be a set valued function as well as $(x_1, y_1) \in gphQ$. Let $r_{x_1} > 0,\ r_{y_1} > 0$. Then Q is metrically regular at $(x_1, y_1)$ on $\mathbb{B}_{r_{x_1}}(x_1) \times \mathbb{B}_{r_{y_1}}(y_1)$ with constant $c$ iff its inverse $Q^{-1}: Y \rightrightarrows 2^X$ is Lipschitz-like at $(y_1, x_1)$ on $\mathbb{B}_{r_{y_1}}(y_1) \times \mathbb{B}_{r_{x_1}}(x_1)$ with constant $c$, that is, for all $y', y'' \in \mathbb{B}_{r_{y_1}}(y_1)$,

$$e\left(Q^{-1}(y') \cap \mathbb{B}_{r_{x_1}}(x_1),\ Q^{-1}(y'')\right) \leq c\|y' - y''\|. \tag{5}$$

**Proof:** Consider metrically the function $Q$ is regular at $(x_1, y_1)$ on $\mathbb{B}_{r_{x_1}}(x_1) \times \mathbb{B}_{r_{y_1}}(y_1)$ with constant $c$. Let $y', y'' \in \mathbb{B}_{r_{y_1}}(y_1)$. We must demonstrate that (5) holds. In order to demonstrate this, let $x \in Q^{-1}(y') \cap \mathbb{B}_{r_{x_1}}(x_1)$. Given that $Q$ is metrically regular at $(x_1, y_1)$ on $\mathbb{B}_{r_{x_1}}(x_1) \times \mathbb{B}_{r_{y_1}}(y_1)$ with constant $c$, we obtain

$$d\left(x, Q^{-1}(y'')\right) \leq c\, d\left(y'', Q(x)\right) \leq c\|y' - y''\|. \tag{6}$$

By definition of access, we have

$$e(Q^{-1}(y') \cap \mathbb{B}_{r_{x_1}}(x_1), Q(y'')) = \sup\left\{d\left(x, Q^{-1}(y'')\right) : x \in Q^{-1}(y') \cap \mathbb{B}_{r_{x_1}}(x_1)\right\}$$

this, together with (6), gives that $e(Q^{-1}(\mathrm{y}') \cap \mathbb{B}_{\mathrm{r}_{x_1}}(x_1), Q(y'')) \leq c\|\mathrm{y}' - y''\|$.

This suggests that (5) is met. On the other has well as, if (5) is true, we have to demonstrate that $Q$ is metrically regular at $(x_1, y_1)$ on $\mathbb{B}_{\mathrm{r}_{x_1}}(x_1) \times \mathbb{B}_{\mathrm{r}_{y_1}}(y_1)$ with constant $c$. To complete this, let $x \in \mathbb{B}_{\mathrm{r}_{x_1}}(x_1)$ as well as $y'' \in \mathbb{B}_{\mathrm{r}_{y_1}}(y_1)$. Since (5) holds with $y' \in \mathbb{B}_{\mathrm{r}_{y_1}}(y_1)$, then let $y' \in Q(x)$. It follows that $x \in Q^{-1}(y') \cap \mathbb{B}_{\mathrm{r}_{x_1}}(x_1)$. Then by the definition of excess, we can write

$$d\big(x, Q^{-1}(y'')\big) \leq e(Q^{-1}(\mathrm{y}') \cap \mathbb{B}_{\mathrm{r}_{x_1}}(x_1), Q(y'')) \leq c\|\mathrm{y}' - y''\|.$$

The inequality mentioned above, we take the absolute minimum in relation to $\mathrm{y}' \in Q(x)$ on each side, we get

$$d\big(x, Q^{-1}(\mathrm{y}')\big) \leq c \inf\{\|\mathrm{y}' - y''\| : \mathrm{y}' \in Q(x)\} = c\, d(y'', Q(x)),$$

that is, $d\big(x, Q^{-1}(\mathrm{y}')\big) \leq c\, d(y'', Q(x))$, for all $x \in \mathbb{B}_{\mathrm{r}_{x_1}}(x_1)$, $y'' \in \mathbb{B}_{\mathrm{r}_{y_1}}(y_1)$.

This implies that $Q$ is metrically regular at $(x_1, y_1)$ on $\mathbb{B}_{\mathrm{r}_{x_1}}(x_1) \times \mathbb{B}_{\mathrm{r}_{y_1}}(y_1)$ with constant $c$. Therefore, the evidence of Lemma1 is completed.

From [24], for metrically regular function, we recall the following Lyusternik-Graves theorem assertion. This theorem establishes the stability of metric regularity of a generalized equation under perturbations as well as is essential to the study of metric regularity. According to the fundamental estimate, a generalized equation $\bar{y} \in Q(x)$ with solution $x = \bar{x}$ may be disturbed by adding a single-valued function $q$ to $Q$, which is Lipschitz continuous with $q(\bar{x}) = 0$, to get a generalized equation $\bar{y} \in (q + Q)(x)$ with solution $x = \bar{x}$. In support of this assertion, we perform that if there is a $t > 0$ such that the set $D \cap \mathbb{B}_t(z)$ is closed, then the set $D \subseteq X$ is locally closed at $z \in D$.

**Lemma 2:** Assume a function $Q: X \rightrightarrows 2^Y$ as well as any $(\bar{x}, \bar{y}) \in \mathrm{gph}Q$ where $\mathrm{gph}Q$ is locally closed. Consider metrically the function $Q$ is regular at $(\bar{x}, \bar{y})$ with constant $\kappa > 0$. Assume also a Lipschitz continuous function $q : X \to Y$ at $\bar{x}$ with Lipschitzs constant $\xi$ such that $\xi < \kappa^{-1}$. Then, metrically the function $q + Q$ is regular at $(\bar{x}, \bar{y} + q(\bar{x}))$ with constant $\frac{\kappa}{1-\kappa\xi}$.

We bring in the following definition of point-based approximation, whose concept is taken from the book of Dontchev and Rockafellar [3, Ch. 6].

**Definition 7: Point-based Approximation:** Choose a sequence of scalars $\{\xi_\kappa\} \subseteq (0, \xi)$. Let $q$ be a function from a subset $U$ of $X$ to $Y$, which is Lipschitz continuous on $U$ with a Lipschitz constant $\nu$. Suppose $c = \xi + \nu$. Consider a class of functions $A : U \times U \rightarrow Y$ such that for each $x, u \in U$, both of the following assertions hold:

a) $\|q(u) - A(x,u)\| \leq \frac{1}{2}c\|x-u\|^2$,
b) With a Lipschitz constant $2\xi$, the function $A(x, \cdot) - A(u, \cdot)$ is Lipschitz continuous on $U$.

The function $A$ then said to be a point-based approximation (PBA in brief) on $U$ for $q$ with constant $c$. In this case, we say that $q$ has a PBA $A$ on $U$ with constant $c$.

The class of functions admitting a PBA includes both smooth as well as non-smooth functions. For instance, when $q$ is Frechet differentiable in $U$ as well as is also Lipschitz continuous on $U$ with modulus $\upsilon$, then the function $A : U \times U \rightarrow Y$, defined by

$$A(x,u) = q(x) + \xi_\kappa(x-u)$$

is a PBA for $q$ with constant $c$ on $U$. Then the first part of Definition 7 becomes $\|q(u) - q(x) - \xi_\kappa(x-u)\| \leq \frac{1}{2}c\|x-u\|^2$.

Moreover, $\|[A(x,x') - A(u,x')] - [A(x,x'') - A(u,x'')]\|$

$$\leq \|A(x,x') - A(x,x'')\| + \|A(u,x'') - A(u,x')\|$$

$$= \|q(x) + \xi_\kappa(x-x') - q(x) - \xi_\kappa(x-x'')\|$$

$$+ \|q(u) + \xi_\kappa(u-x'') - q(u) - \xi_\kappa(u-x')\|$$

$$= \|\xi_\kappa(x'-x'')\| + \|\xi_\kappa(x'-x'')\| \leq 2\xi\|x'-x''\|.$$

This shows that the second part of Definition 7 is Lipschitz continuous on $U$ with modulus $2\xi$.

Suppose $X$ as well as $Y$ are Banach spaces. Let $q: X \rightarrow Y$ be a single-valued function, which is frechet differentiable on $U \subseteq X$ as well as let $Q: X \rightrightarrows 2^Y$ be a set-valued function with closed graph. Let $r_{\bar{x}} > 0, r_{\bar{y}} > 0, \nu > 0$ as well as $\kappa > 0$ be such that $\nu\kappa < 1$. We define

$$r^* = \max\left\{\frac{2r_{\bar{x}}+\kappa r_{\bar{y}}}{1-\nu\kappa}, \frac{2\nu r_{\bar{x}}+r_{\bar{y}}}{1-\nu\kappa}\right\}. \quad (7)$$

Equation (7) makes it clear that $r_{\bar{x}} < r^*$ as well as $r_{\bar{y}} < r^*$.

The G-PPA's convergence study heavily relies on the following lemma, which has been proved by Alom and Rashid in [21].

**Lemma 3:** Consider a set valued function $Q: X \rightrightarrows 2^Y$ with locally closed graph at $(\bar{x}, \bar{y}) \in gphQ$. Let $r^*$ be ensured by (7). Also, assume with constant κ, metrically the function $Q$ is regular at $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{r^*}(\bar{x})$ relative to $\mathbb{B}_{r^*}(\bar{y})$. Consider with Lipschitz constant $\nu$, the function $q: X \to Y$ is Lipschitz continuous upon $\mathbb{B}_{r^*}(\bar{x})$ with $q(\bar{x})=0$. Then, with constant $\frac{\kappa}{1-\nu\kappa}$, metrically the function $q + Q$ is regular at $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{r^*}(\bar{x}) \times \mathbb{B}_{r^*}(\bar{y})$.

The fixed-point lemma for set-valued functions, which was shown by Dontchev and Hagger in [25], brings this section to a close. To demonstrate the existence of any sequence produced by our algorithm, this lemma is absolutely essential. The Banach fixed point lemma is another name for this lemma.

**Lemma 4:** Suppose $\psi : X \rightrightarrows 2^X$ is a set-valued function. Let $\eta_0 \in X,\ r > 0$ as well as $0 < \gamma < 1$ be such that $d(\eta_0, \psi(\eta_0)) < r(1-\gamma)$ (8)

as well as

$$d(x_1, \psi(x_2)) \le e(\psi(x_1) \cap \mathbb{B}_r(\eta_0), \psi(x_2)) \le \gamma \|x_1 - x_2\| \text{ for any } x_1, x_2 \in \mathbb{B}_r(\eta_0). \quad (9)$$

Then there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \psi(x)$, that means $\psi$ has a constant point which belongs to $\mathbb{B}_r(\eta_0)$. We have a point $x \in \mathbb{B}_r(\eta_0)$ such that $x = \psi(x)$ if $\psi$ is one-valued.

## 3. Convergence Analysis of the G-PPA

We assume that $\{\xi_k\}$ is a series of non-negative scalars $\xi_k$ throughout the section. Suppose that $\{\xi_k\} \subseteq (0, \xi)$ as well as $\xi > 0$. An instance of a non-smooth Lipschitz continuous function with constant $\nu$ is $q : U \subseteq X \to Y$. Moreover, using the constant $c = \xi + \nu > 0$, we observe that $q$ has a point-based approximation $A$ on $U$. Let with constant $\kappa$, a set-valued function $Q: X \rightrightarrows 2^Y$ be metrically regular with a closed graph. Assume that $\nu\kappa < 1$. Let $r_{\bar{x}} > 0,\ r_{\bar{y}} > 0,\ \nu > 0$, as well as $\kappa > 0$.

With constant $\frac{\kappa}{1-\nu\kappa}$, metrically the mapping $(q+Q)$ is regular at $(\bar{x},\bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with closed graph. For each $x \in X$, let a set-valued function $P_x : X \rightrightarrows 2^Y$ given by $P_x(\cdot) = A(x,\cdot) + Q(\cdot)$.

Then $D(x) = \{d \in X : 0 \in P_x(x+d)\} = \{d \in X : x + d \in P_x^{-1}(0)\}$. (10)

But, for all $m \in X$ as well as $y \in Y$, we have $m \in P_x^{-1}(y) \Leftrightarrow y \in A(x,m) + Q(m)$. (11)

Specifically, $\bar{x} \in P_x^{-1}(\bar{y})$ for each $(\bar{x},\bar{y}) \in gph(q+Q)$. (12)

Notice that since on $\mathbb{B}_{r_{\bar{x}}}(0) + \bar{x}$, the function $\xi I(\bar{x} - \cdot)$ is Lipschitz continuous, we deduce by Lemma3 that, metrically the function $P_x$ is regular at $(\bar{x},\bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$. Accordingly, for every $y_1, y_2 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, we obtain the following inequality by Lemma 1,

$$e\left(P_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P_{\bar{x}}^{-1}(y_2)\right) \le \frac{\kappa}{1-\kappa(1+\nu)} \|y_1 - y_2\|. \quad (13)$$

Define $\bar{r} = min\{r_{\bar{y}} - \frac{5cr_{\bar{x}}}{2}, \frac{r_{\bar{x}}(1-\kappa(1+\nu+3\xi))}{4\kappa}\}$. (14)

Therefore $\bar{r} > 0 \Leftrightarrow c < min\{\frac{2r_{\bar{y}}}{5r_{\bar{x}}}, \frac{1-\kappa(1-\nu)}{3\kappa}\}$. (15)

The next lemma is key to the convergence analysis of the G-PPA. The proof is a refinement of the proof of [16].

**Lemma 5:** Let $A$ be a point-based approximation of a function $q$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with aconstant $c$. Let metrically the function $P_{\bar{x}}(\cdot)$ be regular at $(\bar{x},\bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$. For all $\bar{r}$ be defined in (14) so that (15) is hold. Let $x \in \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$. So that we have Lipschitz-like function $P_{\bar{x}}^{-1}$at $(\bar{y},\bar{x})$ upon $\mathbb{B}_{r}(\bar{y}) \times \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa(1+\nu+3\xi)}$, such that

$$e\left(P_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}),\ P_{\bar{x}}^{-1}(y_2)\right) \le \frac{\kappa}{1-\kappa(1+\nu+3\xi)} \|y_1 - y_2\| \text{ for any } y_1, y_2 \in \mathbb{B}_{\bar{r}}(\bar{y}).$$

**Proof:** Suppose $t = \frac{\kappa}{1-\kappa(1+\nu)}$. (16)

From the topology embedded in (15) one can write the expression $\xi + \nu < \frac{1-\kappa(1-\nu)}{3\kappa}$ . This suggests the formula is $3\xi\kappa < 1 - \kappa(1 + 2\nu)$. We therefore use this relation in (16) to get $3\xi\kappa < 1$. Suppose that

$$y_1, y_2 \in \mathbb{B}_{\bar{r}}(\bar{y}) \text{ as well as } x' \in P_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}). \quad (17)$$

We will prove that exists such that $\|x' - x''\| \leq \frac{\kappa}{1-\kappa(1+\nu+3\xi)} \|y_1 - y_2\|$.

To do this, we will show that there is a sequence $\{x_\kappa\} \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that

$$y_2 \in A(x, x_{\kappa-1}) - A(\bar{x}, x_{\kappa-1}) + A(\bar{x}, x_\kappa) + Q(x_\kappa), \quad (18)$$

as well as $\|x - x_{\kappa-1}\| \leq t\|y_1 - y_2\|(2\xi t)^{\kappa-2}$ (19)

hold for every $k = 2,3,4, \dots \dots ..$ We will proceed to use mathematical induction over $k$ to continue. Write $\omega_i = y_i - A(x, x') + A(\bar{x}, x')$ for each $i = 1,2.$ (20)

Using $x' \in \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ from (30) as well as $x \in \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ from the statement of Lemma 5 gives us $\|x - x'\| \leq \|x - \bar{x}\| + \|\bar{x} - x'\| \leq r_{\bar{x}}$. (21)

Together with (17), (21) as well as the first affinity in (14) as well as the definition of point-based approximation, we obtain

$$\begin{aligned}\|\omega_i - \bar{y}\| &\leq \|y_i - \bar{y}\| + \|A(x, x') - A(\bar{x}, x')\| \\ &\leq \bar{r} + \|P(x') - A(x, x')\| + \|P(x') - A(\bar{x}, x')\| \\ &\leq \bar{r} + c(\|x - x'\| + \|\bar{x} - x'\|) \leq \bar{r} + c(r_{\bar{x}} + \tfrac{r_{\bar{x}}}{2}) = \bar{r} + \tfrac{3cr_{\bar{x}}}{2} \leq r_{\bar{y}}.\end{aligned}$$

This shows that $\omega_i \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ for each $i = 1, 2$. Let $x' = x_1$. Then from (17), we have $x \in P_{\bar{x}}^{-1}(y_1)$ as well as (11) implies that $y_1 \in A(x, x_1) + Q(x_1)$.

This can be written as $y_1 - A(x, x_1) + A(\bar{x}, x_1) \in A(\bar{x}, x_1) + Q(x_1)$ (22)

as well as thus $\omega_i \in A(\bar{x}, x_1) + Q(x_1)$ follows from (20) as well as (22). Consequently, by (11) $x_1 \in P_{\bar{x}}^{-1}(\omega_1)$. This together with (17) we get

$$x_1 \in P_{\bar{x}}^{-1}(\omega_1) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}).$$

Now, by the fact that at $(\bar{x},\bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y})$, metrically the function $P_{\bar{x}}$ is regular with constant $\frac{\kappa}{1-\kappa(1+\nu)}$, that is, by Lemma 1, (16) as well as (13) we get $x_2\in P_{\bar{x}}^{-1}(\omega_2)$ where $\|x_2-x_1\|\le t\|\omega_1-\omega_2\|=t\|y_1-y_2\|$ for any $\omega_1,\omega_2\in\mathbb{B}_{r_{\bar{y}}}(\bar{y})$.

Also, from (20) along with $x'=x_1$ we have:

$$x_2\in P_{\bar{x}}^{-1}(\omega_2)=P_{\bar{x}}^{-1}(y_2-A(x,x_1)+A(\bar{x},x_1)).$$

This combined with (11) yields $y_2\in A(x,x_1)-A(\bar{x},x_1)+A(\bar{x},x_2)+Q(x_2)$.

Hence (18) as well as (19) are satisfied with built points $x_1,x_2$. Let that (18) as well as (19) are true for $k=1,2,3,\ldots\ldots n$, with the points built in $x_1,x_2,\ldots\ldots x_n$. The induction argument requires us to construct $x_{n+1}$ such that (18) as well as (19) are also true for $k=n+1$. For this reason, we write

$$\omega_i^n=y_2-A(x,x_{n+i-1})+A(\bar{x},x_{n+i-1})\text{ for every } i=0,1. \quad (23)$$

Applying the induction hypothesis as well as the principle of point-based approximation $A$ of $q$, we have

$$\|\omega_0^n-\omega_1^n\|=\|[A(x,x_{n-1})-A(\bar{x},x_{n-1})]-[A(x,x_n)-A(\bar{x},x_n)]\|$$

$$\le 2\xi\|x_n-x_{n-1}\|\le 2\xi t\|y_1-y_2\|(2\xi t)^{n-2}=\|y_1-y_2\|(2\xi t)^{n-1}. \quad (24)$$

So $2\xi t<1$, applying (16), (17), (19) as well as the second relation from (14), we have

$$\|x_n-\bar{x}\|\le\sum_{\kappa=2}^{n}\|x_\kappa-x_{\kappa-1}\|+\|x_1-\bar{x}\|\le\sum_{\kappa=2}^{n}t\|y_1-y_2\|(2\xi t)^{\kappa-2}+\frac{r_{\bar{x}}}{2}$$

$$\le 2t\bar{r}\sum_{\kappa=2}^{n}(2\xi t)^{\kappa-2}+\frac{r_{\bar{x}}}{2}\le\frac{2t\bar{r}}{1-2\xi t}+\frac{r_{\bar{x}}}{2}\le\frac{2\kappa\bar{r}}{1-\kappa(1+\nu+3\xi)}+\frac{r_{\bar{x}}}{2}\le\frac{r_{\bar{x}}}{2}+\frac{r_{\bar{x}}}{2}=r_{\bar{x}}. \quad (25)$$

It shows that $x_n\in\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Hence

$$\|x_n-x\|\le\|x_n-\bar{x}\|+\|\bar{x}-x\|\le r_{\bar{x}}+\frac{r_{\bar{x}}}{2}=\frac{3}{2}r_{\bar{x}}. \quad (26)$$

We use (17), (23), (26) for each $i=0,1$, as well as the first relation from (14) via the definition of point-based approximation, we have

$$\|\omega_i^n-\bar{y}\|\le\|y_2-\bar{y}\|+\|A(x,x_{n+i-1})-A(\bar{x},x_{n+i-1})\|$$

$$\leq \bar{r} + \|P(x_{n+i-1}) - A(x, x_{n+i-1})\| + \|P(x_{n+i-1}) - A(\bar{x}, x_{n+i-1})\|$$

$$\leq \bar{r} + c(\|x - x_{n+i-1}\| + \|\bar{x} - x_{n+i-1}\|) \leq \bar{r} + c(\tfrac{3}{2} r_{\bar{x}} + \bar{r}) \leq \bar{r} + \tfrac{5cr_{\bar{x}}}{2} \leq r_{\bar{y}}.$$

This means that if $\omega_i^n \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, then for all $i = 0, 1$. Having $k = n$, assumption (31) holds, we get $y_2 \in A(x, x_{n-1}) - A(\bar{x}, x_{n-1}) + A(\bar{x}, x_n) + Q(x_n)$.

It can be written as $y_2 - A(x, x_{n-1}) + A(\bar{x}, x_{n-1}) \in A(\bar{x}, x_n) + Q(x_n)$.

Applying (36) it turns into $\omega_0^n \in A(\bar{x}, x_n) + Q(x_n)$ for $i = 0$. This together with (24) as well as (38), gives $x_n \in P_{\bar{x}}^{-1}(\omega_0^n) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x})$.

We own from (13) that there is an element $x_{n+1} \in P_{\bar{x}}^{-1}(\omega_1^n)$ for each $\omega_0^n, \omega_1^n \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$.

$$\|x_{n+1} - x_n\| \leq t\|\omega_0^n - \omega_1^n\|. \tag{27}$$

We have applied (24) in (27), to obtain

$$\|x_{n+1} - x_n\| \leq t\|y_1 - y_2\|(2\xi t)^{n-1}. \tag{28}$$

Now using (23) for $i = 1$ in $x_{n+1} \in P_{\bar{x}}^{-1}(V_1^n)$, we have

$$x_{n+1} \in P_{\bar{x}}^{-1}(y_2 - A(x, x_n) - A(\bar{x}, x_n)),$$

Which along with (11), gives that

$$y_2 \in A(x, x_n) - A(\bar{x}, x_n) + A(\bar{x}, x_{n+1}) + Q(x_{n+1}). \tag{29}$$

Hence, (18) as well as (19) are valid for all, as (28) as well as (29) show that the induction processes have been completed.

As $3\xi t < 1$ from (19) we then deduce that $\{x_\kappa\}$ is a Cauchy sequence as well as therefore converges to some $x''$ that is $x'' = \lim_{k\to\infty} x_\kappa$. Based on the familiarity of $Q$ as well as the limit to (18) fulfilled $y_2 \in A(x, x'') + Q(x'')$ as well as so $x'' \in P_{\bar{x}}^{-1}(y_2)$. Furthermore, from (16) as well as (19), we have

$$\|x' - x''\| \leq \lim_{n\to\infty} sup \sum_{k=2}^{n} \|x_k - x_{k-1}\| \leq \lim_{n\to\infty} sup \sum_{k=2}^{n} (2\xi t)^{n-1} \|y_1 - y_2\|$$

$$\leq \frac{t}{1-2\xi t}\|y_1 - y_2\| = \frac{\kappa}{1-\kappa(1+\nu+3\xi)}\|y_1 - y_2\|.$$

Thus, we have finished the proof of Lemma 5.

To ease our work, we call the map $N_x : X \to Y$ by

$$N_x(\cdot) = A(\bar{x}, \cdot) - A(x, \cdot) \text{ for each } x \in X \tag{30}$$

as well as the set-valued map $\psi_x : X \rightrightarrows 2^X$ by $\psi_x(\cdot) = P_{\bar{x}}^{-1}[N_x(\cdot)]$ (31)

then we have that

$$\|N_x(x') - N_x(x'')\| = \|[A(\bar{x}, x') - A(x, x')] - [A(\bar{x}, x'') - A(x, x'')]\|$$

$$\leq 2\xi\|x' - x''\| \text{ for any } x', x'' \in X. \tag{32}$$

We now present our main theorem, which uses the G-PPA for the generalized non-smooth equation (1) to guarantee the semi-local convergence of any generated sequence based on our beginning point $x_0$.

**Theorem 1:** Suppose $\eta > 1$. Let $q$ be a function that has a point-based approximation $A$ upon $B_{r_{\bar{x}}}(\bar{x})$ with constant $c$. Let at $(\bar{x}, \bar{y})$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$, the map $P_{\bar{x}}(\cdot)$ be metrically regular upon $B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y})$. Let $\bar{r}$ be defined in (14), so that (15) holds. Let $0 < \delta \leq 1$ be such that

(a) $\delta \leq min\left\{\frac{r_{\bar{x}}}{2}, \frac{r_{\bar{y}}}{7c}, \frac{2\bar{r}}{c}\right\}$, (b) $\kappa c(\eta + 5) + \kappa(1 + \nu + 3\xi) \leq 1$, (c) $\|\bar{y}\| < \frac{c\delta^2}{2}$.

Assume $\lim_{x \to \bar{x}} d\big(\bar{y}, A(x, x) + Q(x)\big) = 0.$ (33)

Then, every sequence $\{x_n\}$ created by algorithm 2 starting from $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ for some $\hat{\delta} > 0$, converges linearly to a solution $x^*$ of the generalized non-smooth equation (1), i.e., $x^*$ fulfilled $0 \in q(x^*) + Q((x^*)$.

**Proof:** Under assumption (b), we can write

$$\kappa\xi(\eta + 5) + \kappa(1 + \nu) \leq \kappa c(\eta + 5) + \kappa(1 + \nu) \leq \kappa c(\eta + 5) + \kappa(1 + \nu + 3\xi) \leq 1.$$

The above relationship gives us $\kappa\xi \leq \frac{1-\kappa(1+\nu)}{\eta+5}$ (34)

as well as $\kappa c \leq \frac{1-\kappa(1+\nu)}{\eta+5}$. (35)

Let $z = \frac{\kappa}{2(1-\kappa(1+\nu+3\xi))}$. (36)

Utilizing assumptions (b) in (36) using $\eta > 1$, we obtain

$$\eta zc = \frac{\eta\kappa c}{2(1-\kappa(1+\nu+3\xi))} \leq \frac{\eta}{2(\eta+5)} < 1.$$

It's enough to demonstrate that Algorithm 2 outputs at least one sequence $\{x_n\}$, as well as also that any sequence $\{x_n\}$ generated by Algorithm 2 satisfies the claims below:

For each $n = 0,1,2, \ldots, \|x_n - \bar{x}\| \leq 2\delta$ (37)

as well as $\|x_{n+1} - x_n\| \leq (\eta zc)^{n+1}\delta.$ (38)

We will proceed with using of induction method. For this purpose, we clarify for every $x \in X$, $r_x = \frac{12\kappa}{5(1-\kappa(1+\nu))}(\|\bar{y}\| + \frac{c}{2}\|x - \bar{x}\|^2)$. (39)

By (35) as well as assumption (c) with $\eta > 1$, from (39), we have for every $x \in \mathbb{B}_{2\delta}(\bar{x})$,

$$r_x \leq \frac{12\kappa}{5(1-\kappa(1+\nu))} \cdot \frac{5c\delta^2}{2} \leq \frac{6(\frac{1-\kappa(1+\nu)}{\eta+5})}{1-\kappa(1+\nu)}\delta^2 \leq \frac{6}{\eta+5}\delta \text{ as } \delta^2 \leq \delta$$

$$\leq \delta < 2\delta. \quad (40)$$

Take $0 < \hat{\delta} \leq \delta$ such that

$$d(0, A(x_0, x_0) + Q(x_0)) \leq \frac{c\delta^2}{2} \text{ for each } x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}). \quad (41)$$

From assumption (c), as well as equation (38), there is no evidence that $\hat{\delta}$ exist. We can prove (37) is true for $n = 0$. To show that (38) is also valid for $n = 0$, we require to check either that $x_1$ exists or that $D(x_0) \neq \emptyset$. We will prove this by using Lemma 4 for the map $\psi_{x_0}$ with $\eta_0 = \bar{x}$, $r = r_{x_0}$ as well as $\gamma = \frac{7}{12}$. As we can see from (12) that $\bar{x} \in P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{x_0}}(\bar{x})$. The definition of excess e with the function $\psi_{x_0}$ as well as the relation $\mathbb{B}_{r_{x_0}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ give the following results:

$$d\left(\bar{x}, \psi_{x_0}(\bar{x})\right) \leq e\left(P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{x_0}}(\bar{x}), \psi_{x_0}(\bar{x})\right) \leq e(P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{2\delta}(\bar{x}), P_{\bar{x}}^{-1}[N_{x_0}(\bar{x})])$$

$$\leq e(P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P_{\bar{x}}^{-1}[N_{x_0}(\bar{x})]). \quad (42)$$

Utilizing the definition of point-based approximation $A$ of $q$ with constant $c$, we obtain $\|N_{x_0}(x) - \bar{y}\| = \|A(\bar{x}, x) - A(x_0, x) - \bar{y}\| \leq \|A(\bar{x}, x) - A(x_0, x)\| + \|\bar{y}\|$

$$\leq \|q(x) - A(\bar{x}, x)\| + \|q(x) - A(x_0, x)\| + \|\bar{y}\|$$

$$\leq \frac{c}{2}(\|\bar{x} - x\|^2 + \|x_0 - x\|^2) + \|\bar{y}\|. \tag{43}$$

As $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x})$ as well as $x \in \mathbb{B}_{2\delta}(\bar{x})$, so by the link $7c\delta \leq r_{\bar{y}}$ from assumption (a) as well as assumption (c), we have in (43), we get

$$\|N_{x_0}(x) - \bar{y}\| \leq \frac{c}{2}(\|\bar{x} - x\|^2 + (\|x_0 - \bar{x}\| + \|\bar{x} - x\|)^2) + \|\bar{y}\|$$

$$\leq \frac{c}{2}(4\delta^2 + 9\delta^2) + \frac{c\delta^2}{2} \leq 7c\delta^2 \leq 7c\delta \text{ as } \delta^2 \leq \delta \text{ for } 0 < \delta \leq 1$$

$$\leq r_{\bar{y}}. \tag{44}$$

This gives $N_{x_0}(x) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. (45)

In particular, for $x = \bar{x}$, we obtain from (43) by utilizing inequality $7c\delta \leq r_{\bar{y}}$ derived from assumption (a) that $\|N_{x_0}(\bar{x}) - \bar{y}\| \leq \frac{c}{2}\|x_0 - \bar{x}\|^2 + \|\bar{y}\|$ (46)

$$\leq \frac{c\delta^2}{2} + \frac{c\delta^2}{2} = c\delta^2 \leq c\delta \leq r_{\bar{y}}.$$

This shows that $N_{x_0}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$.

By applying (13), (43) as well as (46), we have from (39) that

$$d(\bar{x}, \psi_{x_0}(\bar{x})) \leq \frac{\kappa}{1-\kappa(1+\nu)}\|\bar{y} - N_{x_0}(\bar{x})\| \leq \frac{\kappa}{1-\kappa(1+\nu)}(\frac{c}{2}\|x_0 - \bar{x}\|^2 + \|\bar{y}\|)$$

$$= \left(1 - \frac{7}{12}\right) r_{x_0} = (1 - \gamma)r.$$

Thus, the first part of Lemma 4 is proved.

Now, we show that the second part of Lemma 4 is true also. To this let $x', x'' \in \mathbb{B}_{r_{x_0}}(\bar{x})$. Now by using $r_{x_0} \leq 2\delta$ from (53) as well as $2\delta \leq r_{\bar{x}}$ from assumption (a), we conclude that $x', x'' \in \mathbb{B}_{r_{x_0}} \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$, as well as by applying (58) we get $N_{x_0}(x'), N_{x_0}(x'') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. As metrically the function $P_{\bar{x}}$ is regular at $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$

relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$, then by Lemma 1, the inequality (13) becomes

$$d\left(x', \psi_{x_0}(x'')\right) \leq e\left(\psi_{x_0}(x') \cap \mathbb{B}_{r_{x_0}}(\bar{x}), \psi_{x_0}(x'')\right) \leq e(\psi_{x_0}(x') \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \psi_{x_0}(x''))$$

$$= e(P_{\bar{x}}^{-1}[N_{x_0}(x')] \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P_{\bar{x}}^{-1}[N_{x_0}(x'')]) \leq \frac{\kappa}{1-\kappa(1+\nu)} \left\|N_{x_0}(x') - N_{x_0}(x'')\right\|. \quad (47)$$

Utilizing (42), (44) with $\eta > 1$ in (47) gives us

$$d\left(x', \psi_{x_0}(x'')\right) \leq e\left(\psi_{x_0}(x') \cap \mathbb{B}_{r_{x_0}}(\bar{x}),\ \psi_{x_0}(x'')\right) \leq \frac{2\xi\kappa}{1-\kappa(1+\nu)} \|x' - x''\|$$

$$\leq \frac{2}{\eta+5} \|x' - x''\| \leq \frac{7}{12} \|x' - x''\| = \gamma \|x' - x''\|.$$

Thus, the second part of Lemma 4 is satisfied. Then we conclude that $\hat{x}_1 \in \mathbb{B}_{r_{x_0}}(\bar{x})$ such that $\hat{x}_1 \in \psi_{x_0}(\hat{x}_1)$, that is, $0 \in A(x_0, \hat{x}_1) + Q(\hat{x}_1)$ as well as therefore $\hat{x}_1 - x_0 \in D(x_0)$. From this it follows that $D(x_0) \neq \emptyset$. Thus $d_0 \in D(x_0)$ can be choose such that

$$\|d_0\| \leq \eta\ d(0, D(x_0)).$$

According to the definition of $D(x_0)$ with respect to the relation $x_1 = x_0 + d_0$ we find from Algorithm 2, we write $D(x_0) = \{d_0 \in X : 0 \in A(x_0,\ x_0 + d_0) + Q(x_0 + d_0)\}$

$$= \{d_0 \in X :\ x_0 + d_0 \in P_{x_0}^{-1}(0)\}. \quad (48)$$

Therefore $d\big(0, D(x_0)\big) = d(x_0, P_{x_0}^{-1}(0))$. (49)

The second step is to verify that (38) holds for $n = 0$. Now $\bar{r} > 0$ by assumption (a). In this case, (14) satisfied (15). The map $P_{\bar{x}}(\cdot)$, according to Lemma 5, is Lipschitz-like at $(\bar{y}, \bar{x})$ upon $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa(1+\nu+3\xi)}$, as metrically the function $P_x(\cdot)$ is regular at $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$. From the first part of relation in (a) as well as first choice of $\hat{\delta}$, we then arrive at $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x}) \subseteq \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$. Thus, we conclude that $P_{x_0}^{-1}(\cdot)$ is Lipschitz-like at $(\bar{y}, \bar{x})$ upon $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa(1+\nu+3\xi)}$. Again, by the relation $\frac{c\delta}{2} \leq \bar{r}$ in assumption (a), we can apply assumption (c) to deduce that $\|\bar{y}\| < \frac{c\delta^2}{2} \leq \frac{c\delta}{2} \leq \bar{r}$.

As a result, $0 \in \mathbb{B}_{\bar{r}}(\bar{y})$. At $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$, metrically the function $P_{x_0}(\cdot)$ is regular with constant $\frac{\kappa}{1-\kappa(1+\nu+3\xi)}$ by applying Lemma 1. Consequently, we have

$$d\left(x_0,\ P_{x_0}^{-1}(0)\right) \leq \frac{\kappa}{1-\kappa(1+\nu+3\xi)} d(0, P_{x_0}(x_0)) \tag{50}$$

In addition, (54) implies that $d\left(0, P_{x_0}(x_0)\right) = d\big(0, A(x_0, x_0) + Q(x_0)\big) \leq \frac{c\delta^2}{2}$.

The use of Algorithm 2 along with (49) as well as (50), we get

$$\|x_1 - x_0\| = \|d_0\| \leq \eta\, d(0, D(x_0)) = \eta\, d(x_0, P_{x_0}^{-1}(0)) \leq \frac{\eta\kappa}{1-\kappa(1+\nu+3\xi)} d(0, P_{x_0}(x_0))$$

$$\leq \frac{\eta\kappa c\delta^2}{2(1-\kappa(1+\nu+3\xi))} \leq \frac{\eta\kappa c\delta}{2(1-\kappa(1+\nu+3\xi))}. \tag{51}$$

Thus, from (51) by applying (36), we obtain $\|x_1 - x_0\| \leq (\eta z c)\delta$.

This proves (38) for $n = 0$. Take the points $x_1, x_2, \dots \dots .., x_k$ for $n = 0,1,2, \dots \dots .., k-1$ such that (37) as well as (51) hold. Thus, $\eta z c \leq 1$ from (37) gives the following inequality for

$$\|x_\kappa - \bar{x}\| \leq \sum_{i=0}^{k-1} \|x_{i+1} - x_i\| + \|x_0 - \bar{x}\| \leq \delta \sum_{i=0}^{k-1} (\eta z c)^{i+1} + \delta \leq \frac{\eta z c}{1-\eta z c}\delta + \delta \leq 2\delta.$$

It becomes $x_k \in \mathbb{B}_{2\delta}(\bar{x})$ as well as so (37) is satisfied for $n = k$.

We therefore need to show that there exists $x_{k+1}$ so that (38) holds for $n = k$. First, we will show that $D(x_k) \neq \emptyset$. We will use Lemma 4 on the map $\psi_{x_\kappa}$ with $\eta_0 = \bar{x}$ , $r = r_{x_k}$ as well as $\gamma = \frac{7}{12}$ to show that $D(x_k) \neq \emptyset$. It suffices to demonstrate that Lemma 4's assumptions (40) as well as (41) are met, the map $\psi_{x_\kappa}$ with $\eta_0 = \bar{x}$ , $r = r_{x_k}$ as well as $\gamma = \frac{7}{12}$. Now $\bar{x} \in P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{x_k}}(\bar{x})$ is clear from (12). Given excess $e$, the function $\psi_{x_0}$ as well as the relation $\mathbb{B}_{r_{x_k}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ satisfy the following:

$$d(\bar{x}, \psi_{x_k}(\bar{x})) \leq e(P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{x_k}}(\bar{x}), \psi_{x_k}(\bar{x})) \leq e(P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{2\delta}(\bar{x}), P_{\bar{x}}^{-1}\big[N_{x_k}(\bar{x})\big])$$

$$\leq e(P_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P_{\bar{x}}^{-1}\big[N_{x_k}(\bar{x})\big]). \tag{52}$$

Applying the definition of point-based approximation $A$ of $q$ with constant $c$ as well as making use of the relation $7c\delta \leq r_{\bar{y}}$ from assumption (a) as well as assumption (c), we have $\|N_{x_k}(\bar{x}) - \bar{y}\| = \|A(\bar{x}, \bar{x}) - A(x_k, \bar{x}) - \bar{y}\| \leq \|A(\bar{x}, \bar{x}) - A(x_k, \bar{x})\| + \|\bar{y}\|$

$$= \|q(\bar{x}) - A(x_k, \bar{x})\| + \|\bar{y}\| \leq \frac{c}{2}\|x_k - \bar{x}\|^2 + \|\bar{y}\|$$

$$\leq 2c\delta^2 + \frac{c\delta^2}{2} = \frac{5c\delta^2}{2} \leq 7c\delta \leq r_{\bar{y}}. \tag{53}$$

This becomes $N_{x_k}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$.

Consequently, by the presumed Lipschitz-like property of $P_{\bar{x}}^{-1}(\cdot)$ with (53), (52) as well as (39), we get

$$\mathrm{d}(\bar{x}, \psi_{x_k}(\bar{x})) \leq \frac{\kappa}{1-\kappa(1+\nu)}\|\bar{y} - N_{x_k}(\bar{x})\| \leq \frac{\kappa}{1-\kappa(1+\nu)}(\frac{c}{2}\|x_\kappa - \bar{x}\|^2 + \|\bar{y}\|)$$

$$= \left(1 - \frac{7}{12}\right) r_{x_k} = (1-\gamma)r.$$

As a result, property (27) of Lemma 4 is valid. We are now tasked with showing that property (28) of Lemma 4 is valid as well. This can be accomplished by allowing $x', x'' \in \mathbb{B}_{r_{x_k}}(\bar{x})$ . Given (40) as well as the relationship $2\delta \leq r_{\bar{x}}$ mentioned in assumption (a) we have $x', x'' \in \mathbb{B}_{r_{x_\kappa}}(\bar{x}) \subseteq \mathbb{B}_{2\delta} \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$. By applying the point-based approximation definition as well as the relationship of $7c\delta \leq r_{\bar{y}}$ from assumption (a), as well as the relation $\|\bar{y}\| < \frac{c\delta^2}{2}$ from assumption (c), we get

$$\|N_{x_k}(x') - \bar{y}\| \leq \|A(\bar{x}, x') - A(x_k, x')\| + \|\bar{y}\|$$

$$\leq \|q(x') - A(\bar{x}, x')\| + \|q(x') - A(x_k, x')\| + \|\bar{y}\|$$

$$\leq \frac{c}{2}(\|\bar{x} - x'\|^2 + \|x_k - x'\|^2) + \|\bar{y}\|$$

$$\leq \frac{c}{2}(\|\bar{x} - x'\|^2 + (\|x_k - \bar{x}\| + \|\bar{x} - x'\|)^2) + \|\bar{y}\|$$

$$\leq \frac{c}{2}(4\delta^2 + 9\delta^2) + \frac{c\delta^2}{2} \leq 7c\delta^2 \leq 7c\delta \text{ as } \delta^2 \leq \delta \text{ for } 0 < \delta \leq 1$$

$$\leq r_{\bar{y}}.$$

This turns into $N_{x_k}(x') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Identically, we can justify that $N_{x_k}(x'') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Again, at $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$, metrically the function $P_{\bar{x}}$ is regular with constant $\frac{\kappa}{1-\kappa(1+\nu)}$. So, by Lemma 1 as well as making use of (42) as well as (44), we have from the inequality (13)

$$d\left(x', \psi_{x_k}(x'')\right) \leq e\left(\psi_{x_k}(x') \cap \mathbb{B}_{r_{x_k}}(\bar{x}),\ \psi_{x_k}(x'')\right) \leq e(\psi_{x_k}(x') \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}),\ \psi_{x_k}(x''))$$
$$= e(P_{\bar{x}}^{-1}[N_{x_k}(x')] \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}),\ P_{\bar{x}}^{-1}[N_{x_k}(x'')]) \leq \frac{\kappa}{1-\kappa(1+\nu)} \|N_{x_k}(x') - N_{x_k}(x'')\|$$
$$\leq \frac{2\xi\kappa}{1-\kappa(1+\nu)} \|x' - x''\| \leq \frac{7}{12} \|x' - x''\| = \gamma \|x' - x''\|.$$

Thus condition (8) of lemma 4 is also verified. After that, we determine that there is a fixed point $\hat{x}_{k+1} \in \mathbb{B}_{r_{x_k}}(\bar{x})$ such that $\hat{x}_{k+1} \in \psi_{x_k}(\hat{x}_{k+1})$ that is, $0 \in A(x_k, \hat{x}_{k+1}) + Q(\hat{x}_{k+1})$ as well as thus $\hat{x}_{k+1} - x_k \in D(x_k)$. That means $D(x_k) \neq \emptyset$ as well as therefore we can choose $D(x_k) \neq \emptyset$ so that $\|d_k\| \leq \eta\, d(0, D(x_k))$.

Write $x_{k+1} = x_k + d_k$. All of $x_{k+1}$ is produced as per Algorithm 2. Furthermore, by the definition of $D(x_k)$, we have

$$D(x_k) = \{d_k \in X : 0 \in A(x_k, x_k + d_k) + Q(x_k + d_k)\}$$
$$= \{d_k \in X :\ x_k + d_k \in P_{x_k}^{-1}(0)\}, \tag{54}$$

as well as so $d\big(0, D(x_k)\big) = d(x_k,\ P_{x_k}^{-1}(0)).$ (55)

Since $x_k \in \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$, we can then conclude that at $(\bar{y}, \bar{x})$ upon $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$, the map $P_{x_k}^{-1}(\cdot)$ is Lipschitz-like with constant $\frac{\kappa}{1-\kappa(1+\nu+3\xi)}$. Hence, with constant $\frac{\kappa}{1-\kappa(1+\nu+3\xi)}$, metrically the function $P_{x_k}(\cdot)$ is regular at $(\bar{x}, \bar{y})$ upon $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ relative to $\mathbb{B}_{\bar{r}}(\bar{y})$ by applying Lemma1. Thus, we have

$$d(x_k, P_{x_k}^{-1}(0)) \leq \frac{\kappa}{1-\kappa(1+\nu+3\xi)} d(0, P_{x_k}(x_k)). \tag{56}$$

Applying Algorithm 2 with (55) as well as (56), yields

$$\|x_{k+1} - x_k\| = \|d_k\| \leq \eta\, d(0, D(x_k)) = \eta\, d(x_k, P_{x_k}^{-1}(0))$$

$$\le \frac{\eta\kappa}{1-\kappa(1+\nu+3\xi)} d(0, P_{x_k}(x_k)) = \frac{\eta\kappa}{1-\kappa(1+\nu+3\xi)} d(0, A(x_k, x_k) + Q(x_k)).$$

In addition, $A(x_{k-1}, x_k) \in Q(x_k)$ as well as using the definition of point-based approximation of $q$ with constant $c$, we arrive at

$$\|x_{k+1} - x_k\| \le \frac{\eta\kappa}{1-\kappa(1+\nu+3\xi)} \|A(x_k, x_k) - A(x_{k-1}, x_k)\|$$

$$= \frac{\eta\kappa}{1-\kappa(1+\nu+3\xi)} \|q(x_k) - A(x_{k-1}, x_k)\| \le \frac{c\eta\kappa}{2(1-\kappa(1+\nu+3\xi))} \|x_k - x_{k-1}\|^2. \quad (57)$$

According to the inductive hypothesis (38) holds for $n = k-1$. Then using (36) into the previous inequality, we have that

$$\|x_{k+1} - x_k\| \le (\eta zc)(\eta zc)^k \delta^2 \le (\eta zc)^{k+1} \delta. \quad (58)$$

It shows that (38) holds true for $n = k$. For all $n = 0, 1, 2, \dots ..$ (37) as well as (38) are satisfied. Thus, all induction procedures are completed. Hence, it proves that the sequence $\{x_n\}$ generated by Algorithm 2 exists. We find $\{x_n\}$ to be a Cauchy sequence according to (38) as well as it will be converging to some $x^*$. By passing to the limit in $0 \in P_{x_n}(x_{n+1})$, one can conclude that $x^*$ is a solution of (1) since the graph of $Q$ is closed. Thus, the proof of Theorem 1 is completed.

Now, if $\bar{x}$ is a answer to (1), which is, $\bar{y} = 0$, Theorem 1 gives the local convergence result of the GPPA for finding a result to the non-smooth generalized equation (1).

**Corollary 1:** Consider $\eta > 1$ as well as $\bar{x}$ to satisfy (1). Let $\hat{r}_{\bar{x}} > 0$ be such that the function $q$ admits a point-based approximation $A$ on $\mathbb{B}_{\hat{r}_{\bar{x}}}(\bar{x})$ with constant $c$. Consider metrically the map $P_{\bar{x}}(\cdot)$ is regular at $(\bar{x}, 0)$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$. Think about that

$$\lim_{x \to \bar{x}} d\big(0, A(x, x) + Q(x)\big) = 0. \quad (59)$$

Then, every sequence $\{x_n\}$ created by algorithm 2 starting from $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ for some $\hat{\delta} > o$, converges linearly to a solution $x^*$ of the generalized non-smooth equation (1), i.e., $x^*$ fulfilled $0 \in q(x^*) + Q(x^*)$.

**Proof:** According to the assumption, metrically the map $P_{\bar{x}}(\cdot)$ is regular at $(\bar{x}, 0)$. Therefore $P_{\bar{x}}(\cdot)$ is metrically regular at $(\bar{x}, 0)$ upon $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_0}(0)$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$ as there exist constants $r_{\bar{x}} > 0,\ r_0 > 0$ as well as $\frac{\kappa}{1-\kappa(1+\nu)} > 0$, that is

$$d(x,\ P_{\bar{x}}^{-1}(y)) \leq \frac{\kappa}{1-\kappa(1+\nu)} d(y,\ P_{\bar{x}}(x)) \text{ for each } x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x}),\ y \in \mathbb{B}_{r_0}(0).$$

Let $0 < r_1 \leq \frac{r_{\bar{x}}}{2},\ 0 < r_2 \leq \frac{r_0}{2}$ as well as $c \in (0,1)$ be such that $1 - \kappa(1 + \nu + 3\xi) > 0$ as well as $2r_2 - 5cr_1 > 0$. Then, define $\bar{r} = \min\{r_2 - \frac{5cr_1}{2},\ \frac{r_1(1-\kappa(1+\nu+3\xi))}{4\kappa}\} > 0.$ (60)

Let $0 < \delta \leq 1$ be such that $\delta \leq min\left\{\frac{r_{\bar{x}}}{2}, \frac{r_{\bar{y}}}{7c}, \frac{2\bar{r}}{c}\right\}$ as well as

$$\kappa c(\eta + 5) + \kappa(1 + \nu + 3\xi) \leq 1.$$

For $\lim_{x\to\bar{x}} d\big(0, A(x,x) + Q(x)\big) = 0$, there exists $0 < \hat{\delta} \leq \delta$ such that $\hat{x} \in \mathbb{B}_{\hat{\delta}}(\bar{x})$, there exists $\hat{y} \in P_{\hat{x}}(\hat{x})$ satisfying $\|\hat{y}\| \leq \frac{c\delta^2}{2}$. Thus $\mathbb{B}_{r_1}(\hat{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ as well as $\mathbb{B}_{r_2}(\hat{y}) \subseteq \mathbb{B}_{r_0}(0)$. We can see by (58) that metrically the map $P_{\bar{x}}(\cdot)$ is regular at $(\hat{x}, \hat{y})$ upon $\mathbb{B}_{r_1}(\hat{x}) \times \mathbb{B}_{r_2}(\hat{y})$ with constant $\frac{\kappa}{1-\kappa(1+\nu)}$). Verifying that all of the assumptions of Theorem1 are satisfied is now the standard exercise. As for finishing the proof of the corollary1, here we need the Theorem1.

## 4. Numerical Test

In order to illustrate the theoretical result (semi-local convergence) of the Gauss-type proximal point method, we consider the following example in one dimension.

**Example 1:** Assume $X = Y = \mathbb{R},\ x_0 = -0.1,\ \xi = 0.1,\ \kappa = 0.2, \nu = 0.4, c = \xi + \nu = 0.5 \in (0, 1)$ as well as $\eta = 2$. Let $q: \mathbb{R} \to \mathbb{R}$ as well as $Q: \mathbb{R} \rightrightarrows \mathbb{R}$ be described by

$$q(x) = \begin{cases} 2x + 3, & if\ x < 0, \\ -2x + 3, & if\ x \geq 0, \end{cases} \quad \text{as well as} \quad Q(x) = \{3x - 2, 8x\}.$$

Then, if $x < 0$, Algorithm 2 produces a series with initial point $x_0 = -0.1$ that converges to $x^* = -0.2$ as well as $x^* = -0.3$ respectively. However, when $x > 0$, Algorithm 2 produces a linear convergent sequence with initial point $x_0 = 0.9$ that converges to $x^* = -1$ as well as $x^* = -0.5$ respectively.

**Solution:** It is evident that $q$ is not differentiable at $x = 0$ as well as hence $q$ is non-smooth function on $\mathbb{R}$. But this function is differentiable on $\mathbb{R} - \{0\}$. We note that

$$(q+Q)(x) = \begin{cases} \{5x+1, 10x+3\}, \ if\ x < 0, \\ \{x+1, 6x+3\}, \ if\ x \geq 0. \end{cases}$$

First, we consider the set-valued mapping $(q+Q)(x) = 5x+1$ for the case $x < 0$. Then the map $q+Q$ is metrically regular at $(-0.1, 0.5) \in gph(q+Q)$. Take into account that $\sup_k \xi_k = \xi = 0.1$. Then from (09), we have that

$$D(\xi_\kappa, x_k) = \{d_k \in \mathbb{R}\ 0 \in A(x_k, x_k+d_k) + Q(x_k+d_k)\}$$
$$=\{d_k \in \mathbb{R} : 0 \in \xi_\kappa d_k + q(x_k+d_k) + Q(x_k+d_k)\}$$
$$= \left\{d_k \in \mathbb{R} :\ d_k = \frac{-10(1+5x_k)}{51}\right\}.$$

However, if $D(\xi_\kappa, x_k) \neq \emptyset$, we get that

$$0 \in \xi_\kappa(x_{k+1} - x_k) + q(x_{k+1}) + Q(x_{k+1}).$$

This becomes $x_{k+1} = \frac{x_k - 10}{51}$. Thus, from (57) as well as (58), we infer that

$$||d_k|| \leq \left(\frac{c\eta\kappa}{2(1-\kappa(1+\nu+3\xi))}\right)^{k+1} ||d_{k-1}||, \text{ where } ||d_k|| = ||x_{k+1} - x_k||.$$

We find that $\frac{c\eta\kappa}{2(1-\kappa(1+\nu+3\xi))} < 1$, for the given values of $c, \eta, \kappa, \nu$ as well as $\xi$, therefore $\left(\frac{c\eta\kappa}{2(1-\kappa(1+\nu+3\xi))}\right)^{k+1} < 1$ for any values of $k$. This demonstrates that the sequence produced by Algorithm 2 meets linearly, supporting the algorithm's semi-local convergence conclusion. The numerical outcomes for the other scenarios can be displayed in a similar manner. This suggests that the solutions to the generalized equation $0 \in q(x) + Q(x)$ are $x^* = -0.2$ as well as $x^* = -0.3$ for $x < 0$ as well as $x^* = -1$ as well as $x^* = -0.5$ for $x > 0$. Tables 1 as well as 2 display the numerical results that were produced using Matlab.

**Table 1:** Numerical results for the case $x < 0$

| Iterations | $(q+Q)(x) = 5x+1$ | | $(q+Q)(x) = 10x+3$ | |
|---|---|---|---|---|
| | $x_k$ | $(q+Q)(x_k)$ | $x_k$ | $(q+Q)(x_k)$ |

| | | | | |
|---|---|---|---|---|
| 1 | -0.1000 | 0.5000 | -0.1000 | 2.0000 |
| 2 | -0.1980 | 0.0098 | -0.2980 | 0.0198 |
| 3 | -0.2000 | 0.0002 | -0.3000 | 0.0002 |
| 4 | -0.2000 | 0.0000 | -0.3000 | 0.0000 |
| 5 | -0.2000 | 0.0000 | -0.3000 | 0.0000 |

**Table 2:** Numerical results for the case $x > 0$

| Iterations | $(q+Q)(x) = x+1$ | | $(q+Q)(x) = 6x+3$ | |
|---|---|---|---|---|
| | $x_k$ | $(q+Q)(x_k)$ | $x_k$ | $(q+Q)(x_k)$ |
| 1 | 0.9000 | 1.9000 | 0.9000 | 8.4000 |
| 2 | -0.8273 | 0.1727 | -0.4770 | 0.1377 |
| 3 | -0.9843 | 0.0157 | -0.4996 | 0.0023 |
| 4 | -0.9986 | 0.0014 | -0.5000 | 0.0000 |
| 5 | -0.9999 | 0.0001 | -0.5000 | 0.0000 |
| 6 | -1.0000 | 0.0000 | ……. | ……. |

The following figures illustrate $(q+Q)(x)$ graphically:

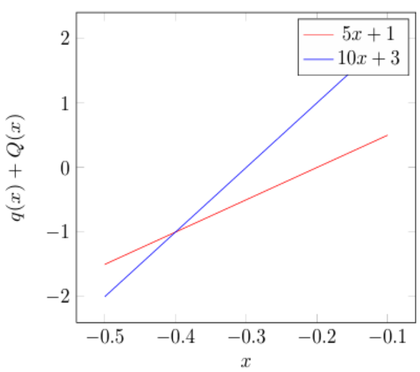


Figure 1

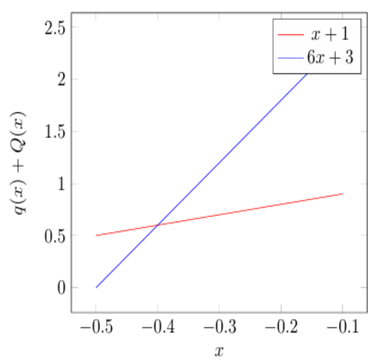


Figure 2

## 5. Concluding Remarks

We have shown both local as well as semi-local convergence findings of the GPPA for solving the non-smooth generalized equation (1) for $\eta > 1$. If $D(\xi_k, x_k)$ is singleton and $\eta = 1$, the results found in [3, Ch.6] and the results presented in this thesis coexist. This outcome expands upon and enhances the outcome of Dontchev and Rockafellar in [3, Ch.6].

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