Polynomially stable of a thermoelastic Timoshenko system with Cattaneo heat conduction law

Abstract: This paper investigates the polynomial stability of a thermoelastic Timoshenko system with Cattaneo's heat conduction law. The system consists of coupled hyperbolic-parabolic equations governing the transverse displacement, rotation angle, temperature, and heat flux. Previous work established the lack of exponential stability regardless of the equal wave speeds (EWS) condition. It is proved in this paper that when the EWS condition is satisfied, the associated C_0-semigroup exhibits polynomial stability. Specifically, it is demonstrated that solutions decay at a rate of $t^{-1/4}$ as $t \to \infty$, with the decay rate uniform for initial data in the domain of the generator. The analysis employs energy methods combined with semigroup theory, leveraging the structural properties induced by the EWS condition to establish polynomial decay estimates. This result extends previous stability analyses and highlights the critical role of wave speed matching in stabilizing Timoshenko systems with second-sound thermal effects.

Keywords: Timoshenko system, Cattaneo heat conduction, polynomial stability

1 Introduction

The following thermoelastic Timoshenko system with Cattaneo heat conduction law is investigated:

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \gamma \theta_x = 0, & x \in (0, L),\ t > 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \gamma \theta = 0, & x \in (0, L),\ t > 0, \\ \rho_3 \theta_t + q_x + \gamma(\varphi_x + \psi)_t = 0, & x \in (0, L),\ t > 0, \\ \tau q_t + \beta q + \theta_x = 0, & x \in (0, L),\ t > 0, \end{cases} \tag{1.1}$$

subject to the boundary conditions

$$\varphi_x(0, t) = \varphi_x(L, t) = \psi(0, t) = \psi(L, t) = \theta(0, t) = \theta(L, t) = 0, \quad t \geq 0, \tag{1.2}$$

and initial data

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), & \varphi_t(x, 0) = \varphi_1(x), & x \in (0, L), \\ \psi(x, 0) = \psi_0(x), & \psi_t(x, 0) = \psi_1(x), & x \in (0, L), \\ \theta(x, 0) = \theta_0(x), & q(x, 0) = q_0(x), & x \in (0, L), \end{cases} \tag{1.3}$$

where $\rho_1, \rho_2, \rho_3, k, b, \gamma, \tau, \beta, L$ are positive constants. Problem (1.1)–(1.3) was studied by M. A. Jorge Silva and R. Racke in [4, Section 3]. They demonstrated that the system fails to achieve exponential stability

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regardless of whether the equality of wave speeds (EWS) condition

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}, \tag{1.4}$$

holds. By introducing a memory term, however, they established exponential stability without requiring the EWS condition (1.4). A natural follow-up question then arises: Is the $\boldsymbol{C_0}$-semigroup $\boldsymbol{\{S(t)\}_{t\geq 0}}$ polynomially stable? The primary objective of this paper is to address this question. We prove that $\{S(t)\}_{t\geq 0}$ is indeed polynomially stable when the EWS condition (1.4) is satisfied.

In [10], the classical Timoshenko beam model is presented as a coupled system governing shear forces and bending moments:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0 & \text{in } (0, L) \times \mathbb{R}^+, \end{cases} \tag{1.5}$$

This model accounts for both shear deformation and rotational inertia effects. Here, the subscripts $(\cdot)_x$ and $(\cdot)_t$ denote partial derivatives with respect to the spatial variable $x \in (0, L)$ and the time variable $t \in \mathbb{R}^+$, respectively. The physical parameters are defined as follows: $\rho_1 = \rho A$ represents the mass per unit length, $b = EI$ is the flexural rigidity, and $\rho_2 = \rho I$ corresponds to the rotational inertia. In these expressions, A denotes the cross-sectional area, ρ the material density, E Young's modulus, and I the area moment of inertia. The dependent variable φ describes the transverse displacement of the beam, while ψ represents the rotation angle of the cross-section due to bending [11].

The literature on Timoshenko systems is extensive (see, e.g., [2, 6, 8]). In recent years, the stability analysis of Timoshenko-type systems has attracted considerable attention, leading to numerous results on uniform and asymptotic energy decay. Soufyane [8] was the first to establish exponential decay for a Timoshenko system with a locally distributed frictional damping acting on only one equation, proving that such decay holds if and only if the wave speeds are equal. Denoting the difference in wave speeds by

$$\chi = \frac{k}{\rho_1} - \frac{b}{\rho_2}, \tag{1.6}$$

a wide range of significant exponential decay results for Timoshenko systems with dissipation in only one equation have been derived under the condition $\chi = 0$. Ammar-Khodja et al. [12] further considered the incorporation of memory effects. Rivera and Fernández [13] investigated Timoshenko systems with past history, imposing appropriate conditions on the relaxation functions. Additional important contributions and related developments can be found in [5, 14–20] and the references therein.

A central focus in the study of thermoelastic diffusion mechanisms has been the stability analysis of Timoshenko systems subject to thermal damping. Fernández Sare and Racke [3] studied a thermoelastic model incorporating Cattaneo-type thermal damping within the bending moment dynamics, resulting in the following evolution system:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \\ \rho_3\theta_t + p_x + \gamma\psi_{xt} = 0, \\ \tau p_t + \beta p + \theta_x = 0. \end{cases} \tag{1.7}$$

Here, the positive constants ρ_3, k, and γ correspond to physical parameters derived from thermoelasticity theory. According to [5], when $\tau = 0$ (i.e., under Fourier's law), the system is exponentially stable if and only if condition (1.6) holds. However, under Dirichlet-Neumann-Dirichlet boundary conditions, Fernández Sare and Racke showed that even when $\chi = 0$, the system fails to be exponentially stable under Cattaneo's law. This surprising finding prompted Santos et al. [7] to conduct a deeper analysis, which led to the following conclusions:

- The classical condition $\chi = 0$ is no longer sufficient to ensure exponential stability in this framework.
- A modified stability criterion emerges through the dimensionless parameter

$$\chi_0 := \left(\tau - \frac{\rho_1}{\rho_3 k}\right)\left(\rho_2 - \frac{b\rho_1}{k}\right) - \frac{\tau\rho_1\gamma^2}{\rho_3 k}.$$

- The system is exponentially stable if and only if $\chi_0 = 0$.
- If $\chi_0 \neq 0$, the energy decays polynomially.

Similarly, the studies conducted in [21–23] examined a shear force-damped thermoelastic system (with $\tau = 0$) under full Dirichlet or mixed Dirichlet-Neumann boundary conditions. It was rigorously established that the system attains exponential stability if and only if $\chi = 0$, while polynomial decay rates occur when $\chi \neq 0$.

Silva and Racke [4] were the first to examine system (1.1)–(1.3). Their analysis established that exponential stability cannot be attained regardless of whether $\chi = 0$ holds, a conclusion similar to that of [3]. Inspired by the perspective of [7], we consider an intriguing question: Does the system exhibit polynomial stability? If so, what should the stability coefficient be—χ, χ_0, or some other value?

This paper is organized as follows. In Section 2, we establish the existence, regularity, and uniqueness of solutions to system (1.1)–(1.3) using semigroup theory (see [9]). In Section 3, we show that system (1.1)–(1.3) is polynomially stable when $\chi = 0$; specifically, the semigroup decays as $t^{-1/4}$.

Returning to system (1.1)–(1.3), we define the spaces

$$L^2_*(0,L) := \left\{ f \in L^2(0,L) \;\middle|\; \int_0^L f(x)\,dx = 0 \right\}, \quad H^1_*(0,L) := H^1(0,L) \cap L^2_*(0,L),$$

and construct the Hilbert space

$$\mathcal{H} := H^1_*(0,L) \times L^2_*(0,L) \times H^1_0(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L),$$

equipped with the inner product

$$(U_1, U_2)_{\mathcal{H}} := \int_0^L \Big[\rho_1 \Phi_1 \Phi_2 + \rho_2 \Psi_1 \Psi_2 + b\psi_{1,x}\psi_{2,x} + k(\varphi_{1,x} + \psi_1)(\varphi_{2,x} + \psi_2) + \rho_3 \theta_1 \theta_2 + \tau q_1 q_2\Big] dx$$

and norm $\|U\|^2_{\mathcal{H}} = (U,U)_{\mathcal{H}}$, where $U_1 = (\varphi_1, \Phi_1, \psi_1, \Psi_1, \theta_1, q_1)^T$, $U_2 = (\varphi_2, \Phi_2, \psi_2, \Psi_2, \theta_2, q_2)^T$, and $U = (\varphi, \Phi, \psi, \Psi, \theta, q)^T \in \mathcal{H}$. Here, $\|\cdot\|$ denotes the standard L^2-norm.

By setting $\Phi = \varphi_t$ and $\Psi = \psi_t$, we reformulate (1.1)–(1.3) as a first-order evolution system:

$$\begin{cases} U_t = \mathcal{A}U, & t > 0, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) =: U_0, \end{cases}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{A}U = \begin{pmatrix} \Phi \\ \dfrac{k}{\rho_1}(\varphi_x + \psi)_x - \dfrac{\gamma}{\rho_1}\theta_x \\ \Psi \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi) + \dfrac{\gamma}{\rho_2}\theta \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\gamma}{\rho_3}(\Phi_x + \Psi) \\ -\dfrac{\beta}{\tau}q - \dfrac{1}{\tau}\theta_x \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ U \in \mathcal{H} \,\middle|\, \Phi \in H^1_*(0,L),\ \varphi_x, \Psi, \theta \in H^1_0(0,L),\ q \in H^1(0,L),\ \varphi, \psi \in H^2(0,L) \right\}.$$

From [4, Theorem 3.1], we have: The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$. However, $\{S(t)\}_{t\geq 0}$ is not exponentially stable, regardless of whether the EWS condition (1.4) holds.

This paper extends these results by proving polynomial stability under the EWS condition:

Theorem 1.1. Assume the EWS condition (1.4) holds. Then the semigroup $\{S(t)\}_{t\geq 0}$ is polynomially stable, satisfying $\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{\sqrt[4]{t}}\|\mathcal{A}U_0\|_{\mathcal{H}}$, $\forall t > 0$ amd $U_0 \in D(\mathcal{A})$,where $C > 0$ is a constant independent of U_0 and t.

2 Proof of Theorem 1.1

In this section, we denote by C, a general positive constant independent of λ, t, U_0, which may change from line to line. The following facts can be found in [4, pages 187-188]:

(F1) $0 \in \rho(\mathcal{A})$, and

(F2) $\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} \leq -\beta\|q\|^2$ for any $U = (\varphi, \Phi, \psi, \Psi, \theta, q)^T \in D(\mathcal{A})$.

Moreover, since $D(\mathcal{A}) \hookrightarrow \mathcal{H}$ compactly, we get

(F3) $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, where $\sigma(\mathcal{A})$ is the spectral set of $\mathcal{A}$, and $\sigma_p(\mathcal{A})$ is the point spectral set of $\mathcal{A}$.

Lemma 2.1. We have $i\mathbb{R} \subset \varrho(\mathcal{A})$, where $\varrho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$.

Proof. We proceed by contradiction. If the conclusion is not hold, by (F3), $i\mathbb{R} \cap \sigma_p(\mathcal{A}) \neq \emptyset$, i.e., $\mathcal{A}$ admits an imaginary eigenvalue $i\lambda$ with $\lambda \in \mathbb{R}$ and a corresponding eigenvector $U = (\varphi, \Phi, \psi, \Psi, \theta, q) \neq 0$ such that $\mathcal{A}U = i\lambda U$, i.e.,

$$\begin{cases} i\lambda\varphi - \Phi = 0, \\ i\lambda\rho_1\Phi - k(\varphi_x + \psi)_x + \gamma\theta_x = 0, \\ i\lambda\psi - \Psi = 0, \\ i\lambda\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi) - \gamma\theta = 0, \\ i\lambda\rho_3\theta + q_x + \gamma(\Phi_x + \Psi) = 0, \\ i\lambda\tau q + \beta q + \theta_x = 0. \end{cases} \tag{2.1}$$

Moreover, by (F1), $\lambda \neq 0$. Then it follows from (F2) that $q = 0$. Consequently, by $(2.1)_6$ (i.e., $\theta_x = 0$) and $\theta \in H_0^1(0,L)$, we get $\theta = 0$. Examining the system $(2.1)_5$, we obtain $\Phi_x + \Psi = 0$. Then it follows the equations $(2.1)_1$ and $(2.1)_3$, that $\varphi_x + \psi = 0$. Since $\theta_x = 0$ and $\lambda \neq 0$, substituting this equality into $(2.1)_2$ forces $\Phi = 0$, which in turn implies $\Psi = 0$ by $\Phi_x + \Psi = 0$ and $\varphi = 0$ by $(2.1)_1$. Returning to $(2.1)_4$, we find $\psi_{xx} = 0$. Then by $\psi \in H^2(0,L) \cap H_0^1(0,L)$, we get $\psi = 0$. So the above analysis shows that $U = 0$, which contradicts $U \neq 0$. □

Let $\lambda \in \mathbb{R}$ and $F = (f^1, \ldots, f^6) \in \mathcal{H}$. Since, by Lemma 2.1, $i\lambda \in \rho(A)$, $U = (\varphi, \Phi, \psi, \Psi, \theta, q) := (i\lambda - \mathcal{A})^{-1}F \in D(\mathcal{A})$ satisfies

$$i\lambda U - \mathcal{A}U = F, \tag{2.2}$$

i.e.,

$$\begin{cases} i\lambda\varphi - \Phi = f^1, \\ i\lambda\rho_1\Phi - k(\varphi_x + \psi)_x + \gamma\theta_x = \rho_1 f^2, \\ i\lambda\psi - \Psi = f^3, \\ i\lambda\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi) - \gamma\theta = \rho_2 f^4, \\ i\lambda\rho_3\theta + q_x + \gamma(\Phi_x + \Psi) = \rho_3 f^5, \\ i\lambda\tau q + \beta q + \theta_x = \tau f^6. \end{cases} \tag{2.3}$$

Lemma 2.2. $\|q\|^2 \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$.

Proof. By applying (F2) and (2.2), we directly derive

$$\beta \int_0^L q^2 dx = \mathrm{Re}\left[i\lambda\|U\|_{\mathcal{H}}^2 - (\mathcal{A}U, U)_{\mathcal{H}}\right] = \mathrm{Re}(i\lambda U - \mathcal{A}U, U)_{\mathcal{H}} = \mathrm{Re}(F, U)_{\mathcal{H}} \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

So Lemma 2.2 follows. □

Lemma 2.3. For any $\epsilon > 0$, there exists a positive constant C_ϵ such that: $\rho_3\|\theta\|^2 \leq \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2$.

Proof. Integrating equation $(2.3)_6$ over $(0,x) \subset (0,L)$, since $\theta(0) = 0$, we obtain

$$i\lambda\tau \int_0^x q(y)dy + \beta \int_0^x q(y)dy + \theta(x) = \tau \int_0^x f^6(y)dy \tag{2.4}$$

Taking the $L^2(0,L)$-inner product of (2.4) with $\theta(x)$ yields:

$$\|\theta\|^2 = \underbrace{-\tau \int_0^L \int_0^x q(y)dy\,(i\lambda\theta(x))dx}_{=:J_1} - \beta \int_0^L \int_0^x q(y)dy\,\theta(x)dx + \tau \int_0^L \int_0^x f^6(y)dy\,\theta(x)dx. \tag{2.5}$$

Using the identity $(2.3)_5$, we can express J_1 as

$$\begin{aligned} J_1 =& -\frac{\tau}{\rho_3} \int_0^L \int_0^x q(y)dy\,\left(\rho_3 f^5 - q_x - \gamma\Phi_x - \gamma\Psi\right) dx \\ =& \frac{\tau}{\rho_3}\left(q(L) + \gamma\Phi(L)\right) \int_0^L q dx - \frac{\tau}{\rho_3} \int_0^L |q|^2 dx \end{aligned}$$

$$- \frac{\tau\gamma}{\rho_3}\int_0^L q\Phi dx + \frac{\tau\gamma}{\rho_3}\int_0^L\int_0^x q(y)dy\Psi(x)dx - \tau\int_0^L\int_0^x q(y)dyf^5(x)dx.$$

Substituting this result into (2.5), we obtain

$$\begin{aligned}\rho_3\|\theta\|^2 =& -\tau\int_0^L |q|^2dx - \tau\gamma\int_0^L q\Phi dx + \tau\gamma\int_0^L\int_0^x q(y)dy\Psi(x)dx - \rho_3\beta\int_0^L\int_0^x q(y)dy\theta(x)dx \\ & - \rho_3\tau\int_0^L\int_0^x q(y)dyf^5(x)dx + \rho_3\tau\int_0^L\int_0^x f^6(y)dy\,\theta(x)dx + \underbrace{\tau\left(q(L)+\gamma\Phi(L)\right)\int_0^L qdx}_{=:J_2}.\end{aligned} \tag{2.6}$$

To estimate J_2, we integrate $(2.3)_5$ over (x, L) to obtain

$$i\lambda\rho_3\int_x^L \theta(s)ds + \int_x^L q_s(s)ds + \gamma\int_x^L (\Phi_s+\Psi)(s)ds = \rho_3\int_x^L f^5(s)ds. \tag{2.7}$$

Then, we have

$$q(L)+\gamma\Phi(L) = q(x)+\gamma\Phi(x) - i\lambda\rho_3\int_x^L\theta(s)ds - \gamma\int_x^L\Psi(s)ds + \rho_3\int_x^L f^5(s)ds. \tag{2.8}$$

Multiplying (2.8) by $\tau\int_0^L q(z)dz$ gives

$$\begin{aligned}J_2 =& \rho_3\tau\int_x^L f^5(s)ds\int_0^L q(z)dz + \tau[q(x)+\gamma\Phi(x)]\int_0^L q(z)dz \\ & - \gamma\tau\int_x^L\Psi(s)ds\int_0^L q(z)dz \underbrace{-\rho_3\int_x^L\theta(s)ds\int_0^L (i\lambda\tau q)(z)dz}_{=:J_3}.\end{aligned}$$

Applying $(2.3)_6$ to J_3 and using $\theta(0)=\theta(L)=0$, we rewrite J_3 as

$$J_3 = -\rho_3\int_x^L\theta(s)ds\int_0^L(\tau f^6 - \beta q - \theta_z)(z)dz = \rho_3\beta\int_x^L\theta(s)ds\int_0^L q(z)dz - \rho_3\tau\int_x^L\theta(s)ds\int_0^L f^6(z)dz.$$

Substituting this into J_2, we derive

$$\begin{aligned}J_2 =& \rho_3\tau\int_x^L f^5(s)ds\int_0^L q(z)dz + \tau[q(x)+\gamma\Phi(x)]\int_0^L q(z)dz \\ & - \gamma\tau\int_x^L\Psi(s)ds\int_0^L q(z)dz + \rho_3\beta\int_x^L\theta(s)ds\int_0^L q(z)dz - \rho_3\tau\int_x^L\theta(s)ds\int_0^L f^6(z)dz\end{aligned}$$

Integrating J_2 over $x\in(0,L)$ and applying Hölder's inequality and Lemma 2.2, we deduce

$$\begin{aligned}L|J_2| \le& C\int_0^L|q(z)|dz\int_0^L\int_0^L|f^5(s)|dsdx + C\int_0^L|q(z)|dz\int_0^L\int_0^L|\Psi(s)|dsdx \\ & + C\int_0^L|q(z)|dz\int_0^L\int_0^L|\theta(s)|dsdx + C\int_0^L|\theta(s)|ds\int_0^L\int_0^L|f^6(s)|dsdx \\ & + C\int_0^L|q(z)|dz\int_0^L|q(x)|dx + C\int_0^L|q(z)|dz\int_0^L|\Phi(x)|dx \\ \le& C\|q\|\|f^5\| + C\|q\|\|\Psi\| + C\|q\|\|\theta\| + C\|\theta\|\|f^6\| + C\|q\|^2 + C\|q\|\|\Phi\|\end{aligned}$$

$$\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|q\|\|U\|_{\mathcal{H}} + C\|q\|\|U\|_{\mathcal{H}} + C\|q\|\|\theta\|.$$

Returning to (2.6), by a simple calculation, we conclude $\rho_3\|\theta\|^2 \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|q\|\|U\|_{\mathcal{H}} + C\|q\|\|U\|_{\mathcal{H}} + C\|q\|\|\theta\|$. Finally, invoking Lemma 2.2 and Young's inequality with $\epsilon > 0$, we establish the validity of this lemma. □

Lemma 2.4. For any $\epsilon > 0$, there exists a positive constant C_ϵ such that: $\|\varphi_x + \psi\|^2 \leq \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}$.

Proof. By equations $(2.3)_1$ and $(2.3)_3$, we rewrite $(2.3)_5$ as

$$i\lambda\rho_3\theta + q_x + i\lambda\gamma(\varphi_x + \psi) = \rho_3 f^5 + \gamma(f^1_x + f^3). \tag{2.9}$$

Multiplying (2.9) by $k(\varphi_x + \psi)$ and integrating over $(0, L)$, since $\varphi_x(L) = \varphi_x(0) = \psi(L) = \psi(0)$, it follows

$$i\lambda k\gamma\|\varphi_x + \psi\|^2 = \underbrace{\int_0^L \left[\rho_3 f^5 + \gamma(f^1_x + f^3)\right] k(\varphi_x + \psi)dx}_{=:J_4} + \underbrace{k\int_0^L q(\varphi_x + \psi)_x dx}_{=:J_5} - i\lambda k\rho_3\int_0^L \theta(\varphi_x + \psi)dx.$$

The term J_4 satisfies $|J_4| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. From equation $(2.3)_2$ and $(2.3)_6$, we derive:

$$J_5 = i\lambda\rho_1\int_0^L q\Phi dx - \rho_1\int_0^L qf^2 dx + \tau\gamma\int_0^L qf^6 dx - \beta\gamma\int_0^L |q|^2 dx - i\lambda\gamma\tau\int_0^L |q|^2 dx$$

which, together with Lemma 2.2, implies the estimate $|J_5| \leq C|\lambda|\|q\|\|\Phi\| + C|\lambda|\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Then we obtain from the above relations that $k\gamma\|\varphi_x + \psi\|^2 \leq C\|q\|\|\Phi\| + C\|\theta\|\|\varphi_x + \psi\| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. So the conclusion follows from the above inequality, Lemmas 2.2, 2.3 and Young's inequality with $\epsilon > 0$. □

Lemma 2.5. Let $|\lambda| \geq 1$. For any $\epsilon > 0$, there exists a positive constant C_ϵ such that: $\rho_1\|\Phi\|^2 \leq \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}$.

Proof. Multiplying equation $(2.3)_2$ by φ, integrating over $(0, L)$, since by $(2.3)_1$, $i\lambda\varphi = \Phi + f^1$, we derive

$$\rho_1\int_0^L \Phi(\Phi + f^1)dx = -k\int_0^L (\varphi_x + \psi)\varphi_x dx - \gamma\int_0^L \theta_x\varphi dx + \rho_1\int_0^L f^2\varphi dx.$$

Then, it follows that

$$\rho_1\|\Phi\|^2 = -k\int_0^L |\varphi_x + \psi|^2 dx \underbrace{-\gamma\int_0^L \theta_x\varphi dx + \rho_1\int_0^L f^2\varphi dx - \rho_1\int_0^L \Phi f^1 dx + k\int_0^L (\varphi_x + \psi)\psi dx}_{=:J_6}. \tag{2.10}$$

Invoking $(2.3)_1$, $(2.3)_3$ and $(2.3)_6$, we express J_6 as

$$\begin{aligned} J_6 :=& -\frac{i}{\lambda}k\int_0^L (\varphi_x + \psi)\left(\Psi + f^3\right)dx + \frac{i\gamma\tau}{\lambda}\int_0^L f^6\left(\Phi + f^1\right)dx \\ &+ \left(\gamma\tau - \frac{i\gamma\beta}{\lambda}\right)\int_0^L q\left(\Phi + f^1\right)dx + \rho_1\int_0^L (f^2\varphi - \Phi f^1)dx. \end{aligned}$$

So for $|\lambda| \geq 1$, J_6 obviously satisfies

$$|J_6| \leq C\left\|\varphi_x + \psi\right\|\left\|U\right\|_{\mathcal{H}} + C\|q\|\|\Phi\| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}.$$

Returning to (2.10) and employing Lemmas 2.2 and 2.4, and repeated applications of Young's inequality with $\epsilon > 0$, we conclude this lemma. □

Lemma 2.6. For any $\epsilon > 0$, there exists a positive constant C_ϵ such that: $\rho_2\|\Psi\|^2 \leq b\|\psi_x\|^2 + \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}$.

Proof. Multiplying equation $(2.3)_4$ by ψ and integrating over $(0, L)$, it follows from the equation $(2.3)_3$ that

$$\rho_2\int_0^L |\Psi|^2\,dx = -b\int_0^L |\psi_x|^2\,dx - k\int_0^L (\varphi_x+\psi)\psi\,dx + \gamma\int_0^L \theta\psi\,dx + \rho_2\int_0^L (f^4\psi - \Psi f^3)\,dx.$$

Then, we have $\rho_2\|\Psi\|^2 \leq b\|\psi_x\|^2 + C\|\varphi_x+\psi\|\|U\|_{\mathcal{H}} + C\|\theta\|\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Applying Lemmas 2.3, 2.4 and Young's inequality with $\epsilon > 0$, we conclude the result. □

Lemma 2.7. Let $|\lambda| \geq 1$. For any $\epsilon > 0$, there exists a positive constant C_ϵ such that: $\|\psi_x\|^2 \leq C|\chi||\lambda|\|\varphi_x + \psi\|\|\Psi\| + C|\lambda|\|q\|\|\psi_x\| + \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}$, where $\chi = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2}$.

Proof. Starting from equations $(2.3)_2$ and $(2.3)_4$, we derive

$$i\lambda(\Phi_x+\Psi) - \frac{k}{\rho_1}(\varphi_x+\psi)_{xx} + \frac{\gamma}{\rho_1}\theta_{xx} - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) - \frac{\gamma}{\rho_2}\theta = f^2_x + f^4. \tag{2.11}$$

Multiplying (2.11) by ψ and integrating by parts over $(0, L)$, by $(2.3)_3$, we get

$$\frac{b}{\rho_2}\|\psi_x\|^2 = -\int_0^L (\Phi_x+\Psi)\Psi dx + \frac{\gamma}{\rho_2}\int_0^L \theta\psi dx - \frac{k}{\rho_2}\int_0^L (\varphi_x+\psi)\psi dx + J_7 + J_8 + J_9, \tag{2.12}$$

where $J_7 := \int_0^L (f^2_x + f^4)\psi dx - \int_0^L (\Phi_x+\Psi)f^3 dx,\quad J_8 = \frac{\gamma}{\rho_1}\int_0^L \theta_x\psi_x dx,\quad J_9 = \frac{k}{\rho_1}\int_0^L (\varphi_x+\psi)\psi_{xx}dx.$

Since, by $(2.3)_{4,1,3}$,

$$\begin{aligned} J_9 &= \frac{k\rho_2}{b\rho_1}\int_0^L (\varphi_x+\psi)(i\lambda\Psi)dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x+\psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x+\psi)\theta dx - \frac{k\rho_2}{b\rho_1}\int_0^L (\varphi_x+\psi)f^4 dx \\ &= \frac{\rho_2 k}{b\rho_1}\int_0^L (\Phi_x+\Psi)\Psi dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x+\psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x+\psi)\theta dx + J_{10}, \end{aligned}$$

where $J_{10} = \frac{\rho_2 k}{b\rho_1}\int_0^L (f^1_x + f^2)\Psi dx - \frac{k\rho_2}{b\rho_1}\int_0^L (\varphi_x+\psi)f^4 dx$, we get

$$\begin{aligned} \frac{b}{\rho_2}\|\psi_x\|^2 = \frac{\rho_2}{b}\chi\int_0^L (\Phi_x+\Psi)\Psi dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x+\psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x+\psi)\theta dx \\ + \frac{\gamma}{\rho_2}\int_0^L \theta\psi dx - \frac{k}{\rho_2}\int_0^L (\varphi_x+\psi)\psi dx + J_7 + J_8 + J_{10}. \end{aligned} \tag{2.13}$$

It is obvious that $|J_7| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. and $|J_{10}| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. For $|\lambda| \geq 1$, we get from the equation $(2.3)_6$ that $J_8 = \frac{\gamma}{\rho_1}\int_0^L (\tau f^6 - i\lambda\tau q - \beta q)\psi_x d \leq C\int_0^L f^6\psi_x dx + C|\lambda|\int_0^L q\psi_x dx \leq C|\lambda|\|q\|\|\psi_x\| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Using above estimates and $(2.3)_{1,3}$, by Young's inequality and Young's inequality with $\varepsilon = \frac{b}{2\rho}$, we have

$$\begin{aligned} \frac{b}{\rho_2}\|\psi_x\|^2 &= \frac{i\lambda\rho_2}{b}\chi\int_0^L (\varphi_x+\psi)\Psi dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x+\psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x+\psi)\theta dx \\ &\quad + \frac{\gamma}{\rho_2}\int_0^L \theta\psi dx - \frac{k}{\rho_2}\int_0^L (\varphi_x+\psi)\psi dx - \frac{\rho_2}{b}\chi\int_0^L (f^1_x + f^3)\Psi dx + J_7 + J_8 + J_{10} \\ &\leq C|\chi||\lambda|\|\varphi_x+\psi\|\|\Psi\| + C\|\varphi_x+\psi\|^2 + C\|\varphi_x+\psi\|\|\theta\| \end{aligned}$$

$$
\begin{aligned}
&+C\|\theta\|\|\psi_x\|+C\|\varphi_x+\psi\|\|\psi_x\|+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+J_7+J_8+J_{10}\\
&\le C|\chi||\lambda|\|\varphi_x+\psi\|\|\Psi\|+C|\lambda|\|q\|\|\psi_x\|+C\|\varphi_x+\psi\|^2+\frac{b}{2\rho_2}\|\psi_x\|^2+C\|\theta\|^2+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},
\end{aligned}
$$

which implies $\|\psi_x\|^2 \le C|\chi||\lambda|\|\varphi_x+\psi\|\|\Psi\|+C|\lambda|\|q\|\|\psi_x\|+C\|\varphi_x+\psi\|^2+C\|\theta\|^2+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Applying Lemmas 2.3 , 2.4 and Young's inequality with $\epsilon > 0$, we conclude the result. □

Proof of Theorem 1.1. Our proof depends on the following results [1]:

(JL) Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. If $i\mathbb{R} \subset \varrho(\mathcal{A})$, then for every fixed $\alpha > 0$, we have $\left\|(i\lambda I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \le C|\lambda|^\alpha$ as $|\lambda| \to +\infty$ if and only if $\left\|S(t)\mathcal{A}^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \le \frac{C}{t^{1/\alpha}}$ as $|\lambda| \to +\infty$.

Since, by our assumption $\chi = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2} = 0$. By using Lemmas 2.7, 2.6 and 2.2, we get, for any $\epsilon > 0$ and $|\lambda| \ge 1$,

$$
\begin{aligned}
b\|\psi_x\|^2+\rho_2\|\Psi\|^2 &\le C|\lambda|\|q\|\|U\|_{\mathcal{H}}+\epsilon\|U\|^2_{\mathcal{H}}+C_\epsilon\|F\|^2_{\mathcal{H}} \le C|\lambda|^2\|q\|^2+\epsilon\|U\|^2_{\mathcal{H}}+C_\epsilon\|F\|^2_{\mathcal{H}}\\
&\le C|\lambda|^2\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\epsilon\|U\|^2_{\mathcal{H}}+C_\epsilon\|F\|^2_{\mathcal{H}} \le \epsilon\|U\|^2_{\mathcal{H}}+C_\epsilon|\lambda|^4\|F\|^2_{\mathcal{H}}.
\end{aligned}
$$

Then by Lemmas 2.2-2.5, we get for any $\epsilon > 0$ and $|\lambda| \ge 1$, $\|U\|^2_{\mathcal{H}} \le \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon|\lambda|^4\|F\|^2_{\mathcal{H}}$ By choose $\epsilon = \frac{1}{2}$, it follows that $\frac{1}{|\lambda|^4}\|U\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}$, which means $\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C|\lambda|^4$. Note Lemma 2.1, our conclusion follows from (JL). □

References

[1] Alexander Borichev and Yuri Tomilov. Optimal polynomial decay of functions and operator semigroups. Math. Ann., 347(2):455–478, 2010.

[2] Luci Harue Fatori, Rodrigo Nunes Monteiro, and Hugo D. Fernández Sare. The Timoshenko system with history and Cattaneo law. Appl. Math. Comput., 228:128–140, 2014.

[3] Hugo D. Fernández Sare and Reinhard Racke. On the stability of damped Timoshenko systems: Cattaneo versus Fourier law. Arch. Ration. Mech. Anal., 194(1):221–251, 2009.

[4] M. A. Jorge Silva and R. Racke. Effects of history and heat models on the stability of thermoelastic Timoshenko systems. J. Differential Equations, 275:167–203, 2021.

[5] Jaime E. Muñoz Rivera and Reinhard Racke. Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability. J. Math. Anal. Appl., 276(1):248–278, 2002.

[6] Belkacem Said-Houari and Aslan Kasimov. Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same. J. Differential Equations, 255(4):611–632, 2013.

[7] M. L. Santos, D. S. Almeida Júnior, and J. E. Muñoz Rivera. The stability number of the Timoshenko system with second sound. J. Differential Equations, 253(9):2715–2733, 2012.

[8] Abdelaziz Soufyane. Stabilisation de la poutre de Timoshenko. C. R. Acad. Sci. Paris Sér. I Math., 328(8):731–734, 1999.

[9] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Semigroups of Linear Operators and Applications to Partial Differential Equations, 1983.

[10] Stephen P Timoshenko. LXVI. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 41(245):744–746, 1921.

[11] M. O. Alves, A. H. Caixeta, M. A. Jorge Silva, J. H. Rodrigues, and D. S. Almeida Júnior. On a Timoshenko system with thermal coupling on both the bending moment and the shear force. J. Evol. Equ., 20(1):295–320, 2020.

[12] F. Ammar-Khodja, A. Benabdallah, J. E. Muñoz Rivera, and R. Racke. Energy decay for Timoshenko systems of memory type. J. Differential Equations, 194(1):82–115, 2003.

[13] Jaime E. Muñoz Rivera and Hugo D. Fernández Sare. Stability of Timoshenko systems with past history. J. Math. Anal. Appl., 339(1):482–502, 2008.

[14] Ali Wehbe and Wael Youssef. Stabilization of the uniform Timoshenko beam by one locally distributed feedback. Appl. Anal., 88(7):1067–1078, 2009.

[15] Aissa Guesmia, Salim A. Messaoudi, and Ali Wehbe. Uniform decay in mildly damped Timoshenko systems with non-equal wave speed propagation. Dynam. Systems Appl., 21(1):133–146, 2012.

[16] Farid Ammar-Khodja, Sebti Kerbal, and Abdelaziz Soufyane. Stabilization of the nonuniform Timoshenko beam. J. Math. Anal. Appl., 327(1):525–538, 2007.

[17] Jaime E. Muñoz Rivera and Hugo D. Fernández Sare. Exponential decay of Timoshenko systems with indefinite memory dissipation. Adv. Differential Equations, 13(7-8):733–752, 2008.

[18] Jaime E. Muñoz Rivera and Reinhard Racke. Timoshenko systems with indefinite damping. J. Math. Anal. Appl., 341(2):1068–1083, 2008.

[19] Luci H. Fatori, Tais O. Saito, Mauricio Sepúlveda, and Renan Takahashi. Energy decay to Timoshenko system with indefinite damping. Math. Methods Appl. Sci., 43(1):225–241, 2020.

[20] Maurizio Grasselli, Vittorino Pata, and Giovanni Prouse. Longtime behavior of a viscoelastic Timoshenko beam. Discrete Contin. Dyn. Syst., 10(1-2):337–348, 2004.

[21] Dilberto da S. Almeida Júnior, M. L. Santos, and J. E. Muñoz Rivera. Stability to 1-D thermoelastic Timoshenko beam acting on shear force. Z. Angew. Math. Phys., 65(6):1233–1249, 2014.

[22] M. S. Alves, M. A. Jorge Silva, T. F. Ma, and J. E. Muñoz Rivera. Invariance of decay rate with respect to boundary conditions in thermoelastic Timoshenko systems. Z. Angew. Math. Phys., 67(3):Art. 70, 16, 2016.

[23] M. S. Alves, M. A. Jorge Silva, T. F. Ma, and J. E. Muñoz Rivera. Non-homogeneous thermoelastic Timoshenko systems. Bull. Braz. Math. Soc. (N.S.), 48(3):461–484, 2017.