

A Study on Explicit Particular Solutions for Second and Third Order Generalized Leonardo-Type Recurrences with Polynomial-Exponential Forcing

Abstract. Recurrence relations offer a versatile framework for analyzing numerical sequences, with applications across both classical and modern branches of mathematics. In earlier work, explicit iterative formulas were established for polynomial-exponential type particular solutions of generalized Leonardo-type sequences. The present article builds on that foundation by presenting concrete examples for orders

$$m = 1, 2,$$

where the input function takes the form

$$C(n) = p(n)d^n,$$

with $p(n) = \sum_{i=0}^s c_i n^i$ a polynomial in n . For such sequences, we construct particular solutions of the form

$$W_n^{(C)} = n^r \left(\sum_{i=0}^s A_i n^i \right) d^n,$$

and illustrate the computation of the coefficients A_i using the established iterative scheme. These examples show how the multiplicity r of the root d in the characteristic equation shapes the structure of the particular solution, and they highlight resonance phenomena in non-homogeneous cases. By working through explicit instances, the paper provides a clear and accessible demonstration of the general theory, strengthening the link between abstract recurrence relations and concrete symbolic computation.

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1. Introduction

Recurrence sequences defined by recurrence relations have long occupied a central position in mathematics, with applications extending to physics, engineering, architecture, biology, computer science, and even artistic

domains. Beneath their seemingly simple construction lies remarkable depth: they embody growth processes, oscillatory phenomena, and symbolic identities. Among the most distinguished second-order recurrences are the Fibonacci, Lucas, Pell, and Jacobsthal sequences. The Fibonacci numbers, in particular, gained lasting fame through Leonardo de Pisa's celebrated rabbit problem in his 1202 treatise *Liber Abaci*. Both Fibonacci and Lucas sequences continue to provide abundant sources of elegant identities, combinatorial interpretations, and analytic connections.

Beyond second-order families, higher-order recurrence sequences play an equally significant role in both theoretical and applied contexts. These generalizations expand the classical framework and reveal deeper algebraic and analytic structures. Third-order examples such as the Tribonacci sequence, fourth-order examples such as the Tetranacci sequence, and fifth-order examples such as the Pentanacci sequence extend the paradigm, each governed by characteristic polynomials whose root configurations determine closed-form solutions. Homogeneous recurrences emphasize the role of characteristic polynomials and root multiplicities, while non-homogeneous recurrences incorporate symbolic input terms whose interaction with the root structure produces resonance phenomena. Collectively, these families form a unified framework that bridges classical recurrence identities with modern symbolic recurrence theory.

We first recall the definition of m -order homogeneous linear recurrence relations.

DEFINITION 1.1. A sequence $\{V_n\}_{n \geq 0}$ is called a homogeneous (linear) recurrence relation order $m \in \mathbb{N}$ if it satisfies

$$V_n = \sum_{k=1}^m a_k V_{n-k} = a_1 V_{n-1} + a_2 V_{n-2} \dots + a_m V_{n-m} \quad (1.1)$$

for

$$m \geq 1$$

with the initial conditions V_0, V_1, \dots, V_{m-1}

and

$$V_n = a_0, \quad (1.2)$$

for

$$m = 0.$$

The recurrence coefficients a_1, a_2, \dots, a_m and the initial conditions V_0, V_1, \dots, V_{m-1} are complex scalars. We allow each coefficient a_i , for $1 \leq i \leq m$, to be identically zero.

The integer m is called the order of the linear recurrence.

The characteristic polynomial of the sequence $(V_n)_{n \geq 0}$ is given by

$$A(z) = z^m - \sum_{k=1}^m a_k z^{m-k} = z^m - a_1 z^{m-1} - a_2 z^{m-2} - \dots - a_{m-1} z - a_m = (z - \theta_1)^{u_1} (z - \theta_2)^{u_2} \dots (z - \theta_v)^{u_v}$$

with distinct $\theta_1, \theta_2, \dots, \theta_v$ and $u_1 + u_2 + \dots + u_v = m$. $\theta_1, \theta_2, \dots, \theta_v$ are called the (characteristic) root of characteristic equation

$$A(z) = z^m - a_1 z^{m-1} - a_2 z^{m-2} - \dots - a_{m-1} z - a_m = (z - \theta_1)^{u_1} (z - \theta_2)^{u_2} \dots (z - \theta_v)^{u_v} = 0. \quad (1.3)$$

For $m \geq 1$, consider the sequence (W_n) defined by the recurrence relation (a **generalized Leonardo-type sequence**)

$$W_n = \sum_{k=1}^m a_k W_{n-k} + p(n)bd^n = \sum_{k=1}^m a_k W_{n-k} + C(n) \quad (1.4)$$

with initial conditions W_0, W_1, \dots, W_{m-1} and the recurrence coefficients a_1, a_2, \dots, a_m are complex scalars or polynomials in $\mathbb{C}[x]$ and with the input function

$$C(n) = p(n)bd^n$$

where

$$p(n) := p(n, x) = \sum_{i=0}^s c_i n^i$$

denotes a polynomial in n of order s , with coefficients belonging to $\mathbb{C}[x]$ or \mathbb{C} and $b \in \mathbb{C}[x]$ or \mathbb{C} , and $d \in \mathbb{C}$ or \mathbb{R} . For more information on generalized Leonardo-type sequences, see Soykan [2] and [1].

We consider the homogeneous recurrence relation (1.1) and its characteristic equation (1.3), corresponding to the sequence (W_n) defined by (1.4).

The particular solution $W_n^{(C)}$ of (1.4) is of the form

$$W_n^{(C)} = n^r \left(\sum_{i=0}^s A_i n^i \right) d^n = n^r \left(A_0 + \sum_{i=1}^s A_i n^i \right) d^n, \quad (1.5)$$

where the coefficients $A_i \in \mathbb{C}[x]$ or \mathbb{C} and r is the multiplicity of d as a root of the characteristic equation (1.3), (if d is not a root of characteristic equation (1.3) then $r = 0$).

We now state a theorem that furnishes explicit iterative formulas for determining the coefficients of the particular solution $W_n^{(C)}$ of (1.4). The construction hinges on whether the parameter d coincides with a characteristic root of (1.3); in cases where d is a root of multiplicity r , this multiplicity dictates the precise modifications required in the iterative scheme.

THEOREM 1.2. [2, p.100, Theorem 5.1] For each $0 \leq i \leq s$, A_i given in (1.5) can be calculated with the iteration as follows:

- If $r = 0$, i.e., none of the roots of the characteristic equation (1.3) equals d , then

$$A_s = -\frac{c_s b d^m}{a_1 d^{m-1} + a_2 d^{m-2} + a_2 d^{m-3} + \dots + a_{m-2} d^2 + a_{m-1} d + a_m - d^m} = -\frac{c_s b d^m}{-d^m + \sum_{j=1}^m a_j d^{m-j}}, \text{ for } n = s$$

and

$$A_n = -\frac{1}{-d^m + \sum_{j=1}^m a_j d^{m-j}} \left(c_n b d^m - \sum_{k=n+1}^s (-1)^{k-n+1} \binom{k}{n} \left(\sum_{j=1}^m j^{k-n} a_j d^{m-j} \right) A_k \right)$$

for $n = s-1, s-2, \dots, 2, 1, 0$.

- If $r > 0$ then

$$A_s = (-1)^{r+1} \frac{c_s b d^m}{\left(\sum_{j=1}^m j^r a_j \times d^{m-j} \right) \binom{s+r}{r}}, \text{ for } n = s$$

and

$$A_n = (-1)^{r+1} \frac{1}{\left(\sum_{j=1}^m j^r a_j \times d^{m-j} \right) \binom{n+r}{r}} \left(c_n b d^m - \sum_{k=n+1}^s (-1)^{k+r-n+1} \binom{k+r}{n} \left(\sum_{j=1}^m j^{k+r-n} a_j \times d^{m-j} \right) A_k \right)$$

for $n = s-1, s-2, \dots, 2, 1, 0$.

In the following sections, we present explicit particular solutions to (1.4) for $m = 1, 2, 3, 4$, where

$$C(n) = p(n)bd^n, \quad p(n) \text{ is a polynomial in } n.$$

We seek solutions of the form

$$W_n^{(C)} = P(n)d^n,$$

where $P(n)$ is itself a polynomial in n .

2. Special Cases

2.1. The Case $m = 1$. Consider the homogeneous relation

$$V_n = a_1 V_{n-1} \tag{2.1}$$

with the same initial condition as V_0 . Suppose that θ_1 is the the roots of characteristic equation

$$z - a_1 = 0 \tag{2.2}$$

associated with (2.1). Note that if the root of (2.2) equals to d then

$$z - a_1 = z - d = 0$$

so that $a_1 = d$ and (2.1) reduces to

$$V_n = dV_{n-1}.$$

We now turn to an example that illustrates the results derived above.

EXAMPLE 2.1. Consider the sequence (W_n) defined by the recurrence relation

$$W_n = a_1 W_{n-1} + p(n)bd^n$$

where $p(n) := p(n, x)$ denotes a polynomial in n of order s , with coefficients belonging to $\mathbb{C}[x]$ or \mathbb{C} and $b \in \mathbb{C}[x]$ or \mathbb{C} , and $d \in \mathbb{C}$ or \mathbb{R} . We seek a particular solution

$$W_n^C = P(n)d^n$$

for the cases $s = 0, 1, 2, 3, 4, 5, 6, 7$ where $P(n)$ is itself a polynomial in n . The order (degree) and coefficients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.2) and W_n^C satisfy

$$W_n^C = a_1 W_{n-1}^C + p(n)bd^n$$

i.e.,

$$P(n)d^n = a_1 P(n-1)d^{n-1} + p(n)bd^n.$$

In each case of s , we consider the homogeneous relation (2.1) and its characteristic equation (2.2), corresponding to the sequence (W_n) with the same initial conditions as W_n , i.e.,

$$V_0 = W_0.$$

We investigate all cases of multiplicity r of d as a root of the characteristic equation (2.2):

(a): $m = 1, s = 0$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + c_0 b d^n.$$

(i): Case $r = 0$, i.e., the root of the characteristic equation does not equal d :

$$W_n^C = A_0 d^n, \quad A_0 = -\frac{c_0 b d}{a_1 - d}.$$

(ii): Case $r = 1$, i.e., the root of the characteristic equation equals d :

$$W_n^C = n A_0 d^n, \quad A_0 = b c_0.$$

(b): $m = 1, s = 1$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + (c_1 n + c_0) b d^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_1 n + A_0) d^n$$

where

$$A_1 = -\frac{c_1 b d}{a_1 - d},$$

$$A_0 = -\frac{1}{a_1 - d} (c_0 b d - a_1 A_1),$$

i.e.,

$$A_1 = -\frac{c_1 b d}{a_1 - d},$$

$$A_0 = -\frac{b d}{(a_1 - d)^2} (-c_0 d + a_1 (c_0 + c_1)).$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n (A_1 n + A_0) d^n$$

where

$$A_1 = \frac{1}{2} b c_1,$$

$$A_0 = \frac{1}{2} b (2c_0 + c_1).$$

(c): $m = 1, s = 2$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + (c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_2 n^2 + A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_2 &= -\frac{c_2bd}{a_1-d}, \\ A_1 &= -\frac{1}{a_1-d}(c_1bd - 2a_1A_2), \\ A_0 &= -\frac{1}{a_1-d}(c_0bd - a_1A_1 + a_1A_2), \end{aligned}$$

i.e.,

$$\begin{aligned} A_2 &= -\frac{c_2bd}{a_1-d}, \\ A_1 &= -\frac{bd}{(a_1-d)^2}(-c_1d + a_1(c_1 + 2c_2)), \\ A_0 &= -\frac{bd}{(a_1-d)^3}(c_0d^2 + a_1^2(c_0 + c_1 + c_2) - a_1(2c_0 + c_1 - c_2)d). \end{aligned}$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n(A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_2 &= \frac{1}{3}bc_2, \\ A_1 &= \frac{1}{2}b(c_1 + c_2), \\ A_0 &= \frac{1}{6}b(6c_0 + 3c_1 + c_2). \end{aligned}$$

(d): $m = 1, s = 3$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + (c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= -\frac{c_3bd}{a_1-d}, \\ A_2 &= -\frac{1}{a_1-d}(c_2bd - 3a_1A_3), \\ A_1 &= -\frac{1}{a_1-d}(c_1bd - 2a_1A_2 + 3a_1A_3), \\ A_0 &= -\frac{1}{a_1-d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3), \end{aligned}$$

i.e.,

$$A_3 = -\frac{c_3bd}{a_1 - d},$$

$$A_2 = -\frac{bd}{(a_1 - d)^2}(-c_2d + a_1(c_2 + 3c_3)),$$

$$A_1 = -\frac{bd}{(a_1 - d)^3}(c_1d^2 + a_1^2(c_1 + 2c_2 + 3c_3) - a_1(2c_1 + 2c_2 - 3c_3)d),$$

$$A_0 = -\frac{bd}{(a_1 - d)^4}(-c_0d^3 + a_1^3(c_0 + c_1 + c_2 + c_3) - a_1^2(3c_0 + 2c_1 - 4c_3)d + a_1(3c_0 + c_1 - c_2 + c_3)d^2).$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$A_3 = \frac{1}{4}bc_3,$$

$$A_2 = \frac{1}{6}b(2c_2 + 3c_3),$$

$$A_1 = \frac{1}{4}b(2c_1 + 2c_2 + c_3),$$

$$A_0 = \frac{1}{6}b(6c_0 + 3c_1 + c_2).$$

(e): $m = 1, s = 4$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + (c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$A_4 = -\frac{c_4bd}{a_1 - d},$$

$$A_3 = -\frac{1}{a_1 - d}(c_3bd - 4a_1A_4),$$

$$A_2 = -\frac{1}{a_1 - d}(c_2bd - 3a_1A_3 + 6a_1A_4),$$

$$A_1 = -\frac{1}{a_1 - d}(c_1bd - 2a_1A_2 + 3a_1A_3 - 4a_1A_4),$$

$$A_0 = -\frac{1}{a_1 - d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3 + a_1A_4),$$

i.e.,

$$A_4 = -\frac{c_4bd}{a_1 - d},$$

$$A_3 = -\frac{bd}{(a_1 - d)^2}(-c_3d + a_1(4c_4 + c_3)),$$

$$A_2 = -\frac{bd}{(a_1 - d)^3}(c_2d^2 + a_1^2(6c_4 + 3c_3 + c_2) + a_1(6c_4 - 3c_3 - 2c_2)d),$$

$$A_1 = -\frac{bd}{(a_1 - d)^4}(-c_1d^3 + a_1^3(4c_4 + 3c_3 + 2c_2 + c_1) + a_1^2(16c_4 - 4c_2 - 3c_1)d + a_1(4c_4 - 3c_3 + 2c_2 + 3c_1)d^2),$$

$$A_0 = -\frac{bd}{(a_1 - d)^5}(c_0d^4 + a_1^4(c_4 + c_3 + c_2 + c_1 + c_0) + a_1^3(11c_4 + 3c_3 - c_2 - 3c_1 - 4c_0)d + a_1^2(11c_4 - 3c_3 - c_2 + 3c_1 + 6c_0)d^2 + a_1(c_4 - c_3 + c_2 - c_1 - 4c_0)d^3).$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n(A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_4 &= \frac{1}{5}bc_4, \\ A_3 &= \frac{1}{4}b(c_3 + 2c_4), \\ A_2 &= \frac{1}{6}b(2c_2 + 3c_3 + 2c_4), \\ A_1 &= \frac{1}{4}b(2c_1 + 2c_2 + c_3), \\ A_0 &= \frac{1}{30}b(30c_0 + 15c_1 + 5c_2 - c_4). \end{aligned}$$

(f): $m = 1, s = 5$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + (c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_5 &= -\frac{c_5bd}{a_1 - d}, \\ A_4 &= -\frac{1}{a_1 - d}(c_4bd - 5a_1A_5), \\ A_3 &= -\frac{1}{a_1 - d}(c_3bd - 4a_1A_4 + 10a_1A_5), \\ A_2 &= -\frac{1}{a_1 - d}(c_2bd - 3a_1A_3 + 6a_1A_4 - 10a_1A_5), \\ A_1 &= -\frac{1}{a_1 - d}(c_1bd - 2a_1A_2 + 3a_1A_3 - 4a_1A_4 + 5a_1A_5), \\ A_0 &= -\frac{1}{a_1 - d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3 + a_1A_4 - a_1A_5), \end{aligned}$$

i. e.,

$$\begin{aligned} A_5 &= -\frac{c_5bd}{a_1 - d}, \\ A_4 &= -\frac{bd}{(a_1 - d)^2}(-c_4d + a_1(c_4 + 5c_5)), \\ A_3 &= -\frac{bd}{(a_1 - d)^3}(c_3d^2 + a_1^2(c_3 + 4c_4 + 10c_5) - 2a_1(c_3 + 2c_4 - 5c_5)d), \\ A_2 &= -\frac{bd}{(a_1 - d)^4}(-c_2d^3 + a_1^3(c_2 + 3c_3 + 6c_4 + 10c_5) - a_1^2(3c_2 + 6c_3 - 40c_5)d + a_1(3c_2 + 3c_3 - 6c_4 + 10c_5)d^2), \end{aligned}$$

$$\begin{aligned}
A_1 &= -\frac{bd}{(a_1-d)^5}(c_1d^4 + a_1^4(c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5) - a_1^3(4c_1 + 6c_2 + 3c_3 - 12c_4 - 55c_5)d + \\
& a_1^2(6c_1 + 6c_2 - 3c_3 - 12c_4 + 55c_5)d^2 - a_1(4c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5)d^3), \\
A_0 &= -\frac{bd}{(a_1-d)^6}(-c_0d^5 + a_1(5c_0 + c_1 - c_2 + c_3 - c_4 + c_5)d^4 - 2a_1^2(5c_0 + 2c_1 - c_2 - c_3 + 5c_4 - 13c_5)d^3 + \\
& 2a_1^3(5c_0 + 3c_1 - 3c_3 + 33c_5)d^2 - a_1^4(5c_0 + 4c_1 + 2c_2 - 2c_3 - 10c_4 - 26c_5)d + a_1^5(c_0 + c_1 + c_2 + c_3 + c_4 + c_5)).
\end{aligned}$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n(A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_5 &= \frac{1}{6}bc_5, \\
A_4 &= \frac{1}{10}b(2c_4 + 5c_5), \\
A_3 &= \frac{1}{12}b(3c_3 + 6c_4 + 5c_5), \\
A_2 &= \frac{1}{6}b(2c_2 + 3c_3 + 2c_4), \\
A_1 &= \frac{1}{12}b(6c_1 + 6c_2 + 3c_3 - c_5), \\
A_0 &= \frac{1}{30}b(30c_0 + 15c_1 + 5c_2 - c_4).
\end{aligned}$$

(g): $m = 1, s = 6$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + (c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_6 &= \frac{c_6bd}{a_1-d}, \\
A_5 &= -\frac{1}{a_1-d}(c_5bd - 6a_1A_6), \\
A_4 &= -\frac{1}{a_1-d}(c_4bd - 5a_1A_5 + 15a_1A_6), \\
A_3 &= -\frac{1}{a_1-d}(c_3bd - 4a_1A_4 + 10a_1A_5 - 20a_1A_6), \\
A_2 &= -\frac{1}{a_1-d}(c_2bd - 3a_1A_3 + 6a_1A_4 - 10a_1A_5 + 15a_1A_6), \\
A_1 &= -\frac{1}{a_1-d}(c_1bd - 2a_1A_2 + 3a_1A_3 - 4a_1A_4 + 5a_1A_5 - 6a_1A_6), \\
A_0 &= -\frac{1}{a_1-d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3 + a_1A_4 - a_1A_5 + a_1A_6),
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_6 &= -\frac{c_6bd}{a_1-d}, \\
A_5 &= -\frac{bd}{(a_1-d)^2}(-c_5d + a_1(c_5 + 6c_6)),
\end{aligned}$$

$$\begin{aligned}
A_4 &= -\frac{bd}{(a_1-d)^3}(c_4d^2 - a_1(2c_4 + 5c_5 - 15c_6)d + a_1^2(c_4 + 5c_5 + 15c_6)), \\
A_3 &= -\frac{bd}{(a_1-d)^4}(-c_3d^3 + a_1(3c_3 + 4c_4 - 10c_5 + 20c_6)d^2 - a_1^2(3c_3 + 8c_4 - 80c_6)d + a_1^3(c_3 + 4c_4 + 10c_5 + 20c_6)), \\
A_2 &= -\frac{bd}{(a_1-d)^5}(c_2d^4 - a_1(4c_2 + 3c_3 - 6c_4 + 10c_5 - 15c_6)d^3 + 3a_1^2(2c_2 + 3c_3 - 2c_4 - 10c_5 + 55c_6)d^2 \\
&\quad - a_1^3(4c_2 + 9c_3 + 6c_4 - 30c_5 - 165c_6)d + a_1^4(c_2 + 3c_3 + 6c_4 + 10c_5 + 15c_6)), \\
A_1 &= -\frac{bd}{(a_1-d)^6}(-c_1d^5 + a_1(5c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6)d^4 - 2a_1^2(5c_1 + 4c_2 - 3c_3 - 4c_4 + 25c_5 - 78c_6) \\
&\quad d^3 + 2a_1^3(5c_1 + 6c_2 - 12c_4 + 198c_6)d^2 - a_1^4(5c_1 + 8c_2 + 6c_3 - 8c_4 - 50c_5 - 156c_6)d + a_1^5(c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 + 6c_6)), \\
A_0 &= -\frac{bd}{(a_1-d)^7}(c_0d^6 - a_1(6c_0 + c_1 - c_2 + c_3 - c_4 + c_5 - c_6)d^5 + a_1^2(15c_0 + 5c_1 - 3c_2 - c_3 + 9c_4 - 25c_5 + 57c_6) \\
&\quad d^4 - 2a_1^3(10c_0 + 5c_1 - c_2 - 4c_3 + 5c_4 + 20c_5 - 151c_6)d^3 + a_1^4(15c_0 + 10c_1 + 2c_2 - 8c_3 - 10c_4 + 40c_5 + 302c_6) \\
&\quad d^2 + a_1^5(-6c_0 - 5c_1 - 3c_2 + c_3 + 9c_4 + 25c_5 + 57c_6)d + a_1^6(c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6)).
\end{aligned}$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n(A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_6 &= \frac{1}{7}bc_6, \\
A_5 &= \frac{1}{6}b(c_5 + 3c_6), \\
A_4 &= \frac{1}{10}b(2c_4 + 5c_5 + 5c_6), \\
A_3 &= \frac{1}{12}b(3c_3 + 6c_4 + 5c_5), \\
A_2 &= \frac{1}{6}b(2c_2 + 3c_3 + 2c_4 - c_6), \\
A_1 &= \frac{1}{12}b(6c_1 + 6c_2 + 3c_3 - c_5), \\
A_0 &= \frac{1}{210}b(210c_0 + 105c_1 + 35c_2 - 7c_4 + 5c_6).
\end{aligned}$$

(h): $m = 1, s = 7$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + (c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (the root does not equal to d):

$$W_n^C = (A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_7 &= -\frac{c_7bd}{a_1-d}, \\
A_6 &= -\frac{1}{a_1-d}(c_6bd - 7a_1A_7), \\
A_5 &= -\frac{1}{a_1-d}(c_5bd - 6a_1A_6 + 21a_1A_7), \\
A_4 &= -\frac{1}{a_1-d}(c_4bd - 5a_1A_5 + 15a_1A_6 - 35a_1A_7), \\
A_3 &= -\frac{1}{a_1-d}(c_3bd - 4a_1A_4 + 10a_1A_5 - 20a_1A_6 + 35a_1A_7), \\
A_2 &= -\frac{1}{a_1-d}(c_2bd - 3a_1A_3 + 6a_1A_4 - 10a_1A_5 + 15a_1A_6 - 21a_1A_7), \\
A_1 &= -\frac{1}{a_1-d}(c_1bd - 2a_1A_2 + 3a_1A_3 - 4a_1A_4 + 5a_1A_5 - 6a_1A_6 + 7a_1A_7), \\
A_0 &= -\frac{1}{a_1-d}(c_0bd - a_1A_1 + a_1A_2 - a_1A_3 + a_1A_4 - a_1A_5 + a_1A_6 - a_1A_7).
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_7 &= -\frac{c_7bd}{a_1-d}, \\
A_6 &= -\frac{bd}{(a_1-d)^2}(-c_6d + a_1(c_6 + 7c_7)), \\
A_5 &= -\frac{bd}{(a_1-d)^3}(c_5d^2 - a_1(2c_5 + 6c_6 - 21c_7)d + a_1^2(c_5 + 6c_6 + 21c_7)), \\
A_4 &= -\frac{bd}{(a_1-d)^4}(-c_4d^3 + a_1(3c_4 + 5c_5 - 15c_6 + 35c_7)d^2 - a_1^2(3c_4 + 10c_5 - 140c_7)d + a_1^3(c_4 + 5c_5 + 15c_6 + 35c_7)), \\
A_3 &= -\frac{bd}{(a_1-d)^5}(c_3d^4 - a_1(4c_3 + 4c_4 - 10c_5 + 20c_6 - 35c_7)d^3 + a_1^2(6c_3 + 12c_4 - 10c_5 - 60c_6 + 385c_7)d^2 - a_1^3(4c_3 + 12c_4 + 10c_5 - 60c_6 - 385c_7)d + a_1^4(c_3 + 4c_4 + 10c_5 + 20c_6 + 35c_7)), \\
A_2 &= -\frac{bd}{(a_1-d)^6}(-c_2d^5 + a_1(5c_2 + 3c_3 - 6c_4 + 10c_5 - 15c_6 + 21c_7)d^4 - 2a_1^2(5c_2 + 6c_3 - 6c_4 - 10c_5 + 75c_6 - 273c_7)d^3 + 2a_1^3(5c_2 + 9c_3 - 30c_5 + 693c_7)d^2 - a_1^4(5c_2 + 12c_3 + 12c_4 - 20c_5 - 150c_6 - 546c_7)d + a_1^5(c_2 + 3c_3 + 6c_4 + 10c_5 + 15c_6 + 21c_7)), \\
A_1 &= -\frac{bd}{(a_1-d)^7}(c_1d^6 - a_1(6c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6 - 7c_7)d^5 + a_1^2(15c_1 + 10c_2 - 9c_3 - 4c_4 + 45c_5 - 150c_6 + 399c_7)d^4 - 2a_1^3(10c_1 + 10c_2 - 3c_3 - 16c_4 + 25c_5 + 120c_6 - 1057c_7)d^3 + a_1^4(15c_1 + 20c_2 + 6c_3 - 32c_4 - 50c_5 + 240c_6 + 2114c_7)d^2 - a_1^5(6c_1 + 10c_2 + 9c_3 - 4c_4 - 45c_5 - 150c_6 - 399c_7)d + a_1^6(c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 + 6c_6 + 7c_7)), \\
A_0 &= -\frac{bd}{(a_1-d)^8}(-c_0d^7 + a_1(7c_0 + c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + c_7)d^6 + a_1^2(-21c_0 - 6c_1 + 4c_2 - 8c_4 + 24c_5 - 56c_6 + 120c_7)d^5 + a_1^3(35c_0 + 15c_1 - 5c_2 - 9c_3 + 19c_4 + 15c_5 - 245c_6 + 1191c_7)d^4 + a_1^4(-35c_0 - 20c_1 + 16c_3 - 80c_5 + 2416c_7)d^3 + a_1^5(21c_0 + 15c_1 + 5c_2 - 9c_3 - 19c_4 + 15c_5 + 245c_6 + 1191c_7)d^2 - a_1^6(7c_0 + 6c_1 + 4c_2 - 8c_4 - 24c_5 - 56c_6 - 120c_7)d + a_1^7(c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7)).
\end{aligned}$$

(ii): Case $r = 1$ (the root equals to d):

$$W_n^C = n(A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_7 &= \frac{1}{8}bc_7, \\
A_6 &= \frac{1}{14}b(2c_6 + 7c_7), \\
A_5 &= \frac{1}{12}b(2c_5 + 6c_6 + 7c_7), \\
A_4 &= \frac{1}{10}b(2c_4 + 5c_5 + 5c_6), \\
A_3 &= \frac{1}{24}b(6c_3 + 12c_4 + 10c_5 - 7c_7), \\
A_2 &= \frac{1}{6}b(2c_2 + 3c_3 + 2c_4 - c_6), \\
A_1 &= \frac{1}{12}b(6c_1 + 6c_2 + 3c_3 - c_5 + c_7), \\
A_0 &= \frac{1}{210}b(210c_0 + 105c_1 + 35c_2 - 7c_4 + 5c_6).
\end{aligned}$$

2.2. The Case $m = 2$. Consider the homogeneous relation

$$V_n = a_1V_{n-1} + a_2V_{n-2} \quad (2.3)$$

with the initial conditions V_0, V_1 . Suppose that θ_1 and θ_2 are roots of the characteristic equation of (2.3):

$$z^2 - a_1z - a_2 = 0. \quad (2.4)$$

Note that if all the roots of (2.4) are equal to d then

$$z^2 - a_1z - a_2 = (z - d)(z - d) = z^2 - 2dz + d^2 = 0$$

so that $a_1 = 2d$, $a_2 = -d^2$ and (2.3) reduces to

$$V_n = 2dV_{n-1} - d^2V_{n-2}.$$

We now turn to an example that illustrates the results derived above.

EXAMPLE 2.2. Consider the sequence (W_n) defined by the recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + p(n)bd^n$$

where $p(n) := p(n, x)$ denotes a polynomial in n of order s , with coefficients belonging to $\mathbb{C}[x]$ or \mathbb{C} and $b \in \mathbb{C}[x]$ or \mathbb{C} , and $d \in \mathbb{C}$ or \mathbb{R} . We seek a particular solution

$$W_n^C = P(n)d^n$$

for the cases $s = 0, 1, 2, 3, 4, 5, 6, 7$ where $P(n)$ is itself a polynomial in n . The order (degree) and coefficients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.4) and W_n^C satisfy

$$W_n^C = a_1W_{n-1}^C + a_2W_{n-2}^C + p(n)bd^n$$

i.e.,

$$P(n)d^n = a_1P(n-1)d^{n-1} + a_2P(n-2)d^{n-2} + p(n)bd^n.$$

In each case of s , we consider the homogeneous relation (2.3) and its characteristic equation (2.4), corresponding to the sequence (W_n) with the same initial conditions as W_n , i.e.,

$$V_0 = W_0, V_1 = W_1.$$

We investigate all cases of multiplicity r of d as a root of the characteristic equation (2.4):

(a): $m = 2, s = 0$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + c_0 b d^n.$$

(i): Case $r = 0$, i.e., none of the roots of the characteristic equation equals d :

$$W_n^C = A_0 d^n, \quad A_0 = -\frac{c_0 b d^2}{a_1 d + a_2 - d^2}.$$

(ii): Case $r = 1$, i.e., exactly one root of the characteristic equation equals d :

$$W_n^C = n A_0 d^n, \quad A_0 = \frac{c_0 b d^2}{a_1 d + 2a_2}.$$

(iii): Case $r = 2$, i.e., all two roots of the characteristic equation equal d :

$$W_n^C = n^2 A_0 d^n, \quad A_0 = \frac{1}{2} b c_0.$$

(b): $m = 2, s = 1$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + (c_1 n + c_0) b d^n.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_1 n + A_0) d^n$$

where

$$A_1 = -\frac{c_1 b d^2}{(a_1 d + a_2 - d^2)},$$

$$A_0 = -\frac{1}{(a_1 d + a_2 - d^2)} (c_0 b d^2 - (a_1 d + 2a_2) A_1),$$

i.e.,

$$A_1 = -\frac{c_1 b d^2}{(a_1 d + a_2 - d^2)},$$

$$A_0 = -\frac{b d^2}{(a_1 d + a_2 - d^2)^2} (-c_0 d^2 + a_1 (c_0 + c_1) d + a_2 (c_0 + 2c_1)).$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_1 n + A_0) d^n$$

where

$$A_1 = \frac{c_1 b d^2}{2(a_1 d + 2a_2)},$$

$$A_0 = \frac{1}{(a_1 d + 2a_2)} (c_0 b d^2 + (a_1 d + 4a_2) A_1),$$

i.e.,

$$A_1 = \frac{c_1 b d^2}{2(a_1 d + 2a_2)},$$

$$A_0 = \frac{b d^2}{2(a_1 d + 2a_2)^2} (a_1(2c_0 + c_1)d + 4a_2(c_0 + c_1)).$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = n^2(A_1 n + A_0)d^n$$

where

$$A_1 = \frac{1}{6} b c_1,$$

$$A_0 = \frac{1}{2} b (c_0 + c_1).$$

(c): $m = 2, s = 2$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + (c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_2 n^2 + A_1 n + A_0) d^n$$

where

$$A_2 = -\frac{c_2 b d^2}{(a_1 d + a_2 - d^2)},$$

$$A_1 = -\frac{1}{(a_1 d + a_2 - d^2)} (c_1 b d^2 - 2(a_1 d + 2a_2) A_2),$$

$$A_0 = -\frac{1}{(a_1 d + a_2 - d^2)} (c_0 b d^2 - (a_1 d + 2a_2) A_1 + (a_1 d + 4a_2) A_2),$$

i.e.,

$$A_2 = \frac{c_2 b d^2}{(a_1 d + a_2 - d^2)},$$

$$A_1 = \frac{b d^2}{(a_1 d + a_2 - d^2)^2} (-c_1 d^2 + a_1 (c_1 + 2c_2) d + a_2 (c_1 + 4c_2)),$$

$$A_0 = -\frac{b d^2}{(a_1 d + a_2 - d^2)^3} (c_0 d^4 + a_1^2 (c_0 + c_1 + c_2) d^2 + a_2^2 (c_0 + 2c_1 + 4c_2) - a_1 (2c_0 + c_1 - c_2) d^3 - 2a_2 (c_0 + c_1 - 2c_2) d^2 + a_1 a_2 (2c_0 + 3c_1 + 3c_2) d).$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_2 n^2 + A_1 n + A_0) d^n$$

where

$$A_2 = \frac{c_2 b d^2}{3(a_1 d + 2a_2)},$$

$$A_1 = \frac{1}{2(a_1 d + 2a_2)} (c_1 b d^2 + 3(a_1 d + 4a_2) A_2),$$

$$A_0 = \frac{1}{(a_1 d + 2a_2)} (c_0 b d^2 + (a_1 d + 4a_2) A_1 - (a_1 d + 8a_2) A_2),$$

i.e.,

$$A_2 = \frac{c_2 b d^2}{3(a_1 d + 2a_2)},$$

$$A_1 = \frac{b d^2}{2(a_1 d + 2a_2)^2} (a_1(c_1 + c_2)d + 2a_2(c_1 + 2c_2)),$$

$$A_0 = \frac{b d^2}{6(a_1 d + 2a_2)^3} (a_1^2(6c_0 + 3c_1 + c_2)d^2 + 8a_2^2(3c_0 + 3c_1 + 2c_2) + 2a_1 a_2(12c_0 + 9c_1 + 2c_2)d).$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = n^2(A_2 n^2 + A_1 n + A_0)d^n$$

where

$$A_2 = \frac{1}{12} b c_2,$$

$$A_1 = \frac{1}{6} b (c_1 + 2c_2),$$

$$A_0 = \frac{1}{12} b (6c_0 + 6c_1 + 5c_2).$$

(d): $m = 2, s = 3$. Consider the sequence (W_n) defined by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + (c_3 n^3 + c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_3 n^3 + A_2 n^2 + A_1 n + A_0) d^n$$

where

$$A_3 = -\frac{c_3 b d^2}{(a_1 d + a_2 - d^2)},$$

$$A_2 = -\frac{1}{(a_1 d + a_2 - d^2)} (c_2 b d^2 - 3(a_1 d + 2a_2) A_3),$$

$$A_1 = -\frac{1}{(a_1 d + a_2 - d^2)} (c_1 b d^2 - 2(a_1 d + 2a_2) A_2 + 3(a_1 d + 4a_2) A_3),$$

$$A_0 = -\frac{1}{(a_1 d + a_2 - d^2)} (c_0 b d^2 - (a_1 d + 2a_2) A_1 + (a_1 d + 4a_2) A_2 - (a_1 d + 8a_2) A_3),$$

i.e.,

$$A_3 = -\frac{c_3 b d^2}{(a_1 d + a_2 - d^2)},$$

$$A_2 = -\frac{b d^2}{(a_1 d + a_2 - d^2)^2} (-c_2 d^2 + a_1(c_2 + 3c_3)d + a_2(c_2 + 6c_3)),$$

$$A_1 = -\frac{b d^2}{(a_1 d + a_2 - d^2)^3} (c_1 d^4 + a_1^2(c_1 + 2c_2 + 3c_3)d^2 + a_2^2(c_1 + 4c_2 + 12c_3) - a_1(2c_1 + 2c_2 - 3c_3)d^3 - 2a_2(c_1 + 2c_2 - 6c_3)d^2 + a_1 a_2(2c_1 + 6c_2 + 9c_3)d),$$

$$A_0 = -\frac{b d^2}{(a_1 d + a_2 - d^2)^4} (-c_0 d^6 + a_1^3(c_0 + c_1 + c_2 + c_3)d^3 + a_2^3(c_0 + 2c_1 + 4c_2 + 8c_3) - a_1^2(3c_0 + 2c_1 - 4c_3)d^4 - a_2^2(3c_0 + 4c_1 - 32c_3)d^2 + a_1(3c_0 + c_1 - c_2 + c_3)d^5 + a_2(3c_0 + 2c_1 - 4c_2 + 8c_3)d^4 - 2a_1 a_2(3c_0 + 3c_1 - c_2 - 9c_3)d^3 + a_1 a_2^2(3c_0 + 5c_1 + 7c_2 + 5c_3)d + a_1^2 a_2(3c_0 + 4c_1 + 4c_2 + 4c_3)d^2).$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{c_3bd^2}{4(a_1d + 2a_2)}, \\ A_2 &= \frac{1}{3(a_1d + 2a_2)}(c_2bd^2 + 6(a_1d + 4a_2)A_3), \\ A_1 &= \frac{1}{2(a_1d + 2a_2)}(c_1bd^2 + 3(a_1d + 4a_2)A_2 - 4(a_1d + 8a_2)A_3), \\ A_0 &= \frac{1}{(a_1d + 2a_2)}(c_0bd^2 + (a_1d + 4a_2)A_1 - (a_1d + 8a_2)A_2 + (a_1d + 16a_2)A_3), \end{aligned}$$

i.e.,

$$\begin{aligned} A_3 &= \frac{c_3bd^2}{4(a_1d + 2a_2)}, \\ A_2 &= \frac{bd^2}{6(a_1d + 2a_2)^2}(a_1(2c_2 + 3c_3)d + 4a_2(c_2 + 3c_3)), \\ A_1 &= \frac{bd^2}{4(a_1d + 2a_2)^3}(a_1^2(2c_1 + 2c_2 + c_3)d^2 + 8a_2^2(c_1 + 2c_2 + 2c_3) + 4a_1a_2(2c_1 + 3c_2 + c_3)d), \\ A_0 &= \frac{bd^2}{6(a_1d + 2a_2)^4}(a_1^3(6c_0 + 3c_1 + c_2)d^3 + 16a_2^3(3c_0 + 3c_1 + 2c_2) + 6a_1a_2^2(12c_0 + 10c_1 + 4c_2 - 3c_3)d + 6a_1^2a_2(6c_0 + 4c_1 + c_2)d^2). \end{aligned}$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = n^2(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{1}{20}bc_3, \\ A_2 &= \frac{1}{12}b(c_2 + 3c_3), \\ A_1 &= \frac{1}{12}b(2c_1 + 4c_2 + 5c_3), \\ A_0 &= \frac{1}{12}b(6c_0 + 6c_1 + 5c_2 + 3c_3). \end{aligned}$$

(e): $m = 2, s = 4$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_4 &= -\frac{c_4bd^2}{(a_1d + a_2 - d^2)}, \\
A_3 &= -\frac{1}{(a_1d + a_2 - d^2)} (c_3bd^2 - 4(a_1d + 2a_2)A_4), \\
A_2 &= -\frac{1}{(a_1d + a_2 - d^2)} (c_2bd^2 - 3(a_1d + 2a_2)A_3 + 6(a_1d + 4a_2)A_4), \\
A_1 &= -\frac{1}{(a_1d + a_2 - d^2)} (c_1bd^2 - 2(a_1d + 2a_2)A_2 + 3(a_1d + 4a_2)A_3 - 4(a_1d + 8a_2)A_4), \\
A_0 &= -\frac{1}{(a_1d + a_2 - d^2)} (c_0bd^2 - (a_1d + 2a_2)A_1 + (a_1d + 4a_2)A_2 - (a_1d + 8a_2)A_3 + (a_1d + 16a_2)A_4),
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_4 &= -\frac{c_4bd^2}{(a_1d + a_2 - d^2)}, \\
A_3 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^2} (-c_3d^2 + a_1(4c_4 + c_3)d + a_2(8c_4 + c_3)), \\
A_2 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^3} (c_2d^4 + a_1^2(6c_4 + 3c_3 + c_2)d^2 + a_2^2(24c_4 + 6c_3 + c_2) + a_1(6c_4 - 3c_3 - 2c_2)d^3 + 2a_2(12c_4 - 3c_3 - c_2)d^2 + a_1a_2(18c_4 + 9c_3 + 2c_2)d), \\
A_1 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^4} (-c_1d^6 + a_1^3(4c_4 + 3c_3 + 2c_2 + c_1)d^3 + a_2^3(32c_4 + 12c_3 + 4c_2 + c_1) + a_1^2(16c_4 - 4c_2 - 3c_1)d^4 + a_2^2(128c_4 - 8c_2 - 3c_1)d^2 + a_1(4c_4 - 3c_3 + 2c_2 + 3c_1)d^5 + a_2(32c_4 - 12c_3 + 4c_2 + 3c_1)d^4 + 6a_1a_2(12c_4 + c_3 - 2c_2 - c_1)d^3 + a_1a_2^2(20c_4 + 21c_3 + 10c_2 + 3c_1)d + a_1^2a_2(16c_4 + 12c_3 + 8c_2 + 3c_1)d^2), \\
A_0 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^5} (c_0d^8 + a_1^4(c_4 + c_3 + c_2 + c_1 + c_0)d^4 + a_2^4(16c_4 + 8c_3 + 4c_2 + 2c_1 + c_0) + a_1^3(11c_4 + 3c_3 - c_2 - 3c_1 - 4c_0)d^5 + 2a_2^3(88c_4 + 12c_3 - 2c_2 - 3c_1 - 2c_0)d^2 + a_1^2(11c_4 - 3c_3 - c_2 + 3c_1 + 6c_0)d^6 + 2a_2^2(88c_4 - 12c_3 - 2c_2 + 3c_1 + 3c_0)d^4 + a_1(c_4 - c_3 + c_2 - c_1 - 4c_0)d^7 + 2a_2(8c_4 - 4c_3 + 2c_2 - c_1 - 2c_0)d^6 + a_1a_2^2(115c_4 + 45c_3 - 5c_2 - 15c_1 - 12c_0)d^3 + 2a_1^2a_2(29c_4 + 9c_3 - c_2 - 6c_1 - 6c_0)d^4 - a_1a_2^3(c_4 - 13c_3 - 11c_2 - 7c_1 - 4c_0)d + a_1^3a_2(5c_4 + 5c_3 + 5c_2 + 5c_1 + 4c_0)d^3 + a_1^2a_2^2(11c_4 + 9c_3 + 11c_2 + 9c_1 + 6c_0)d^2 + a_1a_2(77c_4 - 9c_3 - 7c_2 + 9c_1 + 12c_0)d^5).
\end{aligned}$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_4 &= \frac{c_4bd^2}{5(a_1d + 2a_2)}, \\
A_3 &= \frac{1}{4(a_1d + 2a_2)} (c_3bd^2 + 10(a_1d + 4a_2)A_4), \\
A_2 &= \frac{1}{3(a_1d + 2a_2)} (c_2bd^2 + 6(a_1d + 4a_2)A_3 - 10(a_1d + 8a_2)A_4), \\
A_1 &= \frac{1}{2(a_1d + 2a_2)} (c_1bd^2 + 3(a_1d + 4a_2)A_2 - 4(a_1d + 8a_2)A_3 + 5(a_1d + 16a_2)A_4), \\
A_0 &= \frac{1}{(a_1d + 2a_2)} (c_0bd^2 + (a_1d + 4a_2)A_1 - (a_1d + 8a_2)A_2 + (a_1d + 16a_2)A_3 - (a_1d + 32a_2)A_4),
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_4 &= \frac{c_4bd^2}{5(a_1d + 2a_2)}, \\
A_3 &= \frac{bd^2}{4(a_1d + 2a_2)^2}(a_1(c_3 + 2c_4)d + 2a_2(c_3 + 4c_4)), \\
A_2 &= \frac{bd^2}{6(a_1d + 2a_2)^3}(a_1^2(2c_2 + 3c_3 + 2c_4)d^2 + 8a_2^2(c_2 + 3c_3 + 4c_4) + 2a_1a_2(4c_2 + 9c_3 + 4c_4)d), \\
A_1 &= \frac{bd^2}{4(a_1d + 2a_2)^4}(a_1^3(2c_1 + 2c_2 + c_3)d^3 + 16a_2^3(c_1 + 2c_2 + 2c_3) + 8a_1a_2^2(3c_1 + 5c_2 + 3c_3 - 3c_4)d + \\
&2a_1^2a_2(6c_1 + 8c_2 + 3c_3)d^2), \\
A_0 &= \frac{bd^2}{30(a_1d + 2a_2)^5}(a_1^4(30c_0 + 15c_1 + 5c_2 - c_4)d^4 + 32a_2^4(15c_0 + 15c_1 + 10c_2 - 8c_4) + 4a_1a_2^3(240c_0 + \\
&210c_1 + 100c_2 - 45c_3 - 68c_4)d + 2a_1^3a_2(120c_0 + 75c_1 + 20c_2 - 4c_4)d^3 + 6a_1^2a_2^2(120c_0 + 90c_1 + 30c_2 - \\
&15c_3 + 16c_4)d^2).
\end{aligned}$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = n^2(A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_4 &= \frac{1}{30}bc_4, \\
A_3 &= \frac{1}{20}b(c_3 + 4c_4), \\
A_2 &= \frac{1}{12}b(c_2 + 3c_3 + 5c_4), \\
A_1 &= \frac{1}{12}b(2c_1 + 4c_2 + 5c_3 + 4c_4), \\
A_0 &= \frac{1}{60}b(30c_0 + 30c_1 + 25c_2 + 15c_3 + 3c_4).
\end{aligned}$$

(f): $m = 2, s = 5$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_5 &= -\frac{c_5bd^2}{a_1d + a_2 - d^2}, \\
A_4 &= -\frac{1}{a_1d + a_2 - d^2}(c_4bd^2 - 5(a_1d + 2a_2)A_5), \\
A_3 &= -\frac{1}{a_1d + a_2 - d^2}(c_3bd^2 - 4(a_1d + 2a_2)A_4 + 10(a_1d + 4a_2)A_5), \\
A_2 &= -\frac{1}{a_1d + a_2 - d^2}(c_2bd^2 - 3(a_1d + 2a_2)A_3 + 6(a_1d + 4a_2)A_4 - 10(a_1d + 8a_2)A_5), \\
A_1 &= -\frac{1}{a_1d + a_2 - d^2}(c_1bd^2 - 2(a_1d + 2a_2)A_2 + 3(a_1d + 4a_2)A_3 - 4(a_1d + 8a_2)A_4 + 5(a_1d + 16a_2)A_5), \\
A_0 &= -\frac{1}{a_1d + a_2 - d^2}(c_0bd^2 - (a_1d + 2a_2)A_1 + (a_1d + 4a_2)A_2 - (a_1d + 8a_2)A_3 + (a_1d + 16a_2)A_4 - \\
&(a_1d + 32a_2)A_5),
\end{aligned}$$

i.e.,

$$A_5 = -\frac{c_5bd^2}{a_1d + a_2 - d^2},$$

$$\begin{aligned}
A_4 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^2}(-c_4d^2 + a_1(c_4 + 5c_5)d + a_2(c_4 + 10c_5)), \\
A_3 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^3}(c_3d^4 + a_1^2(c_3 + 4c_4 + 10c_5)d^2 + a_2^2(c_3 + 8c_4 + 40c_5) - 2a_1(c_3 + 2c_4 - 5c_5)d^3 - \\
&\quad 2a_2(c_3 + 4c_4 - 20c_5)d^2 + 2a_1a_2(c_3 + 6c_4 + 15c_5)d), \\
A_2 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^4}(-c_2d^6 + a_1^3(c_2 + 3c_3 + 6c_4 + 10c_5)d^3 + a_2^3(c_2 + 6c_3 + 24c_4 + 80c_5) - a_1^2(3c_2 + \\
&\quad 6c_3 - 40c_5)d^4 - a_2^2(3c_2 + 12c_3 - 320c_5)d^2 + a_1(3c_2 + 3c_3 - 6c_4 + 10c_5)d^5 + a_2(3c_2 + 6c_3 - 24c_4 + 80c_5)d^4 - \\
&\quad 6a_1a_2(c_2 + 3c_3 - 2c_4 - 30c_5)d^3 + a_1a_2^2(3c_2 + 15c_3 + 42c_4 + 50c_5)d + a_1^2a_2(3c_2 + 12c_3 + 24c_4 + 40c_5)d^2), \\
A_1 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^5}(c_1d^8 + a_1^4(c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5)d^4 + a_2^4(c_1 + 4c_2 + 12c_3 + 32c_4 + 80c_5) - \\
&\quad a_1^3(4c_1 + 6c_2 + 3c_3 - 12c_4 - 55c_5)d^5 - 4a_2^3(c_1 + 3c_2 + 3c_3 - 24c_4 - 220c_5)d^2 + a_1^2(6c_1 + 6c_2 - 3c_3 - \\
&\quad 12c_4 + 55c_5)d^6 + 2a_2^2(3c_1 + 6c_2 - 6c_3 - 48c_4 + 440c_5)d^4 - a_1(4c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5)d^7 - 4a_2(c_1 + \\
&\quad c_2 - 3c_3 + 8c_4 - 20c_5)d^6 - a_1a_2^2(12c_1 + 30c_2 + 15c_3 - 180c_4 - 575c_5)d^3 - 2a_1^2a_2(6c_1 + 12c_2 + 3c_3 - \\
&\quad 36c_4 - 145c_5)d^4 + a_1a_2^3(4c_1 + 14c_2 + 33c_3 + 52c_4 - 5c_5)d + a_1^3a_2(4c_1 + 10c_2 + 15c_3 + 20c_4 + 25c_5) \\
&\quad d^3 + a_1^2a_2^2(6c_1 + 18c_2 + 33c_3 + 36c_4 + 55c_5)d^2 + a_1a_2(12c_1 + 18c_2 - 21c_3 - 36c_4 + 385c_5)d^5), \\
A_0 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^6}(-c_0d^{10} + a_1(5c_0 + c_1 - c_2 + c_3 - c_4 + c_5)d^9 - 2a_1^2(5c_0 + 2c_1 - c_2 - c_3 + 5c_4 - \\
&\quad 13c_5)d^8 + 2a_1^3(5c_0 + 3c_1 - 3c_3 + 33c_5)d^7 - a_1^4(5c_0 + 4c_1 + 2c_2 - 2c_3 - 10c_4 - 26c_5)d^6 + a_1^5(c_0 + c_1 + c_2 + \\
&\quad c_3 + c_4 + c_5)d^5 + a_2(5c_0 + 2c_1 - 4c_2 + 8c_3 - 16c_4 + 32c_5)d^8 - 2a_2^2(5c_0 + 4c_1 - 4c_2 - 8c_3 + 80c_4 - 416c_5)d^6 + \\
&\quad 2a_2^3(5c_0 + 6c_1 - 24c_3 + 1056c_5)d^4 - a_2^4(5c_0 + 8c_1 + 8c_2 - 16c_3 - 160c_4 - 832c_5)d^2 + a_2^5(c_0 + 2c_1 + 4c_2 + 8c_3 + \\
&\quad 16c_4 + 32c_5) + 6a_1a_2^2(5c_0 + 5c_1 - c_2 - 13c_3 + 23c_4 + 215c_5)d^5 + 6a_1^2a_2(5c_0 + 4c_1 - c_2 - 5c_3 + 5c_4 + 79c_5) \\
&\quad d^6 - 4a_1a_2^3(5c_0 + 7c_1 + 5c_2 - 14c_3 - 73c_4 - 98c_5)d^3 - 4a_1^3a_2(5c_0 + 5c_1 + 2c_2 - 4c_3 - 16c_4 - 40c_5) \\
&\quad d^5 + a_1a_2^4(5c_0 + 9c_1 + 15c_2 + 21c_3 + 15c_4 - 51c_5)d + a_1^4a_2(5c_0 + 6c_1 + 6c_2 + 6c_3 + 6c_4 + 6c_5)d^4 - \\
&\quad 6a_1^2a_2^2(5c_0 + 6c_1 + 3c_2 - 9c_3 - 27c_4 - 69c_5)d^4 + 2a_1^2a_2^3(5c_0 + 8c_1 + 11c_2 + 11c_3 + 5c_4 + 23c_5)d^2 + \\
&\quad 2a_1^3a_2^2(5c_0 + 7c_1 + 8c_2 + 7c_3 + 8c_4 + 7c_5)d^3 - 4a_1a_2(5c_0 + 3c_1 - 3c_2 + 15c_4 - 72c_5)d^7),
\end{aligned}$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_5 &= \frac{c_5bd^2}{6(a_1d + 2a_2)}, \\
A_4 &= \frac{1}{5(a_1d + 2a_2)}(c_4bd^2 + 15(a_1d + 4a_2)A_5), \\
A_3 &= \frac{1}{4(a_1d + 2a_2)}(c_3bd^2 + 10(a_1d + 4a_2)A_4 - 20(a_1d + 8a_2)A_5), \\
A_2 &= \frac{1}{3(a_1d + 2a_2)}(c_2bd^2 + 6(a_1d + 4a_2)A_3 - 10(a_1d + 8a_2)A_4 + 15(a_1d + 16a_2)A_5), \\
A_1 &= \frac{1}{2(a_1d + 2a_2)}(c_1bd^2 + 3(a_1d + 4a_2)A_2 - 4(a_1d + 8a_2)A_3 + 5(a_1d + 16a_2)A_4 - 6(a_1d + 32a_2)A_5), \\
A_0 &= \frac{1}{(a_1d + 2a_2)}(c_0bd^2 + (a_1d + 4a_2)A_1 - (a_1d + 8a_2)A_2 + (a_1d + 16a_2)A_3 - (a_1d + 32a_2)A_4 + \\
&\quad (a_1d + 64a_2)A_5),
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_5 &= \frac{c_5bd^2}{6(a_1d + 2a_2)}, \\
A_4 &= \frac{bd^2}{10(a_1d + 2a_2)^2}(a_1(2c_4 + 5c_5)d + 4a_2(c_4 + 5c_5)),
\end{aligned}$$

$$\begin{aligned}
A_3 &= \frac{bd^2}{12(a_1d + 2a_2)^3} (a_1^2(3c_3 + 6c_4 + 5c_5)d^2 + 4a_2^2(3c_3 + 12c_4 + 20c_5) + 4a_1a_2(3c_3 + 9c_4 + 5c_5)d), \\
A_2 &= \frac{bd^2}{6(a_1d + 2a_2)^4} (a_1^3(2c_2 + 3c_3 + 2c_4)d^3 + 16a_2^3(c_2 + 3c_3 + 4c_4) + 12a_1a_2^2(2c_2 + 5c_3 + 4c_4 - 5c_5)d + \\
&\quad 12a_1^2a_2(c_2 + 2c_3 + c_4)d^2), \\
A_1 &= \frac{bd^2}{12(a_1d + 2a_2)^5} (a_1^4(6c_1 + 6c_2 + 3c_3 - c_5)d^4 + 32a_2^4(3c_1 + 6c_2 + 6c_3 - 8c_5) + 16a_1a_2^3(12c_1 + 21c_2 + \\
&\quad 15c_3 - 9c_4 - 17c_5)d + 4a_1^3a_2(12c_1 + 15c_2 + 6c_3 - 2c_5)d^3 + 12a_1^2a_2^2(12c_1 + 18c_2 + 9c_3 - 6c_4 + 8c_5)d^2), \\
A_0 &= \frac{bd^2}{30(a_1d + 2a_2)^6} (a_1^5(30c_0 + 15c_1 + 5c_2 - c_4)d^5 + 64a_2^5(15c_0 + 15c_1 + 10c_2 - 8c_4) + 40a_1a_2^4(60c_0 + \\
&\quad 54c_1 + 28c_2 - 9c_3 - 20c_4 + 30c_5)d + 10a_1^4a_2(30c_0 + 18c_1 + 5c_2 - c_4)d^4 + 40a_1^3a_2^2(60c_0 + 48c_1 + 19c_2 - \\
&\quad 9c_3 - 2c_4 + 30c_5)d^2 + 10a_1^2a_2^3(120c_0 + 84c_1 + 26c_2 - 9c_3 + 8c_4 - 15c_5)d^3).
\end{aligned}$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = n^2(A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_5 &= \frac{1}{42}bc_5, \\
A_4 &= \frac{1}{30}b(c_4 + 5c_5), \\
A_3 &= \frac{1}{60}b(3c_3 + 12c_4 + 25c_5), \\
A_2 &= \frac{1}{12}b(c_2 + 3c_3 + 5c_4 + 5c_5), \\
A_1 &= \frac{1}{12}b(2c_1 + 4c_2 + 5c_3 + 4c_4 + c_5), \\
A_0 &= \frac{1}{60}b(30c_0 + 30c_1 + 25c_2 + 15c_3 + 3c_4 - 5c_5).
\end{aligned}$$

(g): $m = 2, s = 6$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_6 &= -\frac{c_6bd^2}{a_1d + a_2 - d^2}, \\
A_5 &= -\frac{1}{a_1d + a_2 - d^2} (c_5bd^2 - 6(a_1d + 2a_2)A_6), \\
A_4 &= -\frac{1}{a_1d + a_2 - d^2} (c_4bd^2 - 5(a_1d + 2a_2)A_5 + 15(a_1d + 4a_2)A_6), \\
A_3 &= -\frac{1}{a_1d + a_2 - d^2} (c_3bd^2 - 4(a_1d + 2a_2)A_4 + 10(a_1d + 4a_2)A_5 - 20(a_1d + 8a_2)A_6), \\
A_2 &= -\frac{1}{a_1d + a_2 - d^2} (c_2bd^2 - 3(a_1d + 2a_2)A_3 + 6(a_1d + 4a_2)A_4 - 10(a_1d + 8a_2)A_5 + 15(a_1d + 16a_2)A_6), \\
A_1 &= -\frac{1}{a_1d + a_2 - d^2} (c_1bd^2 - 2(a_1d + 2a_2)A_2 + 3(a_1d + 4a_2)A_3 - 4(a_1d + 8a_2)A_4 + 5(a_1d + 16a_2)A_5 - \\
&\quad 6A_6(a_1d + 32a_2)), \\
A_0 &= -\frac{1}{a_1d + a_2 - d^2} (c_0bd^2 - (a_1d + 2a_2)A_1 + (a_1d + 4a_2)A_2 - (a_1d + 8a_2)A_3 + (a_1d + 16a_2)A_4 - \\
&\quad (a_1d + 32a_2)A_5 + (a_1d + 64a_2)A_6),
\end{aligned}$$

i. e.,

$$\begin{aligned}
A_6 &= -\frac{c_6bd^2}{a_1d + a_2 - d^2}, \\
A_5 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^2}(-c_5d^2 + a_1(c_5 + 6c_6)d + a_2(c_5 + 12c_6)), \\
A_4 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^3}(c_4d^4 - a_1(2c_4 + 5c_5 - 15c_6)d^3 + a_1^2(c_4 + 5c_5 + 15c_6)d^2 - 2a_2(c_4 + 5c_5 - 30c_6)d^2 + \\
& a_2^2(c_4 + 10c_5 + 60c_6) + a_1a_2(2c_4 + 15c_5 + 45c_6)d), \\
A_3 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^4}(-c_3d^6 + a_1(3c_3 + 4c_4 - 10c_5 + 20c_6)d^5 - a_1^2(3c_3 + 8c_4 - 80c_6)d^4 + a_1^3(c_3 + 4c_4 + \\
& 10c_5 + 20c_6)d^3 + a_2(3c_3 + 8c_4 - 40c_5 + 160c_6)d^4 - a_2^2(3c_3 + 16c_4 - 640c_6)d^2 + a_2^3(c_3 + 8c_4 + 40c_5 + 160c_6) + \\
& a_1a_2^2(3c_3 + 20c_4 + 70c_5 + 100c_6)d + a_1^2a_2(3c_3 + 16c_4 + 40c_5 + 80c_6)d^2 - 2a_1a_2(3c_3 + 12c_4 - 10c_5 - 180c_6)d^3), \\
A_2 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^5}(c_2d^8 - a_1(4c_2 + 3c_3 - 6c_4 + 10c_5 - 15c_6)d^7 + 3a_1^2(2c_2 + 3c_3 - 2c_4 - 10c_5 + 55c_6)d^6 \\
& - a_1^3(4c_2 + 9c_3 + 6c_4 - 30c_5 - 165c_6)d^5 + a_1^4(c_2 + 3c_3 + 6c_4 + 10c_5 + 15c_6)d^4 - 2a_2(2c_2 + 3c_3 - 12c_4 + \\
& 40c_5 - 120c_6)d^6 + 6a_2^2(c_2 + 3c_3 - 4c_4 - 40c_5 + 440c_6)d^4 - 2a_2^3(2c_2 + 9c_3 + 12c_4 - 120c_5 - 1320c_6)d^2 \\
& + a_2^4(c_2 + 6c_3 + 24c_4 + 80c_5 + 240c_6) - 3a_1a_2^2(4c_2 + 15c_3 + 10c_4 - 150c_5 - 575c_6)d^3 - 6a_1^2a_2(2c_2 + 6c_3 + \\
& 2c_4 - 30c_5 - 145c_6)d^4 + a_1a_2^3(4c_2 + 21c_3 + 66c_4 + 130c_5 - 15c_6)d + a_1^3a_2(4c_2 + 15c_3 + 30c_4 + 50c_5 + 75c_6)d^3 \\
& + 3a_1^2a_2^2(2c_2 + 9c_3 + 22c_4 + 30c_5 + 55c_6)d^2 + 3a_1a_2(4c_2 + 9c_3 - 14c_4 - 30c_5 + 385c_6)d^5), \\
A_1 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^6}(-c_1d^{10} + a_1(5c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6)d^9 - 2a_1^2(5c_1 + 4c_2 - 3c_3 - 4c_4 + \\
& 25c_5 - 78c_6)d^8 + 2a_1^3(5c_1 + 6c_2 - 12c_4 + 198c_6)d^7 - a_1^4(5c_1 + 8c_2 + 6c_3 - 8c_4 - 50c_5 - 156c_6)d^6 + a_1^5(c_1 + \\
& 2c_2 + 3c_3 + 4c_4 + 5c_5 + 6c_6)d^5 + a_2(5c_1 + 4c_2 - 12c_3 + 32c_4 - 80c_5 + 192c_6)d^8 - 2a_2^2(5c_1 + 8c_2 - 12c_3 - 32c_4 + \\
& 400c_5 - 2496c_6)d^6 + 2a_2^3(5c_1 + 12c_2 - 96c_4 + 6336c_6)d^4 - a_2^4(5c_1 + 16c_2 + 24c_3 - 64c_4 - 800c_5 - 4992c_6) \\
& d^2 + a_2^5(c_1 + 4c_2 + 12c_3 + 32c_4 + 80c_5 + 192c_6) + 6a_1a_2^2(5c_1 + 10c_2 - 3c_3 - 52c_4 + 115c_5 + 1290c_6)d^5 + \\
& 6a_1^2a_2(5c_1 + 8c_2 - 3c_3 - 20c_4 + 25c_5 + 474c_6)d^6 - 4a_1a_2^3(5c_1 + 14c_2 + 15c_3 - 56c_4 - 365c_5 - 588c_6) \\
& d^3 - 4a_1^3a_2(5c_1 + 10c_2 + 6c_3 - 16c_4 - 80c_5 - 240c_6)d^5 + a_1a_2^4(5c_1 + 18c_2 + 45c_3 + 84c_4 + 75c_5 - 306c_6) \\
& d + a_1^4a_2(5c_1 + 12c_2 + 18c_3 + 24c_4 + 30c_5 + 36c_6)d^4 - 6a_1^2a_2^2(5c_1 + 12c_2 + 9c_3 - 36c_4 - 135c_5 - 414c_6)d^4 + \\
& 2a_1^2a_2^3(5c_1 + 16c_2 + 33c_3 + 44c_4 + 25c_5 + 138c_6)d^2 + 2a_1^3a_2^2(5c_1 + 14c_2 + 24c_3 + 28c_4 + 40c_5 + 42c_6) \\
& d^3 - 4a_1a_2(5c_1 + 6c_2 - 9c_3 + 75c_5 - 432c_6)d^7), \\
A_0 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^7}(c_0d^{12} - a_1(6c_0 + c_1 - c_2 + c_3 - c_4 + c_5 - c_6)d^{11} + a_1^2(15c_0 + 5c_1 - 3c_2 - \\
& c_3 + 9c_4 - 25c_5 + 57c_6)d^{10} - 2a_1^3(10c_0 + 5c_1 - c_2 - 4c_3 + 5c_4 + 20c_5 - 151c_6)d^9 + a_1^4(15c_0 + 10c_1 + \\
& 2c_2 - 8c_3 - 10c_4 + 40c_5 + 302c_6)d^8 + a_1^5(-6c_0 - 5c_1 - 3c_2 + c_3 + 9c_4 + 25c_5 + 57c_6)d^7 + a_1^6(c_0 + \\
& c_1 + c_2 + c_3 + c_4 + c_5 + c_6)d^6 - 2a_2(3c_0 + c_1 - 2c_2 + 4c_3 - 8c_4 + 16c_5 - 32c_6)d^{10} + a_2^2(15c_0 + 10c_1 - \\
& 12c_2 - 8c_3 + 144c_4 - 800c_5 + 3648c_6)d^8 - 4a_2^3(5c_0 + 5c_1 - 2c_2 - 16c_3 + 40c_4 + 320c_5 - 4832c_6) \\
& d^6 + a_2^4(15c_0 + 20c_1 + 8c_2 - 64c_3 - 160c_4 + 1280c_5 + 19328c_6)d^4 - 2a_2^5(3c_0 + 5c_1 + 6c_2 - 4c_3 - \\
& 72c_4 - 400c_5 - 1824c_6)d^2 + a_2^6(c_0 + 2c_1 + 4c_2 + 8c_3 + 16c_4 + 32c_5 + 64c_6) - 2a_1a_2^2(30c_0 + 25c_1 - \\
& 13c_2 - 47c_3 + 179c_4 + 85c_5 - 5353c_6)d^7 - 4a_1^2a_2(15c_0 + 10c_1 - 5c_2 - 8c_3 + 25c_4 + 40c_5 - 755c_6)d^8 + \\
& 2a_1^3a_2(30c_0 + 25c_1 + c_2 - 26c_3 - 17c_4 + 190c_5 + 1171c_6)d^7 + 2a_1a_2^3(30c_0 + 35c_1 + 7c_2 - 91c_3 - 77c_4 + \\
& 1505c_5 + 5887c_6)d^5 - a_1a_2^4(30c_0 + 45c_1 + 43c_2 - 51c_3 - 437c_4 - 1275c_5 + 163c_6)d^3 - 2a_1^4a_2(15c_0 + \\
& 15c_1 + 8c_2 - 6c_3 - 34c_4 - 90c_5 - 202c_6)d^6 + a_1a_2^5(6c_0 + 11c_1 + 19c_2 + 29c_3 + 31c_4 - 19c_5 - 281c_6)d + \\
& a_1^5a_2(6c_0 + 7c_1 + 7c_2 + 7c_3 + 7c_4 + 7c_5 + 7c_6)d^5 + 6a_1^2a_2^2(15c_0 + 15c_1 + c_2 - 27c_3 + c_4 + 225c_5 + 1261c_6) \\
& d^6 - 4a_1^2a_2^3(15c_0 + 20c_1 + 15c_2 - 22c_3 - 111c_4 - 190c_5 - 615c_6)d^4 - 2a_1^3a_2^2(30c_0 + 35c_1 + 21c_2 - 28c_3 -
\end{aligned}$$

$$105c_4 - 280c_5 - 609c_6)d^5 + a_1^2 a_2^4 (15c_0 + 25c_1 + 37c_2 + 43c_3 + 25c_4 - 5c_5 + 337c_6)d^2 + a_1^4 a_2^2 (15c_0 + 20c_1 + 22c_2 + 20c_3 + 22c_4 + 20c_5 + 22c_6)d^4 + 2a_1^3 a_2^3 (10c_0 + 15c_1 + 19c_2 + 18c_3 + 13c_4 + 30c_5 - 11c_6) d^3 + a_1 a_2 (30c_0 + 15c_1 - 17c_2 + 9c_3 + 43c_4 - 25c_5 + 1003c_6)d^9).$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0) d^n$$

where

$$A_6 = \frac{c_6 b d^2}{7(a_1 d + 2a_2)},$$

$$A_5 = \frac{1}{6(a_1 d + 2a_2)} (c_5 b d^2 + 21(a_1 d + 4a_2)A_6),$$

$$A_4 = \frac{1}{5(a_1 d + 2a_2)} (c_4 b d^2 + 15(a_1 d + 4a_2)A_5 - 35(a_1 d + 8a_2)A_6),$$

$$A_3 = \frac{1}{4(a_1 d + 2a_2)} (c_3 b d^2 + 10(a_1 d + 4a_2)A_4 - 20(a_1 d + 8a_2)A_5 + 35(a_1 d + 16a_2)A_6),$$

$$A_2 = \frac{1}{3(a_1 d + 2a_2)} (c_2 b d^2 + 6(a_1 d + 4a_2)A_3 - 10(a_1 d + 8a_2)A_4 + 15(a_1 d + 16a_2)A_5 - 21(a_1 d + 32a_2)A_6),$$

$$A_1 = \frac{1}{2(a_1 d + 2a_2)} (c_1 b d^2 + 3(a_1 d + 4a_2)A_2 - 4(a_1 d + 8a_2)A_3 + 5(a_1 d + 16a_2)A_4 - 6(a_1 d + 32a_2)A_5 + 7(a_1 d + 64a_2)A_6),$$

$$A_0 = \frac{1}{(a_1 d + 2a_2)} (c_0 b d^2 + (a_1 d + 4a_2)A_1 - (a_1 d + 8a_2)A_2 + (a_1 d + 16a_2)A_3 - (a_1 d + 32a_2)A_4 + (a_1 d + 64a_2)A_5 - (a_1 d + 128a_2)A_6),$$

i.e.,

$$A_6 = \frac{c_6 b d^2}{7(a_1 d + 2a_2)},$$

$$A_5 = \frac{b d^2}{6(a_1 d + 2a_2)^2} (a_1(c_5 + 3c_6)d + 2a_2(c_5 + 6c_6)),$$

$$A_4 = \frac{b d^2}{10(a_1 d + 2a_2)^3} (a_1^2(2c_4 + 5c_5 + 5c_6)d^2 + 8a_2^2(c_4 + 5c_5 + 10c_6) + 2a_1 a_2(4c_4 + 15c_5 + 10c_6)d),$$

$$A_3 = \frac{b d^2}{12(a_1 d + 2a_2)^4} (a_1^3(3c_3 + 6c_4 + 5c_5)d^3 + 8a_2^3(3c_3 + 12c_4 + 20c_5) + 12a_1 a_2^2(3c_3 + 10c_4 + 10c_5 - 15c_6)d + 6a_1^2 a_2(3c_3 + 8c_4 + 5c_5)d^2),$$

$$A_2 = \frac{b d^2}{6(a_1 d + 2a_2)^5} (a_1^4(2c_2 + 3c_3 + 2c_4 - c_6)d^4 + 32a_2^4(c_2 + 3c_3 + 4c_4 - 8c_6) + 8a_1 a_2^3(8c_2 + 21c_3 + 20c_4 - 15c_5 - 34c_6)d + 2a_1^3 a_2(8c_2 + 15c_3 + 8c_4 - 4c_6)d^3 + 12a_1^2 a_2^2(4c_2 + 9c_3 + 6c_4 - 5c_5 + 8c_6)d^2),$$

$$A_1 = \frac{b d^2}{12(a_1 d + 2a_2)^6} (a_1^5(6c_1 + 6c_2 + 3c_3 - c_5)d^5 + 64a_2^5(3c_1 + 6c_2 + 6c_3 - 8c_5) + 32a_1 a_2^4(15c_1 + 27c_2 + 21c_3 - 9c_4 - 25c_5 + 45c_6)d + 2a_1^4 a_2(30c_1 + 36c_2 + 15c_3 - 5c_5)d^4 + 8a_1^2 a_2^3(60c_1 + 96c_2 + 57c_3 - 36c_4 - 10c_5 + 180c_6)d^2 + 4a_1^3 a_2^2(60c_1 + 84c_2 + 39c_3 - 18c_4 + 20c_5 - 45c_6)d^3),$$

$$A_0 = \frac{b d^2}{210(a_1 d + 2a_2)^7} (a_1^6(210c_0 + 105c_1 + 35c_2 - 7c_4 + 5c_6)d^6 + 128a_2^6(105c_0 + 105c_1 + 70c_2 - 56c_4 + 160c_6) + 48a_1 a_2^5(840c_0 + 770c_1 + 420c_2 - 105c_3 - 308c_4 + 350c_5 + 860c_6)d + 6a_1^5 a_2(420c_0 + 245c_1 + 70c_2 - 14c_4 + 10c_6)d^5 + 120a_1^2 a_2^4(420c_0 + 350c_1 + 154c_2 - 63c_3 - 56c_4 + 210c_5 - 53c_6)d^2 + 30a_1^4 a_2^2(420c_0 + 280c_1 + 84c_2 - 21c_3 + 14c_4 - 35c_5 + 52c_6)d^4 + 20a_1^3 a_2^3(1680c_0 + 1260c_1 + 448c_2 - 189c_3 + 28c_4 + 315c_5 - 1346c_6) d^3).$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = (c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0) b d^n$$

where

$$\begin{aligned}
A_6 &= \frac{1}{56}bc_6, \\
A_5 &= \frac{1}{42}b(c_5 + 6c_6), \\
A_4 &= \frac{1}{60}b(2c_4 + 10c_5 + 25c_6), \\
A_3 &= \frac{1}{60}b(3c_3 + 12c_4 + 25c_5 + 30c_6), \\
A_2 &= \frac{1}{24}b(2c_2 + 6c_3 + 10c_4 + 10c_5 + 3c_6), \\
A_1 &= \frac{1}{12}b(2c_1 + 4c_2 + 5c_3 + 4c_4 + c_5 - 2c_6), \\
A_0 &= \frac{1}{420}b(210c_0 + 210c_1 + 175c_2 + 105c_3 + 21c_4 - 35c_5 - 25c_6).
\end{aligned}$$

(h): $m = 2, s = 7$. Consider the sequence (W_n) defined by

$$W_n = a_1W_{n-1} + a_2W_{n-2} + (c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0)bd^m.$$

(i): Case $r = 0$ (no root equal to d):

$$W_n^C = (A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^m$$

where

$$\begin{aligned}
A_7 &= -\frac{c_7bd^2}{a_1d + a_2 - d^2}, \\
A_6 &= -\frac{1}{a_1d + a_2 - d^2}(c_6bd^2 - 7(a_1d + 2a_2)A_7), \\
A_5 &= -\frac{1}{a_1d + a_2 - d^2}(c_5bd^2 - 6(a_1d + 2a_2)A_6 + 21(a_1d + 4a_2)A_7), \\
A_4 &= -\frac{1}{a_1d + a_2 - d^2}(c_4bd^2 - 5(a_1d + 2a_2)A_5 + 15(a_1d + 4a_2)A_6 - 35(a_1d + 8a_2)A_7), \\
A_3 &= -\frac{1}{a_1d + a_2 - d^2}(c_3bd^2 - 4(a_1d + 2a_2)A_4 + 10(a_1d + 4a_2)A_5 - 20(a_1d + 8a_2)A_6 + 35(a_1d + 16a_2)A_7), \\
A_2 &= -\frac{1}{a_1d + a_2 - d^2}(c_2bd^2 - 3(a_1d + 2a_2)A_3 + 6(a_1d + 4a_2)A_4 - 10(a_1d + 8a_2)A_5 + 15(a_1d + 16a_2)A_6 - 21(a_1d + 32a_2)A_7), \\
A_1 &= -\frac{1}{a_1d + a_2 - d^2}(c_1bd^2 - 2(a_1d + 2a_2)A_2 + 3(a_1d + 4a_2)A_3 - 4(a_1d + 8a_2)A_4 + 5(a_1d + 16a_2)A_5 - 6(a_1d + 32a_2)A_6 + 7(a_1d + 64a_2)A_7), \\
A_0 &= -\frac{1}{a_1d + a_2 - d^2}(c_0bd^2 - (a_1d + 2a_2)A_1 + (a_1d + 4a_2)A_2 - (a_1d + 8a_2)A_3 + (a_1d + 16a_2)A_4 - (a_1d + 32a_2)A_5 + (a_1d + 64a_2)A_6 - (a_1d + 128a_2)A_7),
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_7 &= -\frac{c_7bd^2}{a_1d + a_2 - d^2}, \\
A_6 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^2}(-c_6d^2 + a_1(c_6 + 7c_7)d + a_2(c_6 + 14c_7)), \\
A_5 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^3}(c_5d^4 - a_1(2c_5 + 6c_6 - 21c_7)d^3 + a_1^2(c_5 + 6c_6 + 21c_7)d^2 - 2a_2(c_5 + 6c_6 - 42c_7)d^2 + a_2^2(c_5 + 12c_6 + 84c_7) + a_1a_2(2c_5 + 18c_6 + 63c_7)d), \\
A_4 &= -\frac{bd^2}{(a_1d + a_2 - d^2)^4}(-c_4d^6 + a_1(3c_4 + 5c_5 - 15c_6 + 35c_7)d^5 - a_1^2(3c_4 + 10c_5 - 140c_7)d^4 + a_1^3(c_4 + 5c_5 + 15c_6 + 35c_7)d^3 + a_2(3c_4 + 10c_5 - 60c_6 + 280c_7)d^4 - a_2^2(3c_4 + 20c_5 - 1120c_7)d^2 +
\end{aligned}$$

$$a_2^3(c_4 + 10c_5 + 60c_6 + 280c_7) + a_1a_2^2(3c_4 + 25c_5 + 105c_6 + 175c_7)d + a_1^2a_2(3c_4 + 20c_5 + 60c_6 + 140c_7)d^2 - 6a_1a_2(c_4 + 5c_5 - 5c_6 - 105c_7)d^3),$$

$$A_3 = -\frac{bd^2}{(a_1d + a_2 - d^2)^5}(c_3d^8 - a_1(4c_3 + 4c_4 - 10c_5 + 20c_6 - 35c_7)d^7 + a_1^2(6c_3 + 12c_4 - 10c_5 - 60c_6 + 385c_7)d^6 - a_1^3(4c_3 + 12c_4 + 10c_5 - 60c_6 - 385c_7)d^5 + a_1^4(c_3 + 4c_4 + 10c_5 + 20c_6 + 35c_7)d^4 - 4a_2(c_3 + 2c_4 - 10c_5 + 40c_6 - 140c_7)d^6 + 2a_2^2(3c_3 + 12c_4 - 20c_5 - 240c_6 + 3080c_7)d^4 - 4a_2^3(c_3 + 6c_4 + 10c_5 - 120c_6 - 1540c_7)d^2 + a_2^4(c_3 + 8c_4 + 40c_5 + 160c_6 + 560c_7) - a_1a_2^2(12c_3 + 60c_4 + 50c_5 - 900c_6 - 4025c_7)d^3 - 2a_1^2a_2(6c_3 + 24c_4 + 10c_5 - 180c_6 - 1015c_7)d^4 + a_1a_2^3(4c_3 + 28c_4 + 110c_5 + 260c_6 - 35c_7)d + a_1^3a_2(4c_3 + 20c_4 + 50c_5 + 100c_6 + 175c_7)d^3 + a_1^2a_2^2(6c_3 + 36c_4 + 110c_5 + 180c_6 + 385c_7)d^2 + a_1a_2(12c_3 + 36c_4 - 70c_5 - 180c_6 + 2695c_7)d^5),$$

$$A_2 = -\frac{bd^2}{(a_1d + a_2 - d^2)^6}(-c_2d^{10} + a_1(5c_2 + 3c_3 - 6c_4 + 10c_5 - 15c_6 + 21c_7)d^9 - 2a_1^2(5c_2 + 6c_3 - 6c_4 - 10c_5 + 75c_6 - 273c_7)d^8 + 2a_1^3(5c_2 + 9c_3 - 30c_5 + 693c_7)d^7 - a_1^4(5c_2 + 12c_3 + 12c_4 - 20c_5 - 150c_6 - 546c_7)d^6 + a_1^5(c_2 + 3c_3 + 6c_4 + 10c_5 + 15c_6 + 21c_7)d^5 + a_2(5c_2 + 6c_3 - 24c_4 + 80c_5 - 240c_6 + 672c_7)d^8 - 2a_2^2(5c_2 + 12c_3 - 24c_4 - 80c_5 + 1200c_6 - 8736c_7)d^6 + 2a_2^3(5c_2 + 18c_3 - 240c_5 + 22176c_7)d^4 - a_2^4(5c_2 + 24c_3 + 48c_4 - 160c_5 - 2400c_6 - 17472c_7)d^2 + a_2^5(c_2 + 6c_3 + 24c_4 + 80c_5 + 240c_6 + 672c_7) + 6a_1a_2^2(5c_2 + 15c_3 - 6c_4 - 130c_5 + 345c_6 + 4515c_7)d^5 + 6a_1^2a_2(5c_2 + 12c_3 - 6c_4 - 50c_5 + 75c_6 + 1659c_7)d^6 - 4a_1a_2^3(5c_2 + 21c_3 + 30c_4 - 140c_5 - 1095c_6 - 2058c_7)d^3 - 4a_1^3a_2(5c_2 + 15c_3 + 12c_4 - 40c_5 - 240c_6 - 840c_7)d^5 + a_1a_2^4(5c_2 + 27c_3 + 90c_4 + 210c_5 + 225c_6 - 1071c_7)d + a_1^4a_2(5c_2 + 18c_3 + 36c_4 + 60c_5 + 90c_6 + 126c_7)d^4 - 6a_1^2a_2^2(5c_2 + 18c_3 + 18c_4 - 90c_5 - 405c_6 - 1449c_7)d^4 + 2a_1^2a_2^3(5c_2 + 24c_3 + 66c_4 + 110c_5 + 75c_6 + 483c_7)d^2 + 2a_1^3a_2^2(5c_2 + 21c_3 + 48c_4 + 70c_5 + 120c_6 + 147c_7)d^3 - 4a_1a_2(5c_2 + 9c_3 - 18c_4 + 225c_6 - 1512c_7)d^7),$$

$$A_1 = -\frac{bd^2}{(a_1d + a_2 - d^2)^7}(c_1d^{12} - a_1(6c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6 - 7c_7)d^{11} + a_1^2(15c_1 + 10c_2 - 9c_3 - 4c_4 + 45c_5 - 150c_6 + 399c_7)d^{10} - 2a_1^3(10c_1 + 10c_2 - 3c_3 - 16c_4 + 25c_5 + 120c_6 - 1057c_7)d^9 + a_1^4(15c_1 + 20c_2 + 6c_3 - 32c_4 - 50c_5 + 240c_6 + 2114c_7)d^8 - a_1^5(6c_1 + 10c_2 + 9c_3 - 4c_4 - 45c_5 - 150c_6 - 399c_7)d^7 + a_1^6(c_1 + 2c_2 + 3c_3 + 4c_4 + 5c_5 + 6c_6 + 7c_7)d^6 - 2a_2(3c_1 + 2c_2 - 6c_3 + 16c_4 - 40c_5 + 96c_6 - 224c_7)d^{10} + a_2^2(15c_1 + 20c_2 - 36c_3 - 32c_4 + 720c_5 - 4800c_6 + 25536c_7)d^8 - 4a_2^3(5c_1 + 10c_2 - 6c_3 - 64c_4 + 200c_5 + 1920c_6 - 33824c_7)d^6 + a_2^4(15c_1 + 40c_2 + 24c_3 - 256c_4 - 800c_5 + 7680c_6 + 135296c_7)d^4 - 2a_2^5(3c_1 + 10c_2 + 18c_3 - 16c_4 - 360c_5 - 2400c_6 - 12768c_7)d^2 + a_2^6(c_1 + 4c_2 + 12c_3 + 32c_4 + 80c_5 + 192c_6 + 448c_7) - 2a_1a_2^2(30c_1 + 50c_2 - 39c_3 - 188c_4 + 895c_5 + 510c_6 - 37471c_7)d^7 - 4a_1^2a_2(15c_1 + 20c_2 - 15c_3 - 32c_4 + 125c_5 + 240c_6 - 5285c_7)d^8 + 2a_1a_2^3(30c_1 + 70c_2 + 21c_3 - 364c_4 - 385c_5 + 9030c_6 + 41209c_7)d^5 + 2a_1^3a_2(30c_1 + 50c_2 + 3c_3 - 104c_4 - 85c_5 + 1140c_6 + 8197c_7)d^7 - a_1a_2^4(30c_1 + 90c_2 + 129c_3 - 204c_4 - 2185c_5 - 7650c_6 + 1141c_7)d^3 - 2a_1^4a_2(15c_1 + 30c_2 + 24c_3 - 24c_4 - 170c_5 - 540c_6 - 1414c_7)d^6 + a_1a_2^5(6c_1 + 22c_2 + 57c_3 + 116c_4 + 155c_5 - 114c_6 - 1967c_7)d + a_1^5a_2(6c_1 + 14c_2 + 21c_3 + 28c_4 + 35c_5 + 42c_6 + 49c_7)d^5 + 6a_1^2a_2^2(15c_1 + 30c_2 + 3c_3 - 108c_4 + 5c_5 + 1350c_6 + 8827c_7)d^6 - 4a_1^2a_2^3(15c_1 + 40c_2 + 45c_3 - 88c_4 - 555c_5 - 1140c_6 - 4305c_7)d^4 - 2a_1^3a_2^2(30c_1 + 70c_2 + 63c_3 - 112c_4 - 525c_5 - 1680c_6 - 4263c_7)d^5 + a_1^2a_2^4(15c_1 + 50c_2 + 111c_3 + 172c_4 + 125c_5 - 30c_6 + 2359c_7)d^2 + a_1^4a_2^2(15c_1 + 40c_2 + 66c_3 + 80c_4 + 110c_5 + 120c_6 + 154c_7)d^4 + 2a_1^3a_2^3(10c_1 + 30c_2 + 57c_3 + 72c_4 + 65c_5 + 180c_6 - 77c_7)d^3 + a_1a_2(30c_1 + 30c_2 - 51c_3 + 36c_4 + 215c_5 - 1530c_6 + 7021c_7)d^9),$$

$$A_0 = -\frac{bd^2}{(a_1d + a_2 - d^2)^8}(-c_0d^{14} + a_1(7c_0 + c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + c_7)d^{13} + a_1^2(-21c_0 - 6c_1 + 4c_2 - 8c_4 + 24c_5 - 56c_6 + 120c_7)d^{12} + a_1^3(35c_0 + 15c_1 - 5c_2 - 9c_3 + 19c_4 + 15c_5 - 245c_6 + 1191c_7)d^{11} +$$

$$\begin{aligned}
& a_1^4(-35c_0 - 20c_1 + 16c_3 - 80c_5 + 2416c_7)d^{10} + a_1^5(21c_0 + 15c_1 + 5c_2 - 9c_3 - 19c_4 + 15c_5 + 245c_6 + 1191c_7) \\
& d^9 - a_1^6(7c_0 + 6c_1 + 4c_2 - 8c_4 - 24c_5 - 56c_6 - 120c_7)d^8 + a_1^7(c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7)d^7 + \\
& a_2(7c_0 + 2c_1 - 4c_2 + 8c_3 - 16c_4 + 32c_5 - 64c_6 + 128c_7)d^{12} - a_2^2(21c_0 + 12c_1 - 16c_2 + 128c_4 - 768c_5 + \\
& 3584c_6 - 15360c_7)d^{10} + a_2^3(35c_0 + 30c_1 - 20c_2 - 72c_3 + 304c_4 + 480c_5 - 15680c_6 + 152448c_7)d^8 \\
& - a_2^4(35c_0 + 40c_1 - 128c_3 + 2560c_5 - 309248c_7)d^6 + a_2^5(21c_0 + 30c_1 + 20c_2 - 72c_3 - 304c_4 + 480c_5 + \\
& 15680c_6 + 152448c_7)d^4 - a_2^6(7c_0 + 12c_1 + 16c_2 - 128c_4 - 768c_5 - 3584c_6 - 15360c_7)d^2 + a_2^7(c_0 + 2c_1 + 4c_2 + \\
& 8c_3 + 16c_4 + 32c_5 + 64c_6 + 128c_7) + a_1a_2^2(105c_0 + 75c_1 - 55c_2 - 93c_3 + 545c_4 - 885c_5 - 6055c_6 + 75507c_7) \\
& d^9 + a_1^2a_2(105c_0 + 60c_1 - 40c_2 - 24c_3 + 152c_4 - 120c_5 - 1960c_6 + 16776c_7)d^{10} - 4a_1a_2^3(35c_0 + 35c_1 - \\
& 5c_2 - 85c_3 + 91c_4 + 1115c_5 - 4565c_6 - 49525c_7)d^7 - 4a_1^3a_2(35c_0 + 25c_1 - 5c_2 - 23c_3 + 19c_4 + 145c_5 - \\
& 245c_6 - 6023c_7)d^9 + a_1a_2^4(105c_0 + 135c_1 + 65c_2 - 297c_3 - 751c_4 + 3015c_5 + 31265c_6 + 61623c_7) \\
& d^5 + a_1^4a_2(105c_0 + 90c_1 + 20c_2 - 72c_3 - 112c_4 + 240c_5 + 2240c_6 + 10128c_7)d^8 - 2a_1a_2^5(21c_0 + 33c_1 + 37c_2 - \\
& 15c_3 - 275c_4 - 1047c_5 - 1883c_6 + 7425c_7)d^3 - 2a_1^5a_2(21c_0 + 21c_1 + 13c_2 - 3c_3 - 35c_4 - 99c_5 - 227c_6 - 483c_7) \\
& d^7 + a_1a_2^6(7c_0 + 13c_1 + 23c_2 + 37c_3 + 47c_4 + 13c_5 - 217c_6 - 1163c_7)d + a_1^6a_2(7c_0 + 8c_1 + 8c_2 + 8c_3 + \\
& 8c_4 + 8c_5 + 8c_6 + 8c_7)d^6 - 2a_1^2a_2^2(105c_0 + 90c_1 - 20c_2 - 144c_3 + 232c_4 + 840c_5 - 3080c_6 - 48504c_7)d^8 + \\
& 2a_1^2a_2^3(105c_0 + 120c_1 + 40c_2 - 216c_3 - 296c_4 + 1800c_5 + 8440c_6 + 40104c_7)d^6 + 2a_1^3a_2^2(105c_0 + 105c_1 + \\
& 25c_2 - 135c_3 - 119c_4 + 585c_5 + 4345c_6 + 18585c_7)d^7 + a_1^2a_2^4(-105c_0 - 150c_1 - 140c_2 + 96c_3 + 856c_4 + \\
& 2040c_5 + 1960c_6 + 18456c_7)d^4 + a_1^4a_2^2(-105c_0 - 120c_1 - 80c_2 + 48c_3 + 256c_4 + 720c_5 + 1600c_6 + 3408c_7)d^6 \\
& - 4a_1^3a_2^3(35c_0 + 45c_1 + 35c_2 - 27c_3 - 157c_4 - 315c_5 - 925c_6 - 1467c_7)d^5 + a_1^2a_2^5(21c_0 + 36c_1 + 56c_2 + \\
& 72c_3 + 56c_4 - 24c_5 + 56c_6 + 2472c_7)d^2 + a_1^5a_2^2(21c_0 + 27c_1 + 29c_2 + 27c_3 + 29c_4 + 27c_5 + 29c_6 + 27c_7) \\
& d^5 + a_1^3a_2^4(35c_0 + 55c_1 + 75c_2 + 79c_3 + 51c_4 + 55c_5 + 315c_6 - 1121c_7)d^3 + a_1^4a_2^3(35c_0 + 50c_1 + 60c_2 + \\
& 56c_3 + 48c_4 + 80c_5 + 176c_7)d^4 - 2a_1a_2(21c_0 + 9c_1 - 11c_2 + 9c_3 + 13c_4 - 111c_5 + 469c_6 - 1671c_7)d^{11}.
\end{aligned}$$

(ii): Case $r = 1$ (exactly one root equal to d):

$$W_n^C = n(A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$A_7 = \frac{c_7bd^2}{8(a_1d + 2a_2)},$$

$$A_6 = \frac{1}{7(a_1d + 2a_2)}(c_6bd^2 + 28(a_1d + 4a_2)A_7),$$

$$A_5 = \frac{1}{6(a_1d + 2a_2)}(c_5bd^2 + 21(a_1d + 4a_2)A_6 - 56(a_1d + 8a_2)A_7),$$

$$A_4 = \frac{1}{5(a_1d + 2a_2)}(c_4bd^2 + 15(a_1d + 4a_2)A_5 - 35(a_1d + 8a_2)A_6 + 70(a_1d + 16a_2)A_7),$$

$$A_3 = \frac{1}{4(a_1d + 2a_2)}(c_3bd^2 + 10(a_1d + 4a_2)A_4 - 20(a_1d + 8a_2)A_5 + 35(a_1d + 16a_2)A_6 - 56(a_1d + 32a_2)A_7),$$

$$A_2 = \frac{1}{3(a_1d + 2a_2)}(c_2bd^2 + 6(a_1d + 4a_2)A_3 - 10(a_1d + 8a_2)A_4 + 15(a_1d + 16a_2)A_5 - 21(a_1d + 32a_2)A_6 + 28(a_1d + 64a_2)A_7),$$

$$A_1 = \frac{1}{2(a_1d + 2a_2)}(c_1bd^2 + 3(a_1d + 4a_2)A_2 - 4(a_1d + 8a_2)A_3 + 5(a_1d + 16a_2)A_4 - 6(a_1d + 32a_2)A_5 + 7(a_1d + 64a_2)A_6 - 8(a_1d + 128a_2)A_7),$$

$$A_0 = \frac{1}{(a_1d + 2a_2)}(c_0bd^2 + (a_1d + 4a_2)A_1 - (a_1d + 8a_2)A_2 + (a_1d + 16a_2)A_3 - (a_1d + 32a_2)A_4 + (a_1d + 64a_2)A_5 - (a_1d + 128a_2)A_6 + (a_1d + 256a_2)A_7),$$

i.e.,

$$\begin{aligned}
A_7 &= \frac{c_7bd^2}{8(a_1d + 2a_2)}, \\
A_6 &= \frac{bd^2}{14(a_1d + 2a_2)^2}(a_1(2c_6 + 7c_7)d + 4a_2(c_6 + 7c_7)), \\
A_5 &= \frac{bd^2}{12(a_1d + 2a_2)^3}(a_1^2(2c_5 + 6c_6 + 7c_7)d^2 + 8a_2^2(c_5 + 6c_6 + 14c_7) + 4a_1a_2(2c_5 + 9c_6 + 7c_7)d), \\
A_4 &= \frac{bd^2}{10(a_1d + 2a_2)^4}(a_1^3(2c_4 + 5c_5 + 5c_6)d^3 + 16a_2^3(c_4 + 5c_5 + 10c_6) + 2a_1a_2^2(12c_4 + 50c_5 + 60c_6 - 105c_7)d + 2a_1^2a_2(6c_4 + 20c_5 + 15c_6)d^2), \\
A_3 &= \frac{bd^2}{24(a_1d + 2a_2)^5}(a_1^4(6c_3 + 12c_4 + 10c_5 - 7c_7)d^4 + 32a_2^4(3c_3 + 12c_4 + 20c_5 - 56c_7) + 16a_1a_2^3(12c_3 + 42c_4 + 50c_5 - 45c_6 - 119c_7)d + 8a_1^3a_2(6c_3 + 15c_4 + 10c_5 - 7c_7)d^3 + 24a_1^2a_2^2(6c_3 + 18c_4 + 15c_5 - 15c_6 + 28c_7)d^2), \\
A_2 &= \frac{bd^2}{6(a_1d + 2a_2)^6}(a_1^5(2c_2 + 3c_3 + 2c_4 - c_6)d^5 + 64a_2^5(c_2 + 3c_3 + 4c_4 - 8c_6) + 16a_1a_2^4(10c_2 + 27c_3 + 28c_4 - 15c_5 - 50c_6 + 105c_7)d + 2a_1^4a_2(10c_2 + 18c_3 + 10c_4 - 5c_6)d^4 + 16a_1^2a_2^3(10c_2 + 24c_3 + 19c_4 - 15c_5 - 5c_6 + 105c_7)d^2 + 2a_1^3a_2^2(40c_2 + 84c_3 + 52c_4 - 30c_5 + 40c_6 - 105c_7)d^3), \\
A_1 &= \frac{bd^2}{12(a_1d + 2a_2)^7}(a_1^6(6c_1 + 6c_2 + 3c_3 - c_5 + c_7)d^6 + 128a_2^6(3c_1 + 6c_2 + 6c_3 - 8c_5 + 32c_7) + 192a_1a_2^5(6c_1 + 11c_2 + 9c_3 - 3c_4 - 11c_5 + 15c_6 + 43c_7)d + 12a_1^5a_2(6c_1 + 7c_2 + 3c_3 - c_5 + c_7)d^5 + 24a_1^2a_2^4(60c_1 + 100c_2 + 66c_3 - 36c_4 - 40c_5 + 180c_6 - 53c_7)d^2 + 12a_1^4a_2^2(30c_1 + 40c_2 + 18c_3 - 6c_4 + 5c_5 - 15c_6 + 26c_7)d^4 + 8a_1^3a_2^3(120c_1 + 180c_2 + 96c_3 - 54c_4 + 10c_5 + 135c_6 - 673c_7)d^3), \\
A_0 &= \frac{bd^2}{210(a_1d + 2a_2)^8}(a_1^7(210c_0 + 105c_1 + 35c_2 - 7c_4 + 5c_6)d^7 + 256a_2^7(105c_0 + 105c_1 + 70c_2 - 56c_4 + 160c_6) + 112a_1a_2^6(840c_0 + 780c_1 + 440c_2 - 90c_3 - 328c_4 + 300c_5 + 920c_6 - 1785c_7)d + 14a_1^6a_2(210c_0 + 120c_1 + 35c_2 - 7c_4 + 5c_6)d^6 + 336a_1^2a_2^5(420c_0 + 360c_1 + 170c_2 - 60c_3 - 84c_4 + 200c_5 + 85c_6 - 1190c_7)d^2 + 42a_1^5a_2^2(420c_0 + 270c_1 + 80c_2 - 15c_3 + 6c_4 - 25c_5 + 40c_6 - 35c_7)d^5 + 280a_1^3a_2^4(420c_0 + 330c_1 + 130c_2 - 54c_3 - 20c_4 + 135c_5 - 215c_6 - 441c_7)d^3 + 280a_1^4a_2^3(210c_0 + 150c_1 + 50c_2 - 18c_3 + 5c_4 + 15c_5 - 85c_6 + 273c_7)d^4).
\end{aligned}$$

(iii): Case $r = 2$ (all two roots equal to d):

$$W_n^C = n^2(A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_7 &= \frac{1}{72}bc_7, \quad A_6 = \frac{1}{56}b(c_6 + 7c_7), \quad A_5 = \frac{1}{84}b(2c_5 + 12c_6 + 35c_7), \\
A_4 &= \frac{1}{60}b(2c_4 + 10c_5 + 25c_6 + 35c_7), \quad A_3 = \frac{1}{120}b(6c_3 + 24c_4 + 50c_5 + 60c_6 + 21c_7), \\
A_2 &= \frac{1}{24}b(2c_2 + 6c_3 + 10c_4 + 10c_5 + 3c_6 - 7c_7), \\
A_1 &= \frac{1}{36}b(6c_1 + 12c_2 + 15c_3 + 12c_4 + 3c_5 - 6c_6 - 5c_7), \\
A_0 &= \frac{1}{420}b(210c_0 + 210c_1 + 175c_2 + 105c_3 + 21c_4 - 35c_5 - 25c_6 + 35c_7).
\end{aligned}$$

References

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