Tchebychev polynomials of second kind on the ellipse and approximations.

Abstract

This paper is concerned with deriving a new system of orthogonal Tchebychev polynomials of second kind with respect to the Lebesgue planar measure concentrated on the ellipse.We study orthogonality,and extremal properties and minimization and Fourier development involving of Tchebychev polynomials of second kind on the ellipse.There are some important properties and certain identities and extremal properties involving these Tchebychev polynomials of second kind on the ellipse.We have used mathematical induction to establish the relation between them.We also present some other utilzations by using Green's and Stoke's formulas to reexpress kernel polynomials.General expressions are found for the kernels polynomials associated to orthonormalized Tchebychev polynomials of second kind on the ellipse.These kernel polynomials can be used to describe the approximation of continuous functions and solving some area extremal problems by Tchebychev polynomials of second kind on the ellipse .They can be used for the representation of the n-th partial sum of the Fourier series expansion of orthonormalized Tchebychev polynomials of second kind in the form of an integral.

Keywords: Unit disk, Conformal mapping ,Area integral,regular functions ,Green's formula , Stoke's formula, Ellipses , Differentiation.

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1 Introduction

Orthogonal polynomials are of considerable importance properties in many branches of science and engineering since they represent an indispensable analytical tool for solving various extremal and minimization and approximation problems and Fourier developments.Associated orthogonal polynomials are a family of polynomials derived from a given set of orthogonal Chebychev polynomials by integrating polynomials or shifting the indices in the recurrence. coefficients. Specically, these associated polynomials are also orthogonal, but with respect to a possibly different measure.

In this paper we study a new system of orthogonal Tchebychev polynomials of second kind $\{U_n(z)\}_{n=0,1,2...}$,with respect to the Lebesgue planar measure concentrated on the ellipse

$D: b^2x^2 + a^2y^2 < a^2b^2$ a system of orthogonal polynomials, given by :

$$U_n(z) = \frac{T'_{n+1}(z)}{n+1} = \frac{\sin\left((n+1)\cos^{-1} z\right)}{\sqrt{1-z^2}} \quad , \quad n = 0, 1, 2....$$

where $T_n(z) = \cos\left(n\cos^{-1} z\right), n = 0, 1, 2, 3....$is a polynomial of degree n . $T_n(z)$ is called the Tchebychev polynomial of degree n .

They satisfies

$$\iint_D U_n(z)\overline{U_m(z)}dxdy = \frac{4(n+1)}{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}\delta_{n,m} \quad , \ (a+b)^2 = \rho, \ n, m = 0, 1, 2....$$

where $\delta_{n,m}$, is the symbol of Kronecker.

The paper will be structured as follows : in section 1 we present some useful terminology as well as some necessary definitions regarding orthogonal Tchebychev polynomials of second kind on the ellipse. In section 2, we present some extremal properties of orthogonal polynomials $\{U_n(z)\}_{n=0,1,2...}$. Second, we give some necessary definitions and basic properties of Fourier polynomials approximability for the polynomials $\{U_n(z)\}_{n=0,1,2...}$best solution for associated extremal problems in respect to them.These orthogonal polynomials can be used to describe the approximation of continuous functions by Tchebychev polynomials of second kind on the ellipse by definite Fourier series and how to compute efficiently such approximations. In addition, some comparisons with some other methods are made.We show a connection between these orthogonal polynomials and Tchebychev polynomials of first kind $\{T_n(z)\}_{n=0,1,2...}$,we derive structures relations between them;we give some necessary definitions and basic extremal and approximations properties of the K-kernel orthogonal polynomials,where $K_n(z, w)$ is the kernel Tchebychev polynomials of second kind on the ellipse $\{U_n(z)\}_{n=0,1,2...}$given by

$$K_n(z, w) = \sum_{k=0}^{n} \frac{\pi\left(\rho^{k+1} - \rho^{-k-1}\right)}{4(k+1)} U_k(z)U_k(w)$$

We conclude the paper with some results concerning polynomials extremum properties of the K_n-kernels polynomials.These kernel polynomials can be used to describe the approximation of continuous functions and solving some area extremal problems by Tchebychev polynomials of second kind on the ellipse .In addition, some comparisons with some other methods are made.

2 Tchebychev polynomials of second kind on the ellipse

Suppose $G \subset \mathbb{C}$ is an arbitrary domain ,f is analytic in G and set

$$L^2(G) = \left\{ f : G \longrightarrow \mathbb{C}, \text{regular in } G \ , \iint_G |f(z)|^2 \, dxdy < \infty, \ z = x + iy \right\} \tag{1}$$

D is the ellipse $b^2x^2+a^2y^2<a^2b^2$, by the conformal mapping $z=\cos w$ the cut ellipse is transformed into R the rectangle $(-ci,-ci+\pi,ci+\pi,ci)$, where

$$a=\cosh c \quad , \quad b=\sinh c. \qquad (c>0)$$

.

we assume that the foci of the ellipse are situated at $z=\pm 1$,that is $a^2-b^2=1$.

The function $T_n(z)=\cos\left(n\cos^{-1}z\right), n=0,1,2,3....$is a polynomial of degree n . $T_n(z)$ is called the Tchebichev polynomial of degree n .The polynomial of degree n

$$U_n(z)=\frac{T'_{n+1}(z)}{n+1}=\frac{\sin\left((n+1)\cos^{-1}z\right)}{\sqrt{1-z^2}} \tag{2}$$

is called the Tchebichev polynomial of second kind.We shall show that the polynomials $U_n(z)$ are orthogonal to each other with respect to the above mentioned ellipse.

Theorem 1 *The orthonormalized polynomials* ,$\{P_n(z)\}_{n=0,1,2,3.....}$

$$P_n(z)=2\sqrt{\frac{n+1}{\pi}}\left(\rho^{n+1}-\rho^{-n-1}\right)^{-\frac{1}{2}}U_n(z) \qquad ,n=0,1,2,3......$$

satisfies

$$\iint_D P_n(z)\overline{P_m(z)}dxdy=\delta_{n,m} \tag{3}$$

$\delta_{n,m}$ *, is the symbol of Kronecker, and*

$$A=\iint_D \left|T'_n(z)\right|^2 dxdy=\frac{n\pi}{4}\left(\rho^n-\rho^{-n}\right) \tag{4}$$

Proof. We apply to the ellipse two cuts from $-a$ to -1 and from 1 to a respectively.Obviously ,these cuts do not affect the value of the area integral

$$A_{n,m}=\iint_D U_n(z)\overline{U_m(z)}dxdy$$

The conformal mapping $z=\cos w$, the cut ellipse is transformed into the rectangle R :$(-ci,-ci+\pi,ci+\pi,ci)$,where $a=\cosh c$,$b=\sinh c$,$c>0$.Since the Jacobien of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)}=\left|\frac{dz}{dw}\right|^2=\left|1-z^2\right| \tag{5}$$

where

$$z=x+iy=\cos w \quad , w=u+iv$$

it follows from (2),(5) that

$$A_{n,m}=\iint_R \sin\left((n+1)w\right)\overline{\sin\left((n+1)w\right)}dudv \tag{6}$$

Let us evaluate $A_{n-1,m-1}$

$$A_{n-1,m-1} = \int_{-c}^{c}\int_{0}^{\pi} \sin\left(n\left(u+iv\right)\right)\overline{\sin\left(m\left(u+iv\right)\right)}dudv$$

$$= \int_{-c}^{c}\int_{0}^{\pi} \sin\left(nu+inv\right)\sin\left(mu-imv\right)dudv$$

In fact

$$\sin\left(nu+inv\right) = \sin nu\cosh nv + i\cos nu\sinh nv$$

and

$$\sin\left(mu-imv\right) = \sin mu\cosh mv - i\cos mu\sinh mv$$

then

$$A_{n-1,m-1} = \int_{-c}^{c}\cosh nv\cosh mvdv\int_{0}^{\pi}\sin nu\sin mudu \tag{7}$$

$$+\int_{-c}^{c}\sinh nv\sinh mvdv\int_{0}^{\pi}\cos nu\cos mudu \tag{8}$$

$$+i\int_{-c}^{c}\sinh nv\cosh mvdv\int_{0}^{\pi}\cos nu\sin mudu \tag{9}$$

$$-i\int_{-c}^{c}\cosh nv\sinh mvdv\int_{0}^{\pi}\sin nu\cos mudu \tag{10}$$

The last two integrals over v vanish because the integrands are odd.The first two integrals vanish if $n \neq m$.For $n = m$ we obtain

$$A_{n-1,n-1} = \int_{-c}^{c}\cosh^2 nvdv\int_{0}^{\pi}\sin^2 nudu + \int_{-c}^{c}\sinh^2 nvdv\int_{0}^{\pi}\cos^2 nudu$$

$$+i\int_{-c}^{c}\sinh nv\cosh nvdv\int_{0}^{\pi}\cos nu\sin n\,\mathrm{u}\,dudv - i\int_{-c}^{c}\cosh nv\sinh nvdv\int_{0}^{\pi}\sin nu\cos nududv$$

After some computations

$$A_{n,n} = \frac{\pi}{2\left(n+1\right)}\sinh 2\left(n+1\right)c$$

or in view $a+b = \cosh c + \sinh c = e^c$

$$A_{n,n} = \frac{\pi}{4\left(n+1\right)}\left(\rho^{n+1}-\rho^{-n-1}\right) \quad , \; \left(a+b\right)^2 = \rho \tag{11}$$

The orthonormalized polynomials are therefore ■

$$P_n\left(z\right) = 2\sqrt{\frac{n+1}{\pi}}\left(\rho^{n+1}-\rho^{-n-1}\right)^{-\frac{1}{2}}U_n\left(z\right)$$

and the theorem is proved.

Remark 2 *Now by* (2),(11)*,we get*

$$\iint\limits_{D} \left|T_n'(z)\right|^2 dxdy = n^2||U_{n-1}||^2 = \frac{n\pi}{4}\left(\rho^n - \rho^{-n}\right)$$

If the ellipse : $D : b^2x^2 + a^2y^2 < a^2b^2$ *and* C *is the closed contour by which* D *is bounded , indeed by Green's formula,[1, 2, 4, 13, 14, 16, 17, 18, 20]*

$$\iint\limits_{D} f(z)\,\overline{g'(z)}dxdy = \frac{1}{2i}\int\limits_{C} f(z)\,\overline{g(z)}dz \tag{12}$$

we have

$$\iint\limits_{D} \frac{\overline{T_{n+1}'(z)}}{z-\xi}dxdy = \frac{n+1}{2i}\int\limits_{C} \frac{\overline{U_n(z)}}{z-\xi}dz - (n+1)\,\pi\overline{U_n(\xi)}$$

3 Extremal problems and approximations

The K-Bergman kernel function associated to orthonormalized polynomials ,$\{P_n(z)\}_{n=0,1,2,3.....}$ is given in [1, 2, 4, 13, 14, 16, 17, 18] by

$$K(z,\xi) = \sum_{n=0}^{\infty} P_n(z)\overline{P_n(\xi)} \quad , \quad z,\xi \in D \tag{13}$$

wich converges absolutely and uniformly in any closed domin which is entirely within D. where

$$K(\xi,\xi) = \sum_{n=0}^{\infty} |P_n(\xi)|^2 \quad , \quad \xi \in D$$

An immediate consequence is the fact that $K(z,\xi)$ is " Hermitian " i.e

$$K(\xi,z) = \overline{K(z,\xi)} \quad z,\xi \in D$$

Given an arbitrary function $f(z)$ of $L^2(D)$,[1, 2, 4, 13, 14, 16, 17, 18] , K has the reproducing kernel property such that

$$f(\xi) = \iint\limits_{D} \overline{K(z,\xi)} f(z)\,dxdy \quad , \quad z = x+iy \quad , \quad f(z) \in L^2(D) \tag{14}$$

Thus by RKP, we have

$$\iint\limits_{D} \overline{K(z,\xi)}dxdy = 1, \qquad z = x+iy$$

and

$$K(\xi,\xi) = \iint\limits_{D} |K(z,\xi)|^2 dxdy \quad , \quad z = x+iy \tag{15}$$

As a consequence of (14),we have

$$\xi^n = \iint\limits_{D} \overline{K(z,\xi)} z^n dxdy \quad , \quad n = 0,1,2...$$

Hence

$$\frac{1}{\eta-\xi}=\iint\limits_{D}\overline{K\left(z,\xi\right)}\frac{1}{\eta-z}dxdy$$

i.e,if

$$\frac{1}{\eta-z}=\sum_{n=0}^{\infty}\frac{z^{n}}{\eta^{n+1}}$$

with then converge uniformly for all z in D , this equal to

$$\frac{1}{\eta-\xi}=\sum_{n=0}^{\infty}\frac{\xi^{n}}{\eta^{k+1}}$$

Let us compute Fourier coeficients of the kernel function in respect to the set of orthonormalized polynomials,$\{P_n\left(z\right)\}_{n=0,1,2,3.....}$If

$$K\left(z,\xi\right)=\sum_{n=0}^{\infty}a_{n}\overline{P_{n}(\xi)}\quad,\qquad z\ ,\xi\in D$$

$\{a_n\}_{n=0,1,2....}$ are the Fourier coeficients of the kernel function in respect to the set $\{P_n\left(z\right)\}_{n=0,1,2,3.....}$We have,in view of (14),

$$a_{n}=\iint\limits_{D}K\left(z,\xi\right)\overline{U_{n}\left(z\right)}dxdy=\overline{\iint\limits_{D}\overline{K\left(z,\xi\right)}U_{n}\left(z\right)dxdy}=\overline{U_{n}\left(\xi\right)}$$

Thus expansion of kernel (13) is true.

Setting $f\left(z\right)=K\left(z,\eta\right)$ in (14), we otain

$$\iint\limits_{D}\overline{K\left(z,\xi\right)}K\left(z,\eta\right)dxdy=K\left(\xi,\eta\right)\quad,\quad z=x+iy \tag{16}$$

$K\left(z,\xi\right)$ solves the extremal problem

$$\gamma=Min\left\{\iint\limits_{D}\left|f\left(z\right)\right|^{2}dxdy,\quad f\left(z\right)\in L^{2}\left(D\right)\quad,\quad f\left(\xi\right)=1\right\} \tag{17}$$

and the value of minimum is

$$\gamma=\frac{1}{K\left(\xi,\xi\right)} \tag{18}$$

Let ξ and η be two distincts points of a domain D and let $L\left(z,\xi\right)$ the kernel function defined by

$$L\left(z,\xi\right)=K\left(z,\xi\right)-\frac{K\left(\eta,\xi\right)}{K\left(\eta,\eta\right)}K\left(z,\eta\right) \tag{19}$$

L has the reproducing kernel property such that

$$f\left(\xi\right)=\iint\limits_{D}\overline{L\left(z,\xi\right)}f\left(z\right)dxdy\quad,\quad z=x+iy\quad,\quad f\left(z\right)\in L^{2}\left(D\right)$$

Thus by RKP, we have

$$K_1(\xi,\xi) = \iint_D |L(z,\xi)|^2\, dxdy \quad , \quad z = x + iy$$

we can show that $L(z,\xi)$ solves the extremal problem

$$\sigma = Min \left\{ \iint_D |f(z)|^2\, dxdy, \quad f(z) \in L^2(D) \quad f(\xi) = 1, f(\eta) = 0 \right\} \tag{20}$$

and the value of minimum is

$$\sigma = \frac{1}{L(\xi,\xi)} \tag{21}$$

The n–th K_n-Bergman kernel function associated to orthonormalized polynomials ,$\{P_n(z)\}_{n=0,1,2,3.....}$ is given in [1, 2, 4, 13, 14, 16, 17, 18] by

$$K_n(z,\xi) = \sum_{k=0}^{n} P_k(z)\overline{P_k(\xi)} \quad , \quad z,\xi \in D \tag{22}$$

wich converges absolutely and uniformly in any closed domin which is entirely within D to K-Bergman kernel function .

An immediate consequence is the fact that $K_n(z,\xi)$ is " Hermitian " i.e

$$K_n(\xi,z) = \overline{K_n(z,\xi)} \quad z,\xi \in D$$

K_n has the reproducing kernel property such that

$$f(\xi) = \iint_D \overline{K_n(z,\xi)} f(z)\, dxdy \quad , \quad z = x + iy \quad , \quad f(z) \in L^2(D)$$

Thus by RKP, we have

$$\iint_D \overline{K_n(z,\xi)} dxdy = 1, \;\; z = x + iy \;, \; n = 0, 1, 2...$$

and

$$K_n(\xi,\xi) = \iint_D |K_n(z,\xi)|^2\, dxdy \quad , z = x + iy$$

Theorem 3 *$\Pi_n(\mathbb{C})$ is the class of algebraic polynomials of degree at most n, with complex coefficients. Let λ and α be fixed complexs constants.If $L_n(z) \in \Pi_n(\mathbb{C}), n = 0, 1, 2, 3...$ Let us denote*

$$I_n = \iint_D |L_n(z)|^2\, dxdy \tag{23}$$

Then we have

$$\mu_n(\lambda) = \min\{I_n \;, L_n(z) \in \Pi_n(\mathbb{C}) \; such\ that \;, \; L_n(\lambda) = \alpha\} = \frac{|\alpha|^2}{K_n(\lambda,\lambda)}$$

if and only if

$$L_n(z) = \alpha \frac{K_n(z,\lambda)}{K_n(\lambda,\lambda)} \tag{24}$$

i.e,

$$L_n(z) = \alpha \frac{\sum_{k=0}^{n} \frac{k+1}{\rho^{k+1}-\rho^{-k-1}} \overline{U_k(\lambda)} U_k(z)}{\sum_{k=0}^{n} \frac{k+1}{\rho^{k+1}-\rho^{-k-1}} |U_k(\lambda)|^2} \quad , \quad (a+b)^2 = \rho \tag{25}$$

and

$$\mu_n(\lambda) = \frac{\pi |\alpha|^2}{4} \frac{1}{\sum_{k=0}^{n} \frac{k+1}{\rho^{k+1}-\rho^{-k-1}} |U_k(\lambda)|^2} \tag{26}$$

Let us denote

$$\pi_n(z_0) = \min \left\{ \left\| Q_n' \right\|_{L^2(D)}^2 : Q_n \in \Pi_n(\mathbb{C}), Q_n(z_0) = 0, \textit{and } Q_n'(z_0) = 1 \right\} \tag{27}$$

Then, the ***Bieberbach*** *polynomial* $B_n(z)$ *,such that*

$$B_n(z) = \frac{\sum_{k=0}^{n-1} \frac{(T_{k+1}(z) - T_{k+1}(z_0)) \overline{U_k(z_0)}}{\rho^{k+1} - \rho^{-k-1}}}{\sum_{k=0}^{n-1} \frac{k+1}{\rho^{k+1}-\rho^{-k-1}} |U_k(z_0)|^2} \tag{28}$$

is the best solution of the extremal problem (27), *and satisfies*

$$\left\| B_n' \right\|_{L^2(D)}^2 = \pi_n(z_0) = \frac{1}{K_{n-1}(z_0, z_0)} \tag{29}$$

Proof. It is possible to express the Fourier polynomial best solution of extremal problem (20), as finite linear combination of orthonormalized polynomials ,$\{P_k(z)\}_{k=0,1,2,3...n.}$.

If $A_0, A_1, A_2....A_n$ are fixed complexs constants .Setting

$$L_n(z) = \sum_{k=0}^{n} A_k P_k(z) \tag{30}$$

Then

$$\iint_D |L_n(z)|^2 \, dxdy = \sum_{k=0}^{n} |A_k|^2 \geq \mu_n(\lambda)$$

this can also be written

$$\sum_{k=0}^{n} |A_k|^2 = \mu_n(\lambda)$$

equivalently

$$A_k = \lambda \overline{P_k(\lambda)} \tag{31}$$

Because

$$\sum_{k=0}^{n} A_k P_k(\lambda) = \alpha$$

imply

$$|\alpha|^2 = \left|\sum_{k=0}^{n} A_k P_k(\lambda)\right|^2 \leq \sum_{k=0}^{n} |A_k|^2 \sum_{k=0}^{n} |P_k(\lambda)|^2$$

Thus

$$\frac{|\alpha|^2}{\sum_{k=0}^{n} |P_k(\lambda)|^2} = \mu_n(\lambda)$$

Setting (31) in (30) we get

$$\mu_n(\lambda) = \iint_D |L_n(z)|^2 \, dxdy = \frac{|\alpha|^2}{\sum_{k=0}^{n} |P_k(\lambda)|^2}$$

if and only if

$$L_n(z) = \alpha \frac{K_n(z,\lambda)}{K_n(\lambda,\lambda)}$$

and the theorem is proved.The **Bieberbach** polynomial $B_n(z)$,deg $B_n \leq n$,associated to orthonormalized polynomials ,$\{P_k(z)\}_{k=0,1,2,3...n.}$,deg $B_n \leq n$,

such that

$$B_n(z) = \frac{\sum_{k=0}^{n-1} \overline{P_k(z_0)} \int_{z_0}^{z} P_k(t)\, dt}{\sum_{k=0}^{n-1} |P_k(z_0)|^2}$$

is the best solution of the extremal problem (27), therefore

$$B_n(z) = \frac{\sum_{k=0}^{n-1} \frac{(T_{k+1}(z) - T_{k+1}(z_0))\, \overline{U_k(z_0)}}{\rho^{k+1} - \rho^{-k-1}}}{\sum_{k=0}^{n-1} \frac{k+1}{\rho^{k+1} - \rho^{-k-1}} |U_k(z_0)|^2}$$

Because

$$B_n'(z) = \frac{\sum_{k=0}^{n-1} P_k(z)\overline{P_k(z_0)}}{\sum_{k=0}^{n-1} |P_k(z_0)|^2}$$

then

$$\|B_n'\|_{L^2(D)}^2 = \frac{1}{\sum_{k=0}^{n-1} |P_k(z_0)|^2}$$

and the theorem is proved. ■

3.1 Green's and Stoke's formulas and interpolations

If the ellipse : $D : b^2x^2+a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.Let $f(z)$ be regular in a domain D.Green's and Stoke's formulas ,[1, 2, 4, 13, 14, 16, 17, 18, 20],is

$$\iint\limits_D f(z)\,\overline{g'(z)}dxdy = \frac{1}{2i}\int\limits_C f(z)\,\overline{g(z)}dz \tag{32}$$

we have

$$\iint\limits_D f(z)dxdy = \frac{1}{2i}\int\limits_C f(z)\overline{z}dz \tag{33}$$

if ξ is a point of D, we have

$$\iint\limits_D \frac{\overline{f'(z)}}{z-\xi}dxdy = \frac{1}{2i}\int\limits_C \frac{\overline{f(z)}}{z-\xi}dz - \pi\overline{f(\xi)} \tag{34}$$

If there exist Schwarz function $S(z)$,such that

$$S(z) = \overline{Z} \quad ,z \in C$$

Suppose that $S(z)$ satisfie the rational expansion :

$$S(z) = \frac{1}{\pi}\sum_{i=1}^{n}\sum_{j=0}^{r_{i-1}} \frac{a_{ij}j!}{(z-z_i)^{j+1}} + g(z)$$

where $z_i \in D$ et a_{ij} $(i = 1, ..., n; j = 0, 1..., r_i - 1$) are fixed complex constants and $g(z)$ is analytic in a domain ellipse D .Let us prove that

$$\iint\limits_D f(z)dxdy = \sum_{k=1}^{n}\sum_{l=1}^{r_{k-1}} a_{kl}f^{(l)}(z) \tag{35}$$

According to (32) and(33), we obtain

$$\begin{aligned}\iint\limits_D f(z)dxdy &= \frac{1}{2i}\int\limits_C f(z)\overline{z}dz \\ &= \frac{1}{2i}\int\limits_C f(z)S(z)dz \\ &= \sum_{k=1}^{n}\sum_{l=1}^{r_{k-1}} a_{kl}\frac{l!}{2\pi i}\oint_{\partial D}\frac{f(z)dz}{(z-z_k)^{l+1}} + \frac{1}{2i}\oint_{\partial D} f(z)g(z)dz\end{aligned}$$

Using Cauchy's formula

$$\iint\limits_D f(z)dxdy = \sum_{k=1}^{n}\sum_{l=1}^{r_{k-1}} a_{kl}f^{(l)}(z)$$

Proposition 4 *Let $z_1, z_2, z_3....z_p$ be n distincts points which are situated in the interior of $D = \{|z| < 1\}$,*

$$I_p = \frac{1}{\pi}\iint\limits_D \left|\frac{1}{z - z_p}\right|^2 dxdy \quad , \quad z = x + iy \tag{36}$$

and

$$J_p = \frac{1}{2\pi}\iint\limits_D \left|\frac{z + z_p}{z - z_p}\right|^2 dxdy \quad , \quad z = x + iy \tag{37}$$

then

$$I_p = \log\left|\frac{1 - |z_p|^2}{|z_p|^2}\right| \tag{38}$$

and

$$J_p = \frac{1}{2} + 2|z_p|^2 \log\left|\frac{1 - |z_p|^2}{|z_p|^2}\right| \tag{39}$$

therefore

$$\iint\limits_D \left(\sum_{p=1}^{m}\left|\frac{1}{z - z_p}\right|^2\right) dxdy = \iint\limits_D \left(\sum_{p=1}^{m}\left|\frac{1}{z - z_p}\right|^2\right) dxdy$$

$$\iint\limits_D \left(\sum_{p=1}^{m}\left|\frac{1}{z - z_p}\right|^2\right) dxdy = \pi \log\left|\prod_{p=1}^{m}\frac{1 - |z_p|^2}{|z_p|^2}\right| \tag{40}$$

and

$$\iint\limits_D \left(\sum_{p=1}^{m}\left|\frac{z + z_p}{z - z_p}\right|^2\right) dxdy = m\pi + 2\pi \log\left|\prod_{p=1}^{m}\frac{1 - |z_p|^2}{|z_p|^2}\right|^{2|z_p|^2} \tag{41}$$

Proof. suppose $z = re^{i\theta} = x + iy$

$$dxdy = rdrd\theta, \quad 0 \le r < 1, \ 0 \le \theta \le 2\pi$$

then

$$I_p = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \frac{rdrd\theta}{(re^{i\theta} - z_p)(re^{-i\theta} - \overline{z_p})}$$

$$= \frac{1}{\pi}\int_0^1 rdr \int_0^{2\pi} \frac{d\theta}{(re^{i\theta} - z_p)(re^{-i\theta} - \overline{z_p})}$$

Setting $w = e^{i\theta}$, $dw = iwd\theta$, $0 \le \theta \le 2\pi$and $|w| = 1$

$$I_p = \frac{1}{\pi}\int_0^1 rdr \int_{C_1} \frac{\frac{dw}{iw}}{(rw - z_p)\left(\frac{r}{w} - \overline{z_p}\right)}$$

i-e

$$I_p = \frac{1}{\pi i}\int_0^1 dr \int_{C_1} \frac{dw}{\left(w - \frac{z_p}{r}\right)(r - w\overline{z_p})}$$

now we state

$$\frac{1}{\pi i}\int\limits_{C_1}\frac{dw}{\left(w-\frac{z_p}{r}\right)\left(r-w\overline{z_p}\right)}=\frac{2r}{r^2-|z_p|^2}$$

hence

$$I_p=\int\limits_0^1\frac{2r}{r^2-|z_p|^2}dr=\log\left|\frac{1-|z_p|^2}{|z_p|^2}\right|$$

then

$$\iint\limits_D\left(\sum_{p=1}^m\left|\frac{1}{z-z_p}\right|^2\right)dxdy=\pi\log\left|\prod_{p=1}^m\frac{1-|z_p|^2}{|z_p|^2}\right|$$

To prove (39)

$$J_p=\frac{1}{2\pi}\iint\limits_D\left|\frac{z+z_p}{z-z_p}\right|^2dxdy\quad,\ z=x+iy$$

suppose $z=re^{i\theta}=x+iy$

$$dxdy=rdrd\theta,\quad 0\leq r<1,\ 0\leq\theta\leq 2\pi$$

then

$$J_p=\frac{1}{2\pi}\int\limits_0^1\int\limits_0^{2\pi}\frac{\left(re^{i\theta}+z_p\right)\left(re^{-i\theta}+\overline{z_p}\right)}{\left(re^{i\theta}-z_p\right)\left(re^{-i\theta}-\overline{z_p}\right)}rdrd\theta$$

$$=\frac{1}{2\pi}\int\limits_0^1 rdr\int\limits_0^{2\pi}\frac{\left(re^{i\theta}+z_p\right)\left(re^{-i\theta}+\overline{z_p}\right)}{\left(re^{i\theta}-z_p\right)\left(re^{-i\theta}-\overline{z_p}\right)}d\theta$$

Setting : $w=e^{i\theta}$, $dw=iwd\theta$, $0\leq\theta\leq 2\pi$ and $|w|=1$, then we have

$$J_p=\frac{1}{2\pi}\int\limits_0^1 rdr\int\limits_{C_1}\frac{\left(rw+z_p\right)\left(r+w\overline{z_p}\right)}{\left(rw-z_p\right)\left(r-w\overline{z_p}\right)}\frac{dw}{iw}$$

i-e

$$J_p=\frac{1}{2\pi i}\int\limits_0^1 dr\int\limits_{C_1}\frac{\left(rw+z_p\right)\left(r+w\overline{z_p}\right)}{\left(w-\frac{z_p}{r}\right)\left(r-w\overline{z_p}\right)}\frac{dw}{w}=\frac{2r\left(r^2+|z_p|^2\right)-r\left(r^2-|z_p|^2\right)}{r^2-|z_p|^2}$$

now we state

$$J_p=\int\limits_0^1 r\frac{r^2+3|z_p|^2}{r^2-|z_p|^2}dr=\int\limits_0^1\left(r+\frac{4r|z_p|^2}{r^2-|z_p|^2}\right)dr$$

hence

$$J_p=\frac{1}{2}+2|z_p|^2\log\left|\frac{1-|z_p|^2}{|z_p|^2}\right|$$

then

$$\iint\limits_D\left(\sum_{p=1}^m\left|\frac{z+z_p}{z-z_p}\right|^2\right)dxdy=m\pi+2\pi\log\left|\prod_{p=1}^m\frac{1-|z_p|^2}{|z_p|^2}\right|^{2|z_p|^2}$$

and the proposition is proved.

Corollary 5 *If the ellipse : $D: b^2x^2 + a^2y^2 < a^2b^2$ and C is the closed contour by which D is bounded.Let $f(z)$ be regular in a domain D.If*

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z) \tag{42}$$

which converges uniformly and absolutely in any closed subdomain of ellipse D.Then

$$a_{n+1} = \frac{i\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8(n+1)^2} \int_C f(z)\overline{U_n(z)}d\overline{z} \quad , \rho = (a+b)^2\,, \ n = 0,1,2.... \tag{43}$$

where $C: b^2x^2 + a^2y^2 = a^2b^2$.

■

Proof. The coefficients a_n are the Fourier coefficients of the function $f'(z)$ corresponding to the orthogonal Tchebychev polynomials of second kind on the ellipse D.

$$(n+1)\,a_{n+1} = \frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{4(n+1)} \iint_D f'(z)\,\overline{U_n(z)}dxdy \quad , \ n = 0,1,2....$$

According to (32) and(33),this may be replaced,

$$a_{n+1} = -\frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8i(n+1)^2} \overline{\int_C \overline{f(z)} U_n(z)\,dz} \quad , z = x + iy \ , n = 0,1,2....$$

i-e

$$a_{n+1} = -\frac{\pi\left(\rho^{n+1} - \rho^{-n-1}\right)}{8i(n+1)^2} \int_C f(z)\,\overline{U_n(z)}d\overline{z} \quad , \ n = 0,1,2....$$

and the corollary is proved. ■

4 Conclusion

In this article,we demonstrate certain identities involving both the Tchebychev polynomials of second kind on the ellipse.This study brings to light some significant results and defines the relationship between these polynomials and Tchebychev polynomials of first kind.General expressions are found for the kernels polynomials relative to orthonormalized Tchebychev polynomials of second kind on the ellipse .We also present some results for these kernel polynomials,they can be used to describe the approximation of continuous functions and solving some areas extremals problems by orthonormalized Tchebychev polynomials of second kind on the ellipse.They can be used for the representation of the n-th partial sum of the Fourier series expansion of orthonormalized Tchebychev polynomials of second kind in the form of an integral.In addition,we look at the practical application of kernels polynomials relative to orthonormalized Tchebychev polynomials of second kind in approximation theory.We use Green's and Stoke's formulas to find the interpolation of area integral of a regular $f(z)$ in a domain of ellipse $D: b^2x^2 + a^2y^2 < a^2b^2$.

It is worth mentioning here that the above-achieved results and analysis are fruitful. Some of their presumed uses are given below :

•The orthonormalized Tchebychev polynomials of second kind on the ellipse and their kernel polynomials are fruitful in approximation theory.

•These orthonormalized Tchebychev polynomials of second kind on the ellipse are fruitful in applied to find the minimum value and the minimizing function for various definite area integrals and solving some areas extremals problems.

•These results strengthen the knowledge of the kernel polynomials associated to the orthonormalized Tchebychev polynomials of second kind on the ellipse.

•They are also beneficial in studying problems connected to solve extremal problem and to describe the approximation of continuous functions by kernel orthonormalized Tchebychev polynomials of second kind on the ellipse .

•They help to study finite linear combinations and defnites summations sequences and calculating general summations.

•These orthogonal polynomials are fruitful in applied to find the interpolation problem, we illustrates that one can use Gaussian quadratures for various definite integrals and solving area extremal problems.

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