UNDER PEER REVIEW

# Polynomially stable of a thermoelastic Timoshenko system with Cattaneo heat conduction law

Abstract: This paper investigates the polynomial stability of a thermoelastic Timoshenko system with Cattaneo's heat conduction law. The system consists of coupled hyperbolic-parabolic equations governing the transverse displacement, rotation angle, temperature, and heat flux. Previous work established the lack of exponential stability regardless of the equal wave speeds (EWS) condition. We prove that when the EWS condition is satisfied, the associated $C_0$-semigroup exhibits polynomial stability. Specifically, we demonstrate that solutions decay at a rate of $t^{-1/4}$ as $t \to \infty$, with the decay rate uniform for initial data in the domain of the generator. The analysis employs energy methods combined with semigroup theory, leveraging the structural properties induced by the EWS condition to establish polynomial decay estimates. This result extends previous stability analyses and highlights the critical role of wave speed matching in stabilizing Timoshenko systems with second-sound thermal effects.

Keywords: Timoshenko system, Cattaneo heat conduction, polynomial stability

## 1 Introduction

We investigate the following thermoelastic Timoshenko system with Cattaneo heat conduction law

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \gamma\theta_x = 0, & x\in(0,L),\ t>0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) - \gamma\theta = 0, & x\in(0,L),\ t>0, \\ \rho_3\theta_t + q_x + \gamma(\varphi_x+\psi)_t = 0, & x\in(0,L),\ t>0, \\ \tau q_t + \beta q + \theta_x = 0, & x\in(0,L),\ t>0, \end{cases} \tag{1.1}$$

subject to the boundary conditions

$$\varphi_x(0,t) = \varphi_x(L,t) = \psi(0,t) = \psi(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad t\geq 0, \tag{1.2}$$

and initial data

$$\begin{cases} \varphi(x,0)=\varphi_0(x), & \varphi_t(x,0)=\varphi_1(x), & x\in(0,L), \\ \psi(x,0)=\psi_0(x), & \psi_t(x,0)=\psi_1(x), & x\in(0,L), \\ \theta(x,0)=\theta_0(x), & q(x,0)=q_0(x), & x\in(0,L), \end{cases} \tag{1.3}$$

where $\rho_1,\rho_2,\rho_3,k,b,\gamma,\tau,\beta,L$ are positive constants. Problem (1.1)–(1.3) was studied by M. A. Jorge Silva and R. Racke in [4, Section 3]. They demonstrated that the system fails to achieve exponential stability, regardless of whether the equality of wave speeds (EWS) condition

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}, \tag{1.4}$$

holds or not. However, by introducing a memory term, they established exponential stability without requiring the EWS condition (1.4). A natural follow-up question arises: Is the $\boldsymbol{C_0}$-semigroup $\boldsymbol{\{S(t)\}_{t\geq 0}}$ polynomially stable? The primary objective of this paper is to address this question. We will prove that $\{S(t)\}_{t\geq 0}$ is indeed polynomially stable when the EWS condition (1.4) is satisfied.

The classical Timoshenko beam model reads as

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \end{cases} \tag{1.5}$$

which has been extensively analyzed in [2, 6, 8] and references therein. It is well-established that when linear damping is present in both equations of (1.5), exponential stability can be achieved without requiring any conditions on wave propagation speeds. However, if damping acts only on one equation, exponential stability holds if and only if the equality of wave speeds (EWS) condition (1.4) is satisfied.

When thermal effects are incorporated, Hugo D. Fernández Sare and Reinhard Racke [3] examined a thermoelastic model with Cattaneo-type thermal damping in the bending moment dynamics, leading to the system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \\ \rho_3\theta_t + p_x + \gamma\psi_{xt} = 0, \\ \tau p_t + \beta p + \theta_x = 0. \end{cases} \tag{1.6}$$

As shown in [5], when $\tau = 0$ (Fourier's law), the system is exponentially stable if and only if the EWS condition (1.4) holds. However, [3] demonstrated that under Cattaneo's law ($\tau > 0$), exponential stability fails even when (1.4) is satisfied. This counterintuitive result motivated Santos et al. [7] to derive a modified stability criterion: exponential stability holds if and only if $\chi_0 := \left(\tau - \frac{\rho_1}{\rho_3 k}\right)\left(\rho_2 - \frac{b\rho_1}{k}\right) - \frac{\tau\rho_1\gamma^2}{\rho_3 k} = 0$, which reduces to (1.4) when $\tau = 0$. Furthermore, polynomial energy decay occurs when $\chi_0 \neq 0$.

Returning to system (1.1)–(1.3), we define the spaces

$$L^2_*(0,L) := \left\{ f \in L^2(0,L) \;\middle|\; \int_0^L f(x)\,dx = 0 \right\}, \quad H^1_*(0,L) := H^1(0,L) \cap L^2_*(0,L),$$

and construct the Hilbert space

$$\mathcal{H} := H^1_*(0,L) \times L^2_*(0,L) \times H^1_0(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L),$$

equipped with the inner product

$$(U_1, U_2)_{\mathcal{H}} := \int_0^L \Big[\rho_1\Phi_1\Phi_2 + \rho_2\Psi_1\Psi_2 + b\psi_{1,x}\psi_{2,x} + k(\varphi_{1,x} + \psi_1)(\varphi_{2,x} + \psi_2) + \rho_3\theta_1\theta_2 + \tau q_1 q_2\Big]\,dx$$

and norm $\|U\|^2_{\mathcal{H}} = (U,U)_{\mathcal{H}}$, where $U_1 = (\varphi_1, \Phi_1, \psi_1, \Psi_1, \theta_1, q_1)^T$, $U_2 = (\varphi_2, \Phi_2, \psi_2, \Psi_2, \theta_2, q_2)^T$, and $U = (\varphi, \Phi, \psi, \Psi, \theta, q)^T \in \mathcal{H}$. Here, $\|\cdot\|$ denotes the standard $L^2$-norm.

By setting $\Phi = \varphi_t$ and $\Psi = \psi_t$, we reformulate (1.1)–(1.3) as a first-order evolution system:

$$\begin{cases} U_t = \mathcal{A}U, & t > 0, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) =: U_0, \end{cases}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{A}U = \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{\gamma}{\rho_1}\theta_x \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{\gamma}{\rho_2}\theta \\ -\frac{1}{\rho_3}q_x - \frac{\gamma}{\rho_3}(\Phi_x + \Psi) \\ -\frac{\beta}{\tau}q - \frac{1}{\tau}\theta_x \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ U \in \mathcal{H} \,\middle|\, \Phi \in H^1_*(0, L),\ \varphi_x, \Psi, \theta \in H^1_0(0, L),\ q \in H^1(0, L),\ \varphi, \psi \in H^2(0, L) \right\}.$$

From [4, Theorem 3.1], we have: The operator $\mathcal{A}$ generates a $C_0$-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$. However, $\{S(t)\}_{t\geq 0}$ is not exponentially stable, regardless of whether the EWS condition (1.4) holds.

This paper extends these results by proving polynomial stability under the EWS condition:

Theorem 1.1. Assume the EWS condition (1.4) holds. Then the semigroup $\{S(t)\}_{t\geq 0}$ is polynomially stable, satisfying $\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{\sqrt[4]{t}}\|\mathcal{A}U_0\|_{\mathcal{H}}$, $\forall t > 0$ amd $U_0 \in D(\mathcal{A})$,where $C > 0$ is a constant independent of $U_0$ and $t$.

## 2 Proof of Theorem 1.1

In this section, we denote by $C$, a general positive constant independent of $\lambda$, $t$, $U_0$, which may change from line to line. The following facts can be found in [4, pages 187-188]:

(F1) $0 \in \rho(\mathcal{A})$, and

(F2) $\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} \leq -\beta\|q\|^2$ for any $U = (\varphi, \Phi, \psi, \Psi, \theta, q)^T \in D(\mathcal{A})$.

Moreover, since $D(\mathcal{A}) \hookrightarrow \mathcal{H}$ compactly, we get

(F3) $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, where $\sigma(\mathcal{A})$ is the spectral set of $\mathcal{A}$, and $\sigma_p(\mathcal{A})$ is the point spectral set of $\mathcal{A}$.

Lemma 2.1. We have $i\mathbb{R} \subset \varrho(\mathcal{A})$, where $\varrho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$.

Proof. We proceed by contradiction. If the conclusion is not hold, by (F3), $i\mathbb{R} \cap \sigma_p(\mathcal{A}) \neq \emptyset$, i.e., $\mathcal{A}$ admits an imaginary eigenvalue $i\lambda$ with $\lambda \in \mathbb{R}$ and a corresponding eigenvector $U = (\varphi, \Phi, \psi, \Psi, \theta, q) \neq 0$ such that $\mathcal{A}U = i\lambda U$, i.e.,

$$\begin{cases} i\lambda\varphi - \Phi = 0, \\ i\lambda\rho_1\Phi - k(\varphi_x + \psi)_x + \gamma\theta_x = 0, \\ i\lambda\psi - \Psi = 0, \\ i\lambda\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi) - \gamma\theta = 0, \\ i\lambda\rho_3\theta + q_x + \gamma(\Phi_x + \Psi) = 0, \\ i\lambda\tau q + \beta q + \theta_x = 0. \end{cases} \tag{2.1}$$

Moreover, by (F1), $\lambda \neq 0$. Then it follows from (F2) that $q = 0$. Consequently, by $(2.1)_6$ (i.e., $\theta_x = 0$) and $\theta \in H_0^1(0, L)$, we get $\theta = 0$. Examining the system $(2.1)_5$, we obtain $\Phi_x + \Psi = 0$. Then it follows the equations $(2.1)_1$ and $(2.1)_3$, that $\varphi_x + \psi = 0$. Since $\theta_x = 0$ and $\lambda \neq 0$, substituting this equality into $(2.1)_2$ forces $\Phi = 0$, which in turn implies $\Psi = 0$ by $\Phi_x + \Psi = 0$ and $\varphi = 0$ by $(2.1)_1$. Returning to $(2.1)_4$, we find $\psi_{xx} = 0$. Then by $\psi \in H^2(0, L) \cap H_0^1(0, L)$, we get $\psi = 0$. So the above analysis shows that $U = 0$, which contradicts $U \neq 0$. □

Let $\lambda \in \mathbb{R}$ and $F = (f^1, \ldots, f^6) \in \mathcal{H}$. Since, by Lemma 2.1, $i\lambda \in \rho(A)$, $U = (\varphi, \Phi, \psi, \Psi, \theta, q) := (i\lambda - \mathcal{A})^{-1}F \in D(\mathcal{A})$ satisfies

$$i\lambda U - \mathcal{A}U = F, \tag{2.2}$$

i.e.,

$$\begin{cases} i\lambda\varphi - \Phi = f^1, \\ i\lambda\rho_1\Phi - k(\varphi_x + \psi)_x + \gamma\theta_x = \rho_1 f^2, \\ i\lambda\psi - \Psi = f^3, \\ i\lambda\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi) - \gamma\theta = \rho_2 f^4, \\ i\lambda\rho_3\theta + q_x + \gamma(\Phi_x + \Psi) = \rho_3 f^5, \\ i\lambda\tau q + \beta q + \theta_x = \tau f^6. \end{cases} \tag{2.3}$$

Lemma 2.2. $\|q\|^2 \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$.

Proof. By applying (F2) and (2.2), we directly derive

$$\beta \int_0^L q^2 dx = \mathrm{Re}\left[i\lambda\|U\|_{\mathcal{H}}^2 - (\mathcal{A}U, U)_{\mathcal{H}}\right] = \mathrm{Re}(i\lambda U - \mathcal{A}U, U)_{\mathcal{H}} = \mathrm{Re}(F, U)_{\mathcal{H}} \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

So Lemma 2.2 follows. □

Lemma 2.3. For any $\epsilon > 0$, there exists a positive constant $C_\epsilon$ such that: $\rho_3\|\theta\|^2 \leq \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2$.

Proof. Integrating equation $(2.3)_6$ over $(0, x) \subset (0, L)$, since $\theta(0) = 0$, we obtain

$$i\lambda\tau \int_0^x q(y)dy + \beta \int_0^x q(y)dy + \theta(x) = \tau \int_0^x f^6(y)dy \tag{2.4}$$

Taking the $L^2(0, L)$-inner product of (2.4) with $\theta(x)$ yields:

$$\|\theta\|^2 = \underbrace{-\tau \int_0^L \int_0^x q(y)dy\,(i\lambda\theta(x))dx}_{=:J_1} - \beta \int_0^L \int_0^x q(y)dy\,\theta(x)dx + \tau \int_0^L \int_0^x f^6(y)dy\,\theta(x)dx. \tag{2.5}$$

Using the identity $(2.3)_5$, we can express $J_1$ as

$$\begin{aligned}J_1 =& -\frac{\tau}{\rho_3}\int_0^L\int_0^x q(y)dy\,\left(\rho_3 f^5 - q_x - \gamma\Phi_x - \gamma\Psi\right)dx\\ =&\frac{\tau}{\rho_3}\left(q(L)+\gamma\Phi(L)\right)\int_0^L q dx - \frac{\tau}{\rho_3}\int_0^L |q|^2 dx\\ &-\frac{\tau\gamma}{\rho_3}\int_0^L q\Phi dx + \frac{\tau\gamma}{\rho_3}\int_0^L\int_0^x q(y)dy\Psi(x)dx - \tau\int_0^L\int_0^x q(y)dy f^5(x)dx.\end{aligned}$$

Substituting this result into (2.5), we obtain

$$\begin{aligned}\rho_3\|\theta\|^2 =& -\tau\int_0^L |q|^2 dx - \tau\gamma\int_0^L q\Phi dx + \tau\gamma\int_0^L\int_0^x q(y)dy\Psi(x)dx - \rho_3\beta\int_0^L\int_0^x q(y)dy\theta(x)dx\\ &-\rho_3\tau\int_0^L\int_0^x q(y)dy f^5(x)dx + \rho_3\tau\int_0^L\int_0^x f^6(y)dy\,\theta(x)dx + \underbrace{\tau\left(q(L)+\gamma\Phi(L)\right)\int_0^L q dx}_{=:J_2}.\end{aligned} \tag{2.6}$$

To estimate $J_2$, we integrate $(2.3)_5$ over $(x, L)$ to obtain

$$i\lambda\rho_3\int_x^L \theta(s)ds + \int_x^L q_s(s)ds + \gamma\int_x^L (\Phi_s+\Psi)(s)ds = \rho_3\int_x^L f^5(s)ds. \tag{2.7}$$

Then, we have

$$q(L)+\gamma\Phi(L) = q(x)+\gamma\Phi(x) - i\lambda\rho_3\int_x^L \theta(s)ds - \gamma\int_x^L \Psi(s)ds + \rho_3\int_x^L f^5(s)ds. \tag{2.8}$$

Multiplying (2.8) by $\tau\int_0^L q(z)dz$ gives

$$\begin{aligned}J_2 =&\rho_3\tau\int_x^L f^5(s)ds\int_0^L q(z)dz + \tau[q(x)+\gamma\Phi(x)]\int_0^L q(z)dz\\ &-\gamma\tau\int_x^L \Psi(s)ds\int_0^L q(z)dz \underbrace{-\rho_3\int_x^L \theta(s)ds\int_0^L (i\lambda\tau q)(z)dz}_{=:J_3}.\end{aligned}$$

Applying $(2.3)_6$ to $J_3$ and using $\theta(0)=\theta(L)=0$, we rewrite $J_3$ as

$$J_3 = -\rho_3\int_x^L \theta(s)ds\int_0^L (\tau f^6 - \beta q - \theta_z)(z)dz = \rho_3\beta\int_x^L \theta(s)ds\int_0^L q(z)dz - \rho_3\tau\int_x^L \theta(s)ds\int_0^L f^6(z)dz.$$

Substituting this into $J_2$, we derive

$$\begin{aligned}J_2 =&\rho_3\tau\int_x^L f^5(s)ds\int_0^L q(z)dz + \tau[q(x)+\gamma\Phi(x)]\int_0^L q(z)dz\\ &-\gamma\tau\int_x^L \Psi(s)ds\int_0^L q(z)dz + \rho_3\beta\int_x^L \theta(s)ds\int_0^L q(z)dz - \rho_3\tau\int_x^L \theta(s)ds\int_0^L f^6(z)dz\end{aligned}$$

Integrating $J_2$ over $x \in (0, L)$ and applying Hölder's inequality and Lemma 2.2, we deduce

$$L|J_2| \leq C\int_0^L |q(z)|dz\int_0^L\int_0^L |f^5(s)|dsdx + C\int_0^L |q(z)|dz\int_0^L\int_0^L |\Psi(s)|dsdx$$

$$
\begin{aligned}
&+C\int_0^L|q(z)|dz\int_0^L\int_0^L|\theta(s)|dsdx+C\int_0^L|\theta(s)|ds\int_0^L\int_0^L|f^6(s)|dsdx\\
&+C\int_0^L|q(z)|dz\int_0^L|q(x)|dx+C\int_0^L|q(z)|dz\int_0^L|\Phi(x)|dx\\
\leq &C\|q\|\|f^5\|+C\|q\|\|\Psi\|+C\|q\|\|\theta\|+C\|\theta\|\|f^6\|+C\|q\|^2+C\|q\|\|\Phi\|\\
\leq &C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|q\|\|U\|_{\mathcal{H}}+C\|q\|\|U\|_{\mathcal{H}}+C\|q\|\|\theta\|.
\end{aligned}
$$

Returning to (2.6), by a simple calculation, we conclude $\rho_3\|\theta\|^2\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|q\|\|U\|_{\mathcal{H}}+C\|q\|\|U\|_{\mathcal{H}}+C\|q\|\|\theta\|$. Finally, invoking Lemma 2.2 and Young's inequality with $\epsilon>0$, we establish the validity of this lemma. □

Lemma 2.4. For any $\epsilon>0$, there exists a positive constant $C_\epsilon$ such that: $\|\varphi_x+\psi\|^2\leq\epsilon\|U\|_{\mathcal{H}}^2+C_\epsilon\|F\|_{\mathcal{H}}^2$.

Proof. By equations $(2.3)_1$ and $(2.3)_3$, we rewrite $(2.3)_5$ as

$$
i\lambda\rho_3\theta+q_x+i\lambda\gamma(\varphi_x+\psi)=\rho_3f^5+\gamma(f_x^1+f^3). \tag{2.9}
$$

Multiplying (2.9) by $k(\varphi_x+\psi)$ and integrating over $(0,L)$, since $\varphi_x(L)=\varphi_x(0)=\psi(L)=\psi(0)$, it follows

$$
i\lambda k\gamma\|\varphi_x+\psi\|^2=\underbrace{\int_0^L\left[\rho_3f^5+\gamma(f_x^1+f^3)\right]k(\varphi_x+\psi)dx}_{=:J_4}+\underbrace{k\int_0^L q(\varphi_x+\psi)_xdx}_{=:J_5}-i\lambda k\rho_3\int_0^L\theta(\varphi_x+\psi)dx.
$$

The term $J_4$ satisfies $|J_4|\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. From equation $(2.3)_2$ and $(2.3)_6$, we derive:

$$
J_5=i\lambda\rho_1\int_0^L q\Phi dx-\rho_1\int_0^L qf^2dx+\tau\gamma\int_0^L qf^6dx-\beta\gamma\int_0^L|q|^2dx-i\lambda\gamma\tau\int_0^L|q|^2dx
$$

which, together with Lemma 2.2, implies the estimate $|J_5|\leq C|\lambda|\|q\|\|\Phi\|+C|\lambda|\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Then we obtain from the above relations that $k\gamma\|\varphi_x+\psi\|^2\leq C\|q\|\|\Phi\|+C\|\theta\|\|\varphi_x+\psi\|+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. So the conclusion follows from the above inequality, Lemmas 2.2, 2.3 and Young's inequality with $\epsilon>0$. □

Lemma 2.5. Let $|\lambda|\geq 1$. For any $\epsilon>0$, there exists a positive constant $C_\epsilon$ such that: $\rho_1\|\Phi\|^2\leq\epsilon\|U\|_{\mathcal{H}}^2+C_\epsilon\|F\|_{\mathcal{H}}^2$.

Proof. Multiplying equation $(2.3)_2$ by $\varphi$, integrating over $(0,L)$, since by $(2.3)_1$, $i\lambda\varphi=\Phi+f^1$, we derive

$$
\rho_1\int_0^L\Phi(\Phi+f^1)dx=-k\int_0^L(\varphi_x+\psi)\varphi_xdx-\gamma\int_0^L\theta_x\varphi dx+\rho_1\int_0^L f^2\varphi dx.
$$

Then, it follows that

$$
\rho_1\|\Phi\|^2=-k\int_0^L|\varphi_x+\psi|^2dx\underbrace{-\gamma\int_0^L\theta_x\varphi dx+\rho_1\int_0^L f^2\varphi dx-\rho_1\int_0^L\Phi f^1dx+k\int_0^L(\varphi_x+\psi)\psi dx}_{=:J_6}. \tag{2.10}
$$

Invoking $(2.3)_1$, $(2.3)_3$ and $(2.3)_6$, we express $J_6$ as

$$
J_6:=-\frac{i}{\lambda}k\int_0^L(\varphi_x+\psi)\left(\Psi+f^3\right)dx+\frac{i\gamma\tau}{\lambda}\int_0^L f^6\left(\Phi+f^1\right)dx
$$

$$+\left(\gamma\tau - \frac{i\gamma\beta}{\lambda}\right)\int_0^L q\left(\Phi + f^1\right)dx + \rho_1\int_0^L (f^2\varphi - \Phi f^1)dx.$$

So for $|\lambda| \geq 1$, $J_6$ obviously satisfies

$$|J_6| \leq C\,\|\varphi_x + \psi\|\,\|U\|_{\mathcal{H}} + C\|q\|\|\Phi\| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}.$$

Returning to (2.10) and employing Lemmas 2.2 and 2.4, and repeated applications of Young's inequality with $\epsilon > 0$, we conclude this lemma. □

Lemma 2.6. For any $\epsilon > 0$, there exists a positive constant $C_\epsilon$ such that: $\rho_2\|\Psi\|^2 \leq b\|\psi_x\|^2 + \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}$.

Proof. Multiplying equation $(2.3)_4$ by $\psi$ and integrating over $(0, L)$, it follows from the equation $(2.3)_3$ that

$$\rho_2\int_0^L |\Psi|^2\,dx = -b\int_0^L |\psi_x|^2\,dx - k\int_0^L (\varphi_x + \psi)\psi\,dx + \gamma\int_0^L \theta\psi\,dx + \rho_2\int_0^L (f^4\psi - \Psi f^3)\,dx.$$

Then, we have $\rho_2\|\Psi\|^2 \leq b\|\psi_x\|^2 + C\|\varphi_x + \psi\|\|U\|_{\mathcal{H}} + C\|\theta\|\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Applying Lemmas 2.3, 2.4 and Young's inequality with $\epsilon > 0$, we conclude the result. □

Lemma 2.7. Let $|\lambda| \geq 1$. For any $\epsilon > 0$, there exists a positive constant $C_\epsilon$ such that: $\|\psi_x\|^2 \leq C|\chi||\lambda|\|\varphi_x + \psi\|\|\Psi\| + C|\lambda|\|q\|\|\psi_x\| + \epsilon\|U\|^2_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}}$, where $\chi = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2}$.

Proof. Starting from equations $(2.3)_2$ and $(2.3)_4$, we derive

$$i\lambda(\Phi_x + \Psi) - \frac{k}{\rho_1}(\varphi_x + \psi)_{xx} + \frac{\gamma}{\rho_1}\theta_{xx} - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{\gamma}{\rho_2}\theta = f^2_x + f^4. \tag{2.11}$$

Multiplying (2.11) by $\psi$ and integrating by parts over $(0, L)$, by $(2.3)_3$, we get

$$\frac{b}{\rho_2}\|\psi_x\|^2 = -\int_0^L (\Phi_x + \Psi)\Psi dx + \frac{\gamma}{\rho_2}\int_0^L \theta\psi dx - \frac{k}{\rho_2}\int_0^L (\varphi_x + \psi)\psi dx + J_7 + J_8 + J_9, \tag{2.12}$$

where $J_7 := \int_0^L (f^2_x + f^4)\psi dx - \int_0^L (\Phi_x + \Psi)f^3 dx$, $\quad J_8 = \frac{\gamma}{\rho_1}\int_0^L \theta_x\psi_x dx$, $\quad J_9 = \frac{k}{\rho_1}\int_0^L (\varphi_x + \psi)\psi_{xx} dx$.

Since, by $(2.3)_{4,1,3}$,

$$\begin{aligned} J_9 &= \frac{k\rho_2}{b\rho_1}\int_0^L (\varphi_x + \psi)(i\lambda\Psi)dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x + \psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x + \psi)\theta dx - \frac{k\rho_2}{b\rho_1}\int_0^L (\varphi_x + \psi)f^4 dx \\ &= \frac{\rho_2 k}{b\rho_1}\int_0^L (\Phi_x + \Psi)\Psi dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x + \psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x + \psi)\theta dx + J_{10}, \end{aligned}$$

where $J_{10} = \frac{\rho_2 k}{b\rho_1}\int_0^L (f^1_x + f^2)\Psi dx - \frac{k\rho_2}{b\rho_1}\int_0^L (\varphi_x + \psi)f^4 dx$, we get

$$\begin{aligned} \frac{b}{\rho_2}\|\psi_x\|^2 = \frac{\rho_2}{b}\chi\int_0^L (\Phi_x + \Psi)\Psi dx + \frac{k^2}{b\rho_1}\int_0^L |\varphi_x + \psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L (\varphi_x + \psi)\theta dx \\ + \frac{\gamma}{\rho_2}\int_0^L \theta\psi dx - \frac{k}{\rho_2}\int_0^L (\varphi_x + \psi)\psi dx + J_7 + J_8 + J_{10}. \end{aligned} \tag{2.13}$$

It is obvious that $|J_7| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. and $|J_{10}| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. For $|\lambda| \geq 1$, we get from the equation $(2.3)_6$ that $J_8 = \frac{\gamma}{\rho_1}\int_0^L(\tau f^6 - i\lambda\tau q - \beta q)\psi_x d \leq C\int_0^L f^6\psi_x dx + C|\lambda|\int_0^L q\psi_x dx \leq C|\lambda|\|q\|\|\psi_x\| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Using above estimates and $(2.3)_{1,3}$, by Young's inequality and Young's inequality with $\varepsilon = \frac{b}{2\rho}$, we have

$$
\begin{aligned}
\frac{b}{\rho_2}\|\psi_x\|^2 &= \frac{i\lambda\rho_2}{b}\chi\int_0^L(\varphi_x+\psi)\Psi dx + \frac{k^2}{b\rho_1}\int_0^L|\varphi_x+\psi|^2 dx - \frac{k\gamma}{b\rho_1}\int_0^L(\varphi_x+\psi)\theta dx \\
&\quad + \frac{\gamma}{\rho_2}\int_0^L\theta\psi dx - \frac{k}{\rho_2}\int_0^L(\varphi_x+\psi)\psi dx - \frac{\rho_2}{b}\chi\int_0^L(f_x^1+f^3)\Psi dx + J_7 + J_8 + J_{10} \\
&\leq C|\chi||\lambda|\|\varphi_x+\psi\|\|\Psi\| + C\|\varphi_x+\psi\|^2 + C\|\varphi_x+\psi\|\|\theta\| \\
&\quad + C\|\theta\|\|\psi_x\| + C\|\varphi_x+\psi\|\|\psi_x\| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + J_7 + J_8 + J_{10} \\
&\leq C|\chi||\lambda|\|\varphi_x+\psi\|\|\Psi\| + C|\lambda|\|q\|\|\psi_x\| + C\|\varphi_x+\psi\|^2 + \frac{b}{2\rho_2}\|\psi_x\|^2 + C\|\theta\|^2 + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},
\end{aligned}
$$

which implies $\|\psi_x\|^2 \leq C|\chi||\lambda|\|\varphi_x+\psi\|\|\Psi\| + C|\lambda|\|q\|\|\psi_x\| + C\|\varphi_x+\psi\|^2 + C\|\theta\|^2 + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Applying Lemmas 2.3 , 2.4 and Young's inequality with $\epsilon > 0$, we conclude the result. □

Proof of Theorem 1.1. Our proof depends on the following results [1]:

(JL) Let $S(t)$ be a bounded $C_0$-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. If $i\mathbb{R} \subset \varrho(\mathcal{A})$, then for every fixed $\alpha > 0$, we have $\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^\alpha$ as $|\lambda| \to +\infty$ if and only if $\left\|S(t)\mathcal{A}^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/\alpha}}$ as $|\lambda| \to +\infty$.

Since, by our assumption $\chi = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2} = 0$. By using Lemmas 2.7, 2.6 and 2.2, we get, for any $\epsilon > 0$ and $|\lambda| \geq 1$,

$$
\begin{aligned}
b\|\psi_x\|^2 + \rho_2\|\Psi\|^2 &\leq C|\lambda|\|q\|\|U\|_{\mathcal{H}} + \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2 \leq C|\lambda|^2\|q\|^2 + \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2 \\
&\leq C|\lambda|^2\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2 \leq \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon|\lambda|^4\|F\|_{\mathcal{H}}^2.
\end{aligned}
$$

Then by Lemmas 2.2-2.5, we get for any $\epsilon > 0$ and $|\lambda| \geq 1$, $\|U\|_{\mathcal{H}}^2 \leq \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon|\lambda|^4\|F\|_{\mathcal{H}}^2$ By choose $\epsilon = \frac{1}{2}$, it follows that $\frac{1}{|\lambda|^4}\|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$, which means $\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^4$. Note Lemma 2.1, our conclusion follows from (JL). □

## References

[1] Alexander Borichev and Yuri Tomilov. Optimal polynomial decay of functions and operator semigroups. Math. Ann., 347(2):455–478, 2010.

[2] Luci Harue Fatori, Rodrigo Nunes Monteiro, and Hugo D. Fernández Sare. The Timoshenko system with history and Cattaneo law. Appl. Math. Comput., 228:128–140, 2014.

[3] Hugo D. Fernández Sare and Reinhard Racke. On the stability of damped Timoshenko systems: Cattaneo versus Fourier law. Arch. Ration. Mech. Anal., 194(1):221–251, 2009.

[4] M. A. Jorge Silva and R. Racke. Effects of history and heat models on the stability of thermoelastic Timoshenko systems. J. Differential Equations, 275:167–203, 2021.

[5] Jaime E. Muñoz Rivera and Reinhard Racke. Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability. J. Math. Anal. Appl., 276(1):248–278, 2002.

[6] Belkacem Said-Houari and Aslan Kasimov. Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same. J. Differential Equations, 255(4):611–632, 2013.

[7] M. L. Santos, D. S. Almeida Júnior, and J. E. Muñoz Rivera. The stability number of the Timoshenko system with second sound. J. Differential Equations, 253(9):2715–2733, 2012.

[8] Abdelaziz Soufyane. Stabilisation de la poutre de Timoshenko. C. R. Acad. Sci. Paris Sér. I Math., 328(8):731–734, 1999.