UNDER PEER REVIEW

Existence of solutions for a mixed local and nonlocal elliptic problem in $\mathbb{R}^N$

Abstract

In this paper, we investigate the existence of solutions to the following equation:

$$-\Delta u + (-\Delta)^s u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda \geq 0$, $\mu \geq 0$, $\mu + \lambda > 0$, $0 < s < 1$, and $2_s^* \leq p < q \leq 2^*$. Here, $2_s^* = \frac{2N}{N-2s}$ and $2^* = \frac{2N}{N-2}$ denote the fractional and Sobolev critical exponents, respectively. We analyze three distinct scenarios regarding the parameters p, q, λ, and μ. By combining the mountain pass theorem with Lions' lemma and the principle of concentration compactness, we establish the existence of a nontrivial solution for each case.

Keywords: Mixed local and nonlocal operators; Sobolev inequality; Principle of concentration compactness

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1 Introduction

In this paper, we study the existence of solutions to the following equation:

$$-\Delta u + (-\Delta)^s u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\lambda \geq 0$, $\mu \geq 0$, $\mu + \lambda > 0$, $0 < s < 1$, and $2_s^* \leq p < q \leq 2^*$. Here, $2_s^* = \frac{2N}{N-2s}$ and $2^* = \frac{2N}{N-2}$ denote the fractional and Sobolev critical exponents, respectively. In the above, $\Delta u = \operatorname{div}(\nabla u)$ is the classical Laplace operator, while $(-\Delta)^s$ represents the fractional Laplacian. For any function $u \in C_0^\infty(\mathbb{R}^N)$, the fractional Laplacian is defined pointwise as

$$(-\Delta)^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(y) - u(x)}{|x-y|^{N+2s}}\, dy,$$

where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius $\varepsilon > 0$.

Equation (1.1) is motivated by the classical sharp Sobolev inequality:

$$S\left(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\right)^{\frac{1}{2^*}} \leq \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{\frac{1}{2}}, \quad 2^* = \frac{2N}{N-2}, \tag{1.2}$$

where S is the optimal constant. Sharp Sobolev inequalities, both in local and nonlocal settings, have been extensively studied and play a fundamental role in the theory of partial differential equations, geometric analysis, and continuum mechanics. The determination of sharp constants and their extremal functions is of particular importance; see, for example, [1, 2, 6, 8, 9, 13, 14, 17, 20–24, 26, 27].

In the local case, let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The classical Sobolev (q,p)-inequality states that

$$S\left(\int_\Omega |u|^q\,dx\right)^{\frac{1}{q}} \le \left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{1}{p}}, \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $1 \le p < N$ and $1 < q < p^* = \frac{Np}{N-p}$. Here, S is the Sobolev constant. A standard approach to prove the attainability of the best constant is to show that the following Dirichlet problem admits a solution $u_q \in W_0^{1,p}(\Omega)$:

$$-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda_q(\Omega)\|u\|_{L^q(\Omega)}^{p-q}|u|^{q-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

We refer to [2, 13, 20, 23] and the references therein for details in the local case. For related results in the nonlocal setting, see [7, 8, 19, 21, 22] and the references therein.

When $q = p^*$ and $\Omega = \mathbb{R}^N$, the minimization problem (1.2) can be solved via symmetrization techniques. We refer to the classical works of Aubin [1] and Talenti [27], as well as the references therein, for further details.

In the context of mixed local and nonlocal operators, Garain et al. [15] considered the following minimization problem defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ with $N \ge 2$:

$$\mu(\Omega) = \inf_{\substack{u \in W_0^{1,p}(\Omega)\\ u \not\equiv 0}} \left\{ \int_\Omega |u|^p\,dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+2s}}\,dx\,dy : \int_\Omega |u|^{1-\delta} f\,dx = 1 \right\}, \tag{1.3}$$

where $0 < \delta < 1 < p < \infty$, $0 < s < 1$, and f is a nonnegative function belonging to $L^m(\Omega) \setminus \{0\}$. The authors established that $\mu(\Omega)$ is attained by a solution of the associated Euler-Lagrange equation:

$$-\Delta_p u + (-\Delta)_p^s u = \frac{f(x)}{u(x)^\delta} \quad \text{in } \Omega, \qquad u > 0 \text{ in } \Omega, \qquad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

In the particular case where $1 - \delta = p$ and $f = 1$, the value $\mu(\Omega)$ corresponds to the first eigenvalue of the mixed operator under the Dirichlet boundary condition. This setting has been extensively studied, see, for instance, [5, 11, 12, 16] and the references therein.

It is well-known that the best constant S in the classical Sobolev inequality is attained in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla u\|_{L^2(\mathbb{R}^N)}$.

In contrast, for the mixed local and nonlocal setting, Biagi et al. [3] showed that the optimal constant $S_{s,q}$ is never attained in the space $X_0(\mathbb{R}^N)$. Here, $S_{s,q}$ is defined by

$$S_{s,q} = \inf_{\substack{u \in X_0(\mathbb{R}^N)\\ u \not\equiv 0}} \left\{ \int_{\mathbb{R}^N} |u|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy : \int_{\mathbb{R}^N} |u|^q dx = 1 \right\}, \tag{1.4}$$

with $q = 2^*$, and the space $X_0(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X_0} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}} = \left(|\nabla u|_2^2 + [u]_{s,2}^2\right)^{\frac{1}{2}},$$

where we denote $|\nabla u|_2^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$ and $[u]_{s,2} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}$.

We note from Lemma 2.1 that the embedding $X_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous, which is a distinctive feature compared to the purely local case. This leads to a natural question regarding whether the Sobolev inequality is attained when $2_s^* \le q < 2^*$. The following results provide a complete answer to this question.

Theorem 1.1. *If $q = 2_s^*$, then the optimal constant $S_{s,q}$ in (1.4) is not attained.*

Theorem 1.2. *If $q \in (2_s^*, 2^*)$, then the optimal constant $S_{s,q}$ in (1.4) is attained.*

As is well known, the study of optimizers for Sobolev-type inequalities is closely related to the existence of solutions to the equation

$$-\Delta u + (-\Delta)^s u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

When $q = 2_s^*$ or $q = 2^*$, the above equation admits no nontrivial solutions, as follows from Theorem 1.1 and [3, Theorem 1.2]. This observation naturally leads us to investigate the more general equation (1.1). We analyze three distinct parameter regimes, and our main results are summarized below.

Theorem 1.3. *For any $p, q \in (2_s^*, 2^*)$, equation (1.1) admits a nontrivial solution for all $\lambda > 0$ and $\mu > 0$.*

Theorem 1.4. *Let $p = 2_s^*$. For any $q \in (2_s^*, 2^*)$ and $\mu > 0$, there exists $\lambda^* > 0$ such that equation (1.1) admits a nontrivial solution for all $\lambda \in (0, \lambda^*)$. Equivalently, for any $q \in (2_s^*, 2^*)$ and $\lambda > 0$, there exists $\mu^* > 0$ such that equation (1.1) admits a nontrivial solution for all $\mu > \mu^*$.*

Theorem 1.5. *Let $q = 2^*$. $\lambda + \mu$ is small enough. For any $p \in (2_s^*, 2^*)$ and $\lambda > 0$, there exists $\mu^{**} > 0$ such that equation (1.1) admits a nontrivial solution for all $\mu \in (0, \mu^{**})$.*

Plan of the paper. The structure of this paper is as follows. Section 2 contains the preliminary results. Section 3 addresses the proofs of Theorems 1.1 and 1.2. Section 4 deals with the proofs of Theorems 1.3 and 1.4. In Section 5, we prove a mixed-type concentration-compactness result. Finally, Section 6 establishes the proof of Theorem 1.5.

2 Preliminaries

Lemma 2.1. *The embedding $X_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $q \in [2_s^*, 2^*]$, but it is not compact in this range.*

Proof. From the definition of $X_0(\mathbb{R}^N)$, the embeddings

$$X_0(\mathbb{R}^N) \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \quad \text{and} \quad X_0(\mathbb{R}^N) \hookrightarrow D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$$

are clearly continuous. Consequently, we have

$$S_2 \|u\|_{2^*}^2 \le \|\nabla u\|_2^2 \le \|u\|^2 \quad \text{and} \quad S_1 \|u\|_{2_s^*}^2 \le [u]_{s,2}^2 \le \|u\|^2, \tag{2.1}$$

where

$$S_1 = \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,2}^2}{\|u\|_{2_s^*}^2} \quad \text{and} \quad S_2 = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

Inequality (2.1) implies that $u \in L^{2_s^*}(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$. By the interpolation inequality, we obtain

$$\|u\|_r \leq \|u\|_{2_s^*}^{\theta} \|u\|_{2^*}^{1-\theta}, \tag{2.2}$$

for any $r \in [2_s^*, 2^*]$, where θ satisfies

$$\frac{1}{r} = \frac{\theta}{2_s^*} + \frac{1-\theta}{2^*}.$$

Combining (2.1) and (2.2) yields

$$\|u\|_r \leq \frac{1}{S_2^{(1-\theta)/2}} \cdot \frac{1}{S_1^{\theta/2}} \|u\|. \tag{2.3}$$

This completes the proof of the continuity of the embedding.

To show that the embedding is not compact, it suffices to construct a bounded sequence in $X_0(\mathbb{R}^N)$ that has no convergent subsequence in $L^q(\mathbb{R}^N)$ for any $q \in [2_s^*, 2^*]$. Note that the norm $\|\cdot\|$ is translation-invariant. For any nonzero $u \in X_0(\mathbb{R}^N)$ and any vector $y \in \mathbb{R}^N \setminus \{0\}$, define the sequence

$$u_k(x) = u(x + ky), \quad k \in \mathbb{N}.$$

Then $\{u_k\}$ is bounded in $X_0(\mathbb{R}^N)$, but is not precompact in $L^q(\mathbb{R}^N)$ for any $q \in [2_s^*, 2^*]$. □

Lemma 2.2. *Let $r > 0$ and $2_s^* \leq q < 2^*$. Suppose $\{u_n\}$ is a bounded sequence in $X_0(\mathbb{R}^N)$ such that*

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n|^q \, dx \to 0 \quad \text{as } n \to \infty.$$

Then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for every $p \in (2_s^*, 2^*)$.

Proof. Let $q < t < 2^*$ and $u \in X_0(\mathbb{R}^N)$. By the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
\|u\|_{L^t(B(y,r))} &\leq \|u\|_{L^q(B(y,r))}^{1-\lambda} \|u\|_{L^{2^*}(B(y,r))}^{\lambda} \\
&\leq c \, \|u\|_{L^q(B(y,r))}^{1-\lambda} \left(\int_{B(y,r)} |\nabla u|^2 dx \right)^{\lambda/2},
\end{aligned}$$

where $\lambda = \frac{t-q}{2^*-q} \cdot \frac{2^*}{t}$. Choosing $\lambda = 2/t$, we obtain

$$\int_{B(y,r)} |u|^t dx \leq c^t \, \|u\|_{L^q(B(y,r))}^{(1-\lambda)t} \int_{B(y,r)} |\nabla u|^2 dx.$$

Now, covering $\mathbb{R}^N$ by balls of radius r in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we find

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^t dx &\leq (N+1) c^t \sup_{y \in \mathbb{R}^N} \left[\int_{B(y,r)} |u|^q dx \right]^{(1-\lambda)t/q} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&\leq (N+1) c^t \sup_{y \in \mathbb{R}^N} \left[\int_{B(y,r)} |u|^q dx \right]^{(1-\lambda)t/q} \|u\|^2.
\end{aligned}$$

Under the assumptions of the lemma, it follows from the Sobolev and Hölder inequalities that $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for $2_s^* < t < 2^*$. Since $2_s^* \leq s \leq 2^*$, $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2_s^* < p < 2^*$. □

Definition 2.1. *We say that u is a weak solution of problem (1.1) if $u(x)$ satisfies*

$$\int_{\mathbb{R}^N} |\nabla u|^{2-2} \nabla u \nabla v dx + \iint_{\mathbb{R}^{2N}} \mathcal{A}u(x,y)(v(x)-v(y))\,dxdy = \lambda \int_{\mathbb{R}^N} |u|^{p-2} u(x) v(x)\,dx + \mu \int_{\mathbb{R}^N} |u|^{q-2} u(x) v(x)\,dx \tag{2.4}$$

for all $v \in X_0(\mathbb{R}^N)$, where for simplicity

$$\mathcal{A}u(x,y) = \frac{u(x)-u(y)}{|x-y|^{N+2s}}. \tag{2.5}$$

It is clear that the energy functional $J_{\lambda,\mu} : X_0 \to \mathbb{R}$ associated with problem (1.1), defined by

$$J_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p\,dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q\,dx,$$

is well defined on the space $X_0(\mathbb{R}^N)$. Moreover, its Fréchet derivative at u in the direction v is given by

$$\langle J'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^N} |\nabla u|^{2-2} \nabla u \nabla v dx + \iint_{\mathbb{R}^{2N}} \mathcal{A}u(x,y)(v(x)-v(y))\,dxdy - \lambda \int_{\mathbb{R}^N} |u|^{p-2} u(x) v(x)\,dx - \mu \int_{\mathbb{R}^N} |u|^{q-2} u(x) v(x)\,dx$$

for any $v \in X_0(\mathbb{R}^N)$. One can verify that $J_{\lambda,\mu} \in C^1(X_0,\mathbb{R})$, and thus the critical points of $J_{\lambda,\mu}$ correspond precisely to the weak solutions of problem (1.1).

Definition 2.2. *A sequence $\{u_n\}_n \subset X_0(\mathbb{R}^N)$ is called a $(PS)_c$ sequence, if $J_{\lambda,\mu}(u_n) \to c$ and $J'_{\lambda,\mu}(u_n) \to 0$.*

Lemma 2.3. *If $\{u_n\}_n$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$, then $\{u_n\}_n$ is bounded in $X_0(\mathbb{R}^N)$ and $c \geq 0$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence and $2 < p < q$, there exists $n_0 > 0$ such that for all $n \geq n_0$,

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 \leq J_{\lambda,\mu}(u_n) - \frac{1}{p}\langle J'_{\lambda,\mu}(u_n), u_n\rangle \leq c + o(1). \tag{2.6}$$

We conclude that $\{u_n\}$ is bounded in $X_0(\mathbb{R}^N)$. Passing to the limit in (2.6), we deduce that $c \geq 0$, which completes the proof. □

3 The best constant

In this section, we first present the proof of Theorem 1.1. To this end, we begin by determining the sharp constant in (1.4) for the case $q = 2_s^*$, defined as

$$S_{s,q} := \inf\left\{\|u\|^2 : u \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{H}(\mathbb{R}^N)\right\}, \tag{3.1}$$

where $\mathcal{H}(\mathbb{R}^N)$ denotes the unit sphere in $L^{2_s^*}(\mathbb{R}^N)$, that is,

$$\mathcal{H}(\mathbb{R}^N) := \left\{u \in L^{2_s^*}(\mathbb{R}^N) : \|u\|_{2_s^*} = 1\right\}. \tag{3.2}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $X_0(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$, and in view of the continuous embedding, we also have

$$S_{s,q} := \inf\left\{\|u\|^2 : u \in X_0(\mathbb{R}^N) \cap \mathcal{H}(\mathbb{R}^N)\right\}. \tag{3.3}$$

we need the following lemma.

Lemma 3.1. *Let $s \in (0,1)$. Then the identity $S_{s,q} = S_1 = \tilde{S}_1$ holds, where $\tilde{S}_1$ is defined as*

$$\tilde{S}_1 := \inf\{[u]_{s,2}^2 : u \in C_0^\infty\left(\mathbb{R}^N\right) \cap \mathcal{H}\left(\mathbb{R}^N\right)\}.$$

Proof. We now show that $S_{s,q} = \tilde{S}_1$. By density arguments, it follows directly that $S_{s,q} = S_1 = \tilde{S}_1$. Since $\|u\| \geq [u]_{s,2}$ for all $u \in C_0^\infty(\mathbb{R}^N)$, we have $S_{s,q} \geq \tilde{S}_1$.

To establish the reverse inequality, consider any $u \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{H}(\mathbb{R}^N)$ and define the rescaled function $u_k := k^{\frac{N-2s}{2}} u(kx)$. One readily verifies that

$$\|u_k\|_{2_s^*} = \|u\|_{2_s^*} \quad \text{and} \quad [u_k]_{s,2} = [u]_{s,2}.$$

From the definition of $S_{s,q}$, it follows that

$$S_{s,q} \leq \|u_k\|^2 = \|\nabla u_k\|_2^2 + [u_k]_{s,2}^2 = k^{2-2s}\|\nabla u\|_2^2 + [u]_{s,2}^2.$$

Letting $k \to 0$, we obtain $S_{s,q} \leq [u]_{s,2}^2$. Since u is arbitrary, we conclude that

$$S_{s,q} \leq \inf\left\{[u]_{s,2}^2 : u \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{H}(\mathbb{R}^N)\right\} = \tilde{S}_1,$$

and hence $S_{s,q} = S_1 = \tilde{S}_1$. $\square$

We now proceed to the proof of Theorem 1.1.

Proof of Theprem 1.1. We proceed by contradiction. Assume that there exists a nonzero function $u_0 \in X_0\left(\mathbb{R}^N\right)$ such that $\|u_0\|_{2_s^*} = 1$ and

$$\|u_0\|^2 = \|\nabla u_0\|_2^2 + [u_0]_{s,2}^2 = S_{s,q} = S_1.$$

By the embedding $X_0(\mathbb{R}^N) \hookrightarrow D^{1,2}\left(\mathbb{R}^N\right)$, we have

$$S_1 \leq [u_0]_{s,2}^2 \leq \|\nabla u_0\|_2^2 + [u_0]_{s,2}^2 = S_{s,q} = S_1,$$

which yields $\|\nabla u_0\|_2^2 = 0$. Therefore, u_0 must be constant throughout $\mathbb{R}^N$, contradicting the assumption $\|u_0\|_{2_s^*} = 1$. $\square$

In order to prove Theorem 1.2, we employ the constrained minimization method. Consider a minimizing sequence $\{u_n\}_n \subset X_0\left(\mathbb{R}^N\right)$ satisfying

$$\|u_n\|_q = 1, \quad \|u_n\|^2 \to S_{s,q}, \quad n \to \infty. \tag{3.4}$$

If necessary, we may extract a subsequence. By Lemma 2.3, we may assume that $u_n \rightharpoonup u$ in $X_0\left(\mathbb{R}^N\right)$, which implies

$$\|u\|^2 \leq \varliminf_{n\to\infty} \|u_n\|^2 = S_{s,q}.$$

Therefore, u is a minimizer provided that $\|u\|_q = 1$. However, we only know that $\|u\|_q \le 1$. Indeed, for any $v \in X_0(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$, the translated function $v^y(x) := v(x+y)$ satisfies

$$\|v^y\|_q = \|v\|_q, \quad \|v^y\| = \|v\|.$$

Consequently, the problem is invariant under the non-compact group of translations.

Proof of Theorem 1.2. Since $\|u\|_q = 1$ and $q \in (2_s^*, 2^*)$, Lemma 2.2 implies that

$$\delta := \varliminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^{2_s^*} > 0.$$

Passing to a subsequence if necessary, we may assume the existence of $(y_n) \subset \mathbb{R}^N$ such that $\int_{B(y_n,1)} |u_n|^{2_s^*} > \frac{\delta}{2}$. Define $v_n := u_n^{y_n}$. Then $\|v_n\|_q = 1$, $\|v_n\|^2 \to S_{s,q}$ and $\int_{B(0,1)} |v_n|^{2_s^*} > \frac{\delta}{2}$. Since $\{v_n\}$ is bounded in $X_0(\mathbb{R}^N)$, we may assume, passing to a subsequence if necessary,

$$v_n \rightharpoonup v \text{ in } X_0(\mathbb{R}^N), \quad v_n \to v \text{ in } L_{loc}^{2_s^*}(\mathbb{R}^N), \quad v_n \to v \text{ a.e. on } \mathbb{R}^N.$$

By the Brézis-Lieb Lemma,

$$1 = \|v\|_q^q + \lim_{n\to\infty} \|w_n\|_q^q,$$

where $w_n := v_n - v$. Hence we have

$$\begin{aligned} S_{s,q} = \lim_{n\to\infty} \|v_n\|^2 &= \|v\|^2 + \lim_{n\to\infty} \|w_n\|^2 \\ &\ge S_{s,q} \left[\|v\|_q^2 + \left(1 - \|v\|_q^q\right)^{\frac{2}{q}} \right]. \end{aligned}$$

Since $\int_{B(0,1)} |v_n|^{2_s^*} > \frac{\delta}{2}$ and $v \neq 0$, we obtain $\|v\|_q^q = 1$, and thus

$$\|v\|^2 = S_{s,q} = \lim_{n\to\infty} \|v_n\|^2.$$

The proof is complete. □

4 Proof of Theorem 1.3 and Theorem 1.4

To prove Theorems 1.3 and 1.4, we require the following two lemmas.

Lemma 4.1. *If $\{u_n\}_n$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$, then, passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup 0$ in $X_0(\mathbb{R}^N)$. Then, for every $2_s^* \le p < q < 2^*$, we have*

$$c \ge \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right) \left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}} \quad \textit{or} \quad c = 0.$$

Proof. From Lemma 2.3, we know that $\{u_n\}_n$ is bounded in $X_0(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$ in $X_0(\mathbb{R}^N)$, we have

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^q \to 0, \quad n \to \infty.$$

By Lemma 2.2, we obtain $\|u_n\|_q^q \to 0$.

Since $\{u_n\}_n$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ and $\|u_n\|_q^q \to 0$, it follows that

$$\begin{aligned} c = J_{\lambda,\mu}(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u_n|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u_n|^q dx + o(1) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u_n|^p dx + o(1), \end{aligned} \tag{4.1}$$

and

$$0 = \|u_n\|^2 - \lambda\int_{\mathbb{R}^N} |u_n|^p dx - \mu\int_{\mathbb{R}^N} |u_n|^q dx + o(1) = \|u_n\|^2 - \lambda\int_{\mathbb{R}^N} |u_n|^p dx + o(1). \tag{4.2}$$

From (4.2) and the Sobolev inequality, we obtain

$$\|u_n\|_p \geq \left(\frac{S_{s,p}}{\lambda}\right)^{\frac{1}{p-2}} \quad \text{or} \quad \|u_n\|_p = 0. \tag{4.3}$$

Then (4.1) and (4.3) imply that

$$c \geq \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}} \quad \text{or} \quad c = 0.$$

This completes the proof. □

Lemma 4.2. *For each $\lambda, \mu > 0$, there exist $\vartheta > 0$ and $\rho > 0$ such that $J_{\lambda,\mu}(u) \geq \vartheta$ for all $u \in X_0(\mathbb{R}^N)$ with $\|u\| = \rho$.*

Proof. It follows from the Sobolev inequality that

$$\begin{aligned} J_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda S_{s,p}^{-\frac{p}{2}}}{p}\|u\|^p - \frac{\mu S_{s,q}^{-\frac{q}{2}}}{q}\|u\|^q. \end{aligned} \tag{4.4}$$

Define the function

$$g(t) := \frac{1}{2} - \frac{\lambda S_{s,p}^{-\frac{p}{2}}}{p} t^{p-2} - \frac{\mu S_{s,q}^{-\frac{q}{2}}}{q} t^{q-2} \quad \text{for all } t \geq 0.$$

Clearly, $\lim_{t\to 0^+} g(t) = \frac{1}{2} > 0$ since $p, q > 2$. Taking $\rho := \|u\|$ sufficiently small such that

$$\frac{\lambda S_{s,p}^{-\frac{p}{2}}}{p}\rho^{p-2} + \frac{\mu S_{s,q}^{-\frac{q}{2}}}{q}\rho^{q-2} < \frac{1}{2},$$

we obtain

$$J(u) \geq g(\rho)\rho^2 =: \vartheta.$$

This concludes the proof. □

We are now ready to prove Theorems 1.3 and 1.4. Our goal is to identify a specific $(PS)_c$ sequence $\{u_n\}_n$ for which the value of c is less than $\left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}}$. By Lemma 4.1, we can conclude that $u_n \rightharpoonup u$ with $u \neq 0$.

Proof of Theorem 1.3. We verify the assumptions of the mountain pass theorem. By Theorem 1.2, there exists $v_0 \in X_0(\mathbb{R}^N)$ such that $\|v_0\|^2 = S_{s,p}$ and $\|v_0\|_p = 1$. For any $t \geq 0$, we have

$$J_{\lambda,\mu}(tv_0) = \frac{1}{2}t^2\|v_0\|^2 - \frac{\lambda}{p}t^p \int_{\mathbb{R}^N} |v_0|^p dx - \frac{\mu}{q}t^q \int_{\mathbb{R}^N} |v_0|^q dx.$$

Since $2 < 2_s^* \leq p < q < 2^*$, there exists $e := t_0 v_0$ satisfying $\|e\| > \rho$, $J_{\lambda,\mu}(e) \leq 0$ and

$$\max_{t\geq 0} J_{\lambda,\mu}(te) < \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}}.$$

Define

$$c_{\lambda,\mu} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X_0(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$.

By Lemma 4.2, we obtain $\vartheta \leq c_{\lambda,\mu} < \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}}$. Therefore, we obtain a $(PS)_c$ sequence $\{u_n\}_n$ satisfying $u_n \rightharpoonup u$ with $u \neq 0$, and the level c satisfies $c < \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}}$. One can verify that $J'_{\lambda,\mu}(u) = 0$ and $J_{\lambda,\mu}(u) = c_{\lambda,\mu}$. Hence, u is a nontrivial solution to equation (1.1). $\square$

Proof of Theorem 1.4. Similarly, we verify the assumptions of the mountain pass theorem. By Theorem 1.2, there exists $\tilde{v}_0 \in X_0(\mathbb{R}^N)$ such that $\|\tilde{v}_0\|^2 = S_{s,q}$ and $\|\tilde{v}_0\|_q = 1$. For any $t \geq 0$, we have

$$J_{\lambda,\mu}(t\tilde{v}_0) = \frac{1}{2}t^2\|\tilde{v}_0\|^2 - \frac{\lambda}{p}t^p \int_{\mathbb{R}^N} |\tilde{v}_0|^p dx - \frac{\mu}{q}t^q \int_{\mathbb{R}^N} |\tilde{v}_0|^q dx.$$

Since $2 < 2_s^* \leq p < q < 2^*$, there exists $\tilde{e} := \tilde{t}_0 \tilde{v}_0$ satisfying $\|\tilde{e}\| > \rho$, $J_{\lambda,\mu}(\tilde{e}) \leq 0$ and

$$\max_{t\geq 0} J_{\lambda,\mu}(t\tilde{e}) < \left(\frac{\mu}{2} - \frac{\mu}{q}\right)\left(\frac{S_{s,q}}{\mu}\right)^{\frac{q}{q-2}}.$$

Define

$$\tilde{c}_{\lambda,\mu} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X_0(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = \tilde{e}\}$.

By Lemma 4.2, we obtain $\vartheta \leq \tilde{c}_{\lambda,\mu} < \left(\frac{\mu}{2} - \frac{\mu}{q}\right)\left(\frac{S_{s,q}}{\mu}\right)^{\frac{q}{q-2}}$. For every $\mu > 0$, we can choose λ^* such that

$$\left(\frac{\mu}{2} - \frac{\mu}{q}\right)\left(\frac{S_{s,q}}{\mu}\right)^{\frac{q}{q-2}} \leq \left(\frac{\lambda^*}{2} - \frac{\lambda^*}{p}\right)\left(\frac{S_{s,p}}{\lambda^*}\right)^{\frac{p}{p-2}} \leq \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}},$$

for any $\lambda \in (0, \lambda^*)$. Alternatively, for every $\lambda > 0$, we can choose μ^* such that

$$\left(\frac{\mu}{2} - \frac{\mu}{q}\right)\left(\frac{S_{s,q}}{\mu}\right)^{\frac{q}{q-2}} \leq \left(\frac{\mu^*}{2} - \frac{\mu^*}{q}\right)\left(\frac{S_{s,q}}{\mu^*}\right)^{\frac{q}{q-2}} \leq \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}},$$

for any $\mu > \mu^*$. Therefore, we obtain a $(PS)_c$ sequence $\{\tilde{u}_n\}_n$ satisfying $\tilde{u}_n \rightharpoonup \tilde{u}$ with $\tilde{u} \neq 0$, and the level c satisfies $c < \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda}\right)^{\frac{p}{p-2}}$. One can verify that $J'_{\lambda,\mu}(\tilde{u}) = 0$ and $J_{\lambda,\mu}(\tilde{u}) = \tilde{c}_{\lambda,\mu}$. Hence, $\tilde{u}$ is a nontrivial solution to (1.1). $\square$

5 Mixed version of Principle of concentration Compactness

When $q = 2^*$, to determine the threshold value c, we investigate the concentration-compactness principle for mixed local and nonlocal operators.

Let

$$C_c(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) : \mathrm{supp}(u) \text{ is a compact subset of } \mathbb{R}^N\},$$

and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\eta|_\infty = \sup_{x\in\mathbb{R}^N} |\eta(x)|$. As is well known, a finite measure on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$. For a measure ϱ, we define the norm

$$\|\varrho\| = \sup_{\substack{\eta\in C_0(\mathbb{R}^N)\\ |\eta|_\infty=1}} |(\varrho,\eta)|,$$

where $(\varrho,\eta) = \int_{\mathbb{R}^N} \eta d\varrho$.

Definition 5.1. *Let $\mathcal{M}\left(\mathbb{R}^N\right)$ denote the finite nonnegative Borel measure space on $\mathbb{R}^N$. For any $\varrho \in \mathcal{M}\left(\mathbb{R}^N\right)$ the equation $\varrho\left(\mathbb{R}^N\right) = \|\varrho\|$ holds. We say that $\varrho_n \rightharpoonup \varrho$ in $\mathcal{M}\left(\mathbb{R}^N\right)$ if $(\varrho_n,\eta) \to (\varrho,\eta)$ holds for all $\eta \in C_0\left(\mathbb{R}^N\right)$ as $n \to \infty$.*

Theorem 5.1. *Let $\{u_n\}_n \subset X_0(\mathbb{R}^N)$ with upper bound $C > 0$ for all $n \geq 1$ and*

$$\begin{aligned}
&u_n \rightharpoonup u && \text{in } X_0(\mathbb{R}^N),\\
&|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy \rightharpoonup \xi && \text{weakly in } \mathcal{M}(\mathbb{R}^N),\\
&|u_n|^\tau \rightharpoonup \nu && \text{weakly in } \mathcal{M}(\mathbb{R}^N).
\end{aligned}$$

Then

$$\begin{aligned}
&\xi = |\nabla u|^2 + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy + \sum_{j\in\Lambda} \xi_j\delta_{x_j} + \tilde{\xi}, && \xi(\mathbb{R}^N) \leq C^2,\\
&\nu = |u|^\tau + \sum_{j\in\Lambda} \nu_j\delta_{x_j}, && \nu(\mathbb{R}^N) \leq S^\tau C^2.
\end{aligned}$$

where Λ is at most countable, sequence $\{\xi_j\}_j, \{\nu_j\}_j \subset \mathbb{R}_0^+, \{x_j\}_j \subset \mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at x_j, $\tilde{\xi}$ is a non-atomic measure,

$$\nu(\mathbb{R}^N) \leq S_{s,\tau}^{-\tau/2}\xi(\mathbb{R}^N)^{\tau/2}, \quad \nu_j \leq S_{s,\tau}^{-\tau/2}\xi_j^{\tau/2}, \text{ for all } j \in \Lambda,$$

$\tau \in [2_s^*, 2^*]$ and $S_{s,\tau}$ is the best constant.

We need the following lemma.

Lemma 5.1. *Assume that $\{u_n\}_n \subset X_0(\mathbb{R}^N)$ is the sequence given by Theorem 5.1, let $x_0 \in \mathbb{R}^N$ fixed and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_1(0)$, $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$, and $|\nabla\varphi| \leq 2$. Set $\varphi_\varepsilon = \varphi((x-x_0)/\varepsilon)$ for all $x \in \mathbb{R}^N$. Then we have*

$$\lim_{\varepsilon\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2|\nabla\varphi_\varepsilon|^2 = 0, \tag{5.1}$$

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy = 0. \tag{5.2}$$

Proof. we first prove (5.1),

$$\int_{\mathbb{R}^N} |u_n|^2|\nabla\varphi_\varepsilon|^2dx \le \int\limits_{B_{2\varepsilon}(x_0)\setminus B_\varepsilon(x_0)} \frac{2^2}{\varepsilon^2}|u_n|^2dx = \int\limits_{B_2(x_0)\setminus B_1(x_0)} 2^2\varepsilon^{N-2}|u_n|^2dx \le 2^2\varepsilon^{N-2}C\|u_n\|^2,$$

so the proof of (5.1) is complete as $n\to\infty$, $\varepsilon\to 0$ by $2<N$, and $\{u_n\}_n$ is bounded in $X_0(\mathbb{R}^N)$. we can refer to [28, Lemma 2.3] for the proof of (5.2), so here we omit it. □

Proof of Theorem 5.1. We will divide our proof into four parts.
Part 1. We first prove that $\xi(\mathbb{R})\le C^2$ and $\nu(\mathbb{R}^N)\le S_{s,\tau}^{-\tau/2}\xi(\mathbb{R}^N)^{\tau/2}$, $\nu_j\le S_{s,\tau}^{-\tau/2}\xi_j^{\tau/2}$, for all $j\in\Lambda$.
For $\varrho>0$, take $\eta\in C_0^\infty(B_{2\varrho}(0))$ satisfying $0\le\eta\le 1$ and $\eta\equiv 1$ on $B_\varrho(0)$. Then

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx\to\int_{\mathbb{R}^N}\eta(x)\ d\xi$$

as $n\to\infty$. Since $\|u_n\|\le C$, we have

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx\le\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\iint\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx\le C^2.$$

Hence, $\xi(B_\varrho(0))\le\int_{\mathbb{R}^N}\eta\ d\xi\le C^2$, let $\varrho\to\infty$, we get that $\xi(\mathbb{R}^N)\le C^2$. Similarly, we can obtain $\nu(\mathbb{R}^N)\le S_{s,\tau}^{-\tau/2}\xi(\mathbb{R}^N)^{\tau/2}$, $\nu_j\le S_{s,\tau}^{-\tau/2}\xi_j^{\tau/2}$, since $\int_{\mathbb{R}^N}|u|^q\,dx\le S_{s,\tau}^{-\tau/2}\,\|u_n\|^{\tau/2}$.

Part 2. We claim that

$$\xi=|\nabla u|^2+\int\limits_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy+\sum_{j\in\Lambda}\xi_j\delta_{x_j}+\tilde{\xi},$$

where $\{x_j\}_j\subset\mathbb{R}^N$, $\{\xi_j\}_j\subset[0,\infty)$, Λ is an at most countable set, $\tilde{\xi}$ is a nonnegative non-atomic measure and δ_{x_j} is the Dirac mass at x_j.
Take $0\le\eta\in C_0(\mathbb{R}^N)$ and set

$$K(u)=\int_{\mathbb{R}^N}\Big(|\nabla u|^2+\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx,$$

It is easy to verify K is a continuously differential convex functional on $X_0(\mathbb{R}^N)$, so K is weakly lower semicontinuous on $X_0(\mathbb{R}^N)$. Thus,

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx\ge\int_{\mathbb{R}^N}\Big(|\nabla u|^2+\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx,$$

using $|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\rightharpoonup\xi$ we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx=\int_{\mathbb{R}^N}\eta\ d\xi.$$

So

$$\int_{\mathbb{R}^N}\eta\ d\xi\ge\int_{\mathbb{R}^N}\Big(|\nabla u|^2+\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy\Big)\eta(x)dx.$$

The arbitrariness of $\eta \in C_0(\mathbb{R}^N)$ with $\eta \geq 0$ implies that

$$\xi \geq |\nabla u|^2 + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy.$$

Then we obtain

$$\xi - \left(|\nabla u|^2 + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy\right) = \sum_{j\in\Lambda} \xi_j \delta_{x_j} + \tilde{\xi}.$$

Part 3. We show that $\nu = |u|^\tau + \sum_{j\in\Lambda} \nu_j \delta_{x_j}$, where $\{x_j\}_j \subset \mathbb{R}^N$, $\{\nu_j\}_j \subset [0,\infty)$.
Take $\eta \in C_0(\mathbb{R}^N)$. Since $u_n \rightharpoonup u$ weakly in $X_0\left(\mathbb{R}^N\right)$, there exists a subsequence still denoted by $\{u_n\}_n$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$. It follows from the boundedness of $\{u_n\}_n$ in $L^\tau\left(\mathbb{R}^N\right)$ and the Brézis-Lieb lemma that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|u_n|^\tau - |u_n - u|^\tau\right) \eta dx = \int_{\mathbb{R}^N} |u|^\tau \eta dx.$$

Set $\bar{\nu} = \nu - |u|^\tau$. Since $\int_{\mathbb{R}^N} |u_n|^\tau \eta(x) dx \to \int_{\mathbb{R}^N} \eta \, d\nu$ as $n \to \infty$, it follows that

$$\int_{\mathbb{R}^N} \eta \, d\bar{\nu} = \int_{\mathbb{R}^N} \eta d\nu - \int_{\mathbb{R}^N} |u|^\tau \eta dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^\tau \eta dx, \tag{5.3}$$

so that $|u_n|^\tau \rightharpoonup \bar{\nu}\ \mathcal{M}(\mathbb{R}^N)$. Moreover,

$$\bar{\nu} = \nu - |u|^\tau = \sum_{j\in\Lambda'} \nu_j \delta_{y_j} + \tilde{\nu}.$$

Next we should prove that the atom of ν is that of ξ and $\tilde{\nu} = 0$. Let $x_0 \in \mathbb{R}^N$ fixed and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_1(0)$, $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$, and $|\nabla\varphi| \leq 2$. Set $\varphi_\varepsilon = \varphi(\frac{x-x_0}{\varepsilon})$ for all $x \in R^N$. Then

$$\int_{\mathbb{R}^N} |u_n \varphi_\varepsilon|^\tau dx = \int_{\mathbb{R}^N} |u_n|^\tau \varphi_\varepsilon^\tau dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon^\tau d\nu \text{ as } n \to \infty,$$

and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon^\tau d\nu \to \nu\left(\{x_0\}\right) \text{ as } \varepsilon \to 0.$$

Similarly, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy\right) \varphi_\varepsilon^p \, dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, d\xi \text{ as } n \to \infty,$$

and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, d\xi \to \xi(\{x_0\}) \text{ as } \varepsilon \to 0.$$

Hence, we obtain

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n \varphi_\varepsilon|^\tau dx \to \nu\left(\{x_0\}\right), \tag{5.4}$$

and

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \varphi_\varepsilon^2 \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2 \varphi_\varepsilon^2}{|x-y|^{N+2s}} dxdy = \xi(\{x_0\}). \tag{5.5}$$

Recall the following Young inequality:

$$|\zeta_1+\zeta_2|^2 \leq (1+\beta)^{2-1}|\zeta_1|^2+(1+1/\beta)^{2-1}|\zeta_2|^2, \tag{5.6}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$ and $\beta>0$. By the definition of $S_{s,\tau}$ and inequality (5.6), we can obtain

$$\begin{aligned}
\int_{\mathbb{R}^N}|u_n\varphi_\varepsilon|^\tau\,dx \leq & S_{s,\tau}^{-\tau/2}\Big(\int_{\mathbb{R}^N}|\nabla(u_n\varphi_\varepsilon)|^2dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)\varphi_\varepsilon(x)-u_n(y)\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\tau/2}\\
\leq & S_{s,\tau}^{-\tau/2}\Big[\Big(\int_{\mathbb{R}^N}|u_n\nabla\varphi_\varepsilon|^2dx+\iint_{\mathbb{R}^{2N}}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2|u_n(y)|^2}{|x-y|^{N+2s}}dxdy\Big)\\
&+(1+1/\beta)^{2-1}\Big(\int_{\mathbb{R}^N}|\nabla u_n\varphi_\varepsilon|^2dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2|\varphi_\varepsilon(x)|^2}{|x-y|^{N+2s}}dxdy\Big)\Big]^{\tau/2}
\end{aligned} \tag{5.7}$$

for any $\beta>0$. By (5.4), (5.5), (5.7) and Lemma 5.1, we have

$$\nu(\{x_0\}) \leq S_{s,\tau}^{-\frac{\tau}{2}}(1+1/\beta)^{(2-1)\frac{\tau}{2}}\xi(\{x_0\})^{\frac{\tau}{2}}.$$

Then letting $\beta\to\infty$, we get

$$\nu(\{x_0\}) \leq S_{s,\tau}^{-\frac{\tau}{2}}\xi(\{x_0\})^{\frac{\tau}{2}}. \tag{5.8}$$

Hence, the arbitrariness of x_0 implies that the atom of ν is that of ξ, that is $\{y_j : j\in\Lambda'\}\subset\{x_j : j\in\Lambda\}$. So we obtain that

$$\nu-|u|^\tau=\sum_{j\in\Lambda}\nu_j\delta_{x_j}+\tilde{\nu}.$$

Next we prove that $\tilde{\nu}=0$. Assume $\bar{u}_n=u_n-u$, then we have $\bar{u}_n\rightharpoonup 0$ weakly in $X_0(\mathbb{R}^N)$. Hence there exists a subsequence of $\{\bar{u}_n\}_n$ still denoted by $\{\bar{u}_n\}_n$ such that

$$|\nabla\bar{u}_n|^2+\int_{\mathbb{R}^N}\frac{|\bar{u}_n(x)-\bar{u}_n(y)|^2}{|x-y|^{N+2s}}dy\rightharpoonup\bar{\xi}\ \text{ weakly in } \mathcal{M}(\mathbb{R}^N).$$

For any $0<r<R$ take $\eta\in C_0^\infty(B_R(x_0))$ satisfying $0\leq\eta\leq 1$ and $\eta\equiv 1$ on $B_r(x_0)$. It follows from the definition of $S_{s,\tau}$ that

$$\begin{aligned}
&\int_{B_R(x_0)}|\bar{u}_n|^\tau\,\eta^\tau dx=\int_{B_R(x_0)}|\bar{u}_n\eta|^\tau\,dx\\
&\leq S_{s,\tau}^{-\frac{\tau}{2}}\left(\int_{B_R(x_0)}|\nabla(\bar{u}_n\eta)|^2dx+\int_{B_R(x_0)}\int_{B_R(x_0)}\frac{|\bar{u}_n(x)\eta(x)-\bar{u}_n(y)\eta(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{\tau}{2}}\\
&\leq S_{s,\tau}^{-\frac{\tau}{2}}\Big[2\Big(\int_{B_R(x_0)}|\bar{u}_n\nabla\eta|^2dx+\int_{B_R(x_0)}\int_{B_R(x_0)}\frac{|\eta(x)-\eta(y)|^2|\bar{u}_n(y)|^2}{|x-y|^{N+2s}}dxdy\Big)\\
&+2\Big(\int_{B_R(x_0)}|\nabla\bar{u}_n\eta|^2dx+\int_{B_R(x_0)}\int_{B_R(x_0)}\frac{|\bar{u}_n(x)-\bar{u}_n(y)|^2|\eta(x)|^2}{|x-y|^{N+2s}}dxdy\Big)\Big]^{\frac{\tau}{2}}.
\end{aligned} \tag{5.9}$$

Note that

$$|\eta(x)-\eta(y)|^2 \leq (\|\eta\|_{C^1}+2)^2\min\{1,|x-y|^2\}$$

for all $x, y \in B_R(x_0)$. Hence, by the compact embedding $X_0(B_R(x_0)) \hookrightarrow L^2(B_R(x_0))$, we can deduce that $\bar{u}_n \to 0$ strongly in $L^2(B_R(x_0))$. Moreover, we have

$$\begin{aligned}
&\int\limits_{B_R(x_0)} \int\limits_{B_R(x_0)} \frac{|\eta(x)-\eta(y)|^2|\bar{u}_n(y)|^2}{|x-y|^{N+2s}} dxdy \\
\le &(\|\eta\|_{C^1}+2)^2 \int\limits_{B_R(x_0)} \min\{1, |x-y|^2\}|x-y|^{-N-2s}dx \int\limits_{B_R(x_0)} |\bar{u}_n(y)|^2 dy \\
\le &C \int\limits_{B_R(x_0)} |\bar{u}_n(y)|^2 dy \to 0 \text{ as } n\to\infty,
\end{aligned}$$

so that

$$\lim_{n\to\infty} \int\limits_{B_R(x_0)} \int\limits_{B_R(x_0)} \frac{|\eta(x)-\eta(y)|^2|\bar{u}_n(y)|^2}{|x-y|^{N+2s}} dxdy = 0. \tag{5.10}$$

Note that

$$\limsup_{n\to\infty} \int\limits_{B_R(x_0)} |\eta(x)|^2 \Big(|\nabla \bar{u}_n|^2 + \int\limits_{B_R(x_0)} \frac{|\bar{u}_n(x)-\bar{u}_n(y)|^2}{|x-y|^{N+2s}} dy\Big) dx \le \int\limits_{\mathbb{R}^N} \eta^2 \, d\bar{\xi} \le \int\limits_{\overline{B_R(x_0)}} d\bar{\xi} = \bar{\xi}(\overline{B_R(x_0)}). \tag{5.11}$$

Inserting (5.10) and (5.11) into (5.9), we get

$$\bar{\nu}(\overline{B_r(x_0)}) \le \int_{\mathbb{R}^N} \eta^\tau \, d\bar{\nu} = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\bar{u}_n|^\tau \, \eta^\tau dx \le S_{s,\tau}^{-\frac{\tau}{2}} \Big(2 \int\limits_{B_R(x_0)} |\bar{u}_n \nabla \eta|^2 dx + 2\bar{\xi}(\overline{B_R(x_0)})\Big)^{\frac{\tau}{2}}. \tag{5.12}$$

Letting $r \to \mathbb{R}^-$, we have $\int\limits_{B_R(x_0)} |\bar{u}_n \nabla \eta|^2 dx \to 0$, then we get that

$$\bar{\nu}(\overline{B_r(x_0)}) \le S_{s,\tau}^{-\frac{\tau}{2}} \Big(2\bar{\xi}(\overline{B_R(x_0)})\Big)^{\frac{\tau}{2}}. \tag{5.13}$$

This means that $\bar{\nu}$ is absolutely continuous with respect to $\bar{\xi}$. Hence the Radon-Nikodym theorem implies that there exists a function $h \in L^1(R^N, \bar{\xi})$ such that $d\bar{\nu} = hd\bar{\xi}$. Then we derive from Lebesgue's differential theorem and (5.13) that

$$\begin{aligned}
h(x_0) = \lim_{\mathbb{R}\to 0} \frac{\bar{\nu}(\overline{B_R(x_0)})}{\bar{\xi}(\overline{B_R(x_0)})} &\le S_{s,\tau}^{-\frac{\tau}{2}} \cdot 2^{\frac{\tau}{2}} \lim_{\mathbb{R}\to 0} \bar{\xi}(\overline{B_r(x_0)})^{\frac{\tau}{2}-1} \\
&= S_{s,\tau}^{-\frac{\tau}{2}} \cdot 2^{\frac{\tau}{2}} \bar{\xi}(\{x_0\})^{\frac{\tau}{2}-1}.
\end{aligned} \tag{5.14}$$

Now we show that $\tilde{\nu} = 0$. Let $x \in \mathbb{R}^N \setminus \{x_j : j \in \Lambda\}$. If $h(x) \ne 0$, then by (5.14) we know that $\bar{\xi}(\{x\}) \ne 0$, thus $\bar{\nu}(\{x\}) \ne 0$. Note that (5.3) implies that $\bar{\nu}$ and ν has the same atoms, so that x is an atom of ξ, which is a contradiction. Hence $h \equiv 0$ on $\mathbb{R}^N \setminus \{x_j : j \in \Lambda\}$. Moreover, by $d\bar{\nu} = hd\bar{\xi}$, we obtain $\bar{\nu} = 0$ on $\mathbb{R}^N \setminus \{x_j : j \in \Lambda\}$. In conclusion, $\tilde{\nu} = 0$, since $\tilde{\nu}$ is a non-atomic measure.

Part 4: In the end, we prove

$$\nu(\mathbb{R}^N) \le S_{s,\tau}^{-\frac{\tau}{2}} 2^{\frac{\tau}{2}} \xi(\mathbb{R}^N)^{\frac{\tau}{2}}, \ \ \nu_j \le S_{s,\tau}^{-\frac{\tau}{2}} \xi_j^{\frac{\tau}{2}} \ \textbf{for all } j \in \Lambda.$$

Take $\eta \in C_0^\infty(B_{2R}(0))$ satisfying $0 \le \eta \le 1,\ \eta \equiv 1$ on $B_R(0)$ and $|\nabla \eta| \le 2/R$. Let $\beta > 0$, then we make a similar discussion as in **Part 3** to obtain:

$$\begin{aligned}\int_{\mathbb{R}^N} |\bar{u}_n|^\tau \eta^\tau dx \le & S_{s,\tau}^{-\tau/2}\Big(\int_{\mathbb{R}^N} |\nabla(u_n\eta)|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)\varphi_\varepsilon(x) - u_n(y)\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy\Big)^{\tau/2} \\ \le & S_{s,\tau}^{-\tau/2}\Big[\Big(\int_{\mathbb{R}^N} |u_n \nabla \eta|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s}} dxdy\Big) \\ & + (1+1/\beta)\Big(\int_{\mathbb{R}^N} |\nabla u_n \eta|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2 |\eta(x)|^2}{|x-y|^{N+2s}} dxdy\Big)\Big]^{\tau/2} \end{aligned} \tag{5.15}$$

Using a proof similar to Lemma 5.1, we have

$$\lim_{\mathbb{R}\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 |\nabla \eta|^2 = 0,$$
$$\lim_{\mathbb{R}\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 |\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}} dxdy = 0.$$

Letting $R \to \infty,\ n \to \infty$ in (5.15), we obtain

$$\int_{\mathbb{R}^N} \eta^\tau\ d\nu \le S_{s,\tau}^{-\frac{\tau}{2}} \left(\left(1 + \frac{1}{\beta}\right) \int_{\mathbb{R}^N} \eta^2\ d\xi \right)^{\frac{\tau}{2}}.$$

Letting $\beta \to \infty$, we have

$$\int_{\mathbb{R}^N} \eta^\tau\ d\nu \le S_{s,\tau}^{-\frac{\tau}{2}} \left(\int_{\mathbb{R}^N} \eta^2\ d\xi \right)^{\frac{\tau}{2}}. \tag{5.16}$$

Combining (5.16) with $\nu(\overline{B_R(x_0)}) \le \int_{\mathbb{R}^N} \eta^\tau\ d\nu$ and let $R \to \infty$, we get

$$\nu(\mathbb{R}^N) \le S_{s,\tau}^{-\frac{\tau}{2}} \xi(\mathbb{R}^N)^{\frac{\tau}{2}}.$$

With a similar discussion to (5.8), we can conclude that

$$\nu_j \le S_{s,\tau}^{-\frac{\tau}{2}} \xi_j^{\frac{\tau}{2}} \text{ for each } j \in \Lambda.$$

Indeed, this fact follows by replacing φ_ε in **Part 3** with $\varphi_{\varepsilon,j} = \varphi(\frac{x-x_j}{\varepsilon})$. Thus, the proof is complete. □

Theorem 5.2. *Let $\{u_n\}_n \subset X^{1,2}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\begin{cases} |\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy \rightharpoonup \xi & \text{weakly in } \mathcal{M}(\mathbb{R}^N), \\ |u_n|^\tau \rightharpoonup \nu & \text{weakly in } \mathcal{M}(\mathbb{R}^N), \end{cases}$$

and define

$$\xi_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{x\in\mathbb{R}^N:|x|>R} \Big(|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy\Big) dx,$$
$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{x\in\mathbb{R}^N:|x|>R} |u_n|^\tau\, dx.$$

Then the quantities ξ_∞ *and* ν_∞ *are well defined and satisfy*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)dx=\xi(\mathbb{R}^N)+\xi_\infty$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^\tau\,dx=\nu(\mathbb{R}^N)+\nu_\infty.$$

Moreover, the following inequality holds:

$$\nu_\infty\le S_{s,\tau}^{-\frac{\tau}{2}}\xi_\infty^{\frac{\tau}{2}}.$$

Proof. Assume $\chi\in C^\infty(\mathbb{R}^N)$ such that $0\le\chi\le1$, $\chi=0$ in $B_1(0)$, $\chi=1$ in $\mathbb{R}^N\setminus B_2(0)$, and $|\nabla\chi|\le 2$. For any $R>0$, we define $\chi_R(x)=\chi(x/R)$. Then

$$\begin{aligned}\int_{x\in\mathbb{R}^N:|x|>2R}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)dx&\le\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\chi_R^2\,dx\\&\le\int_{\{x\in\mathbb{R}^N:|x|>R\}}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)dx.\end{aligned}$$

This means that

$$\xi_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\chi_R^2\,dx.$$

Similarly,

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^\tau\,\chi_R^\tau\,dx.$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)dx\\&=\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)\chi_R^2\,dx+\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)(1-\chi_R^2)\,dx.\end{aligned}\tag{5.17}$$

By the definition in Theorem 5.2, we have

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)(1-\chi_R^2)\,dx\to\int_{\mathbb{R}^N}(1-\chi_R^2)\,d\xi,$$

as $n\to\infty$. So we get

$$\xi(\mathbb{R}^N)=\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)(1-\chi_R^2)\,dx.$$

By (5.17), we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\Big)dx$$

$$
\begin{aligned}
&= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy\Big) dx \\
&= \lim_{R\to\infty} \limsup_{n\to\infty} \Big[\int_{\mathbb{R}^N} \Big(|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy\Big) \chi_R^2 \, dx \\
&+ \int_{\mathbb{R}^N} \Big(|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy\Big) (1-\chi_R^2) \, dx \Big] \\
&= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla u_n|^2 + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dy\Big) \chi_R^2 \, dx + \xi(\mathbb{R}^N) \\
&= \xi_\infty + \xi(\mathbb{R}^N).
\end{aligned}
$$

Similarly, we can obtain that

$$
\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\tau \, dx = \nu(\mathbb{R}^N) + \nu_\infty.
$$

In the end, we prove the last inequality:

$$
\nu_\infty \le S_{s,\tau}^{-\frac{\tau}{2}} \xi_\infty^{\frac{\tau}{2}}.
$$

By (5.6), we deduce that

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n(x)\chi_R(x)|^\tau dx \\
\le& S_{s,\tau}^{-\frac{\tau}{2}} \Big(\int_{\mathbb{R}^N} |\nabla(u_n\chi_R)|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)\chi_R(x)-u_n(y)\chi_R(y)|^2}{|x-y|^{N+2s}} dxdy \Big)^{\frac{\tau}{2}} \\
\le& S_{s,\tau}^{-\frac{\tau}{2}} \Big[(1+\beta) \Big(\int_{\mathbb{R}^N} |u_n \nabla \chi_R|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|\chi_R(x)-\chi_R(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s}} dxdy \Big) \\
&+ (1+1/\beta) \Big(\int_{\mathbb{R}^N} |\nabla u_n \chi_R|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2 |\chi_R(x)|^2}{|x-y|^{N+2s}} dxdy \Big) \Big]^{\frac{\tau}{2}}.
\end{aligned}
$$

Then we claim that

$$
\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 |\nabla \chi_R|^2 = 0, \tag{5.18}
$$

$$
\lim_{R\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 |\chi_R(x)-\chi_R(y)|^2}{|x-y|^{N+2s}} dxdy = 0. \tag{5.19}
$$

It is easy to see that

$$
\int_{\mathbb{R}^N} |u_n|^2 |\nabla \chi_R|^2 dx \le \int_{B_{2R}(0)\setminus B_R(0)} \left|\frac{2}{R}\right|^2 |u_n|^2 dx \le C \left|\frac{2}{R}\right|^2 \|u_n\|^2,
$$

so the (5.18) is hold as $n\to\infty,\ R\to\infty$ since $X^{1,2}(B_{2R}(0)\setminus B_R(0)) \hookrightarrow L^2(B_{2R}(0)\setminus B_R(0))$ and $\{u_n\}_n$ is bounded. The proof of (5.19) we can refer to the proof of [28, Theorem 2.4], so here we omit it. By (5.18) and (5.19), we obtain

$$
\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n(x)\chi_R|^\tau dx
$$

$$\leq S_{s,\tau}^{-\frac{\tau}{2}}\left((1+1/\beta)\lim_{R\to\infty}\limsup_{n\to\infty}\Big(\int_{\mathbb{R}^N}|\nabla u_n\chi_R|^2dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2|\chi_R(x)|^2}{|x-y|^{N+2s}}dxdy\Big)\right)^{\frac{\tau}{2}}$$
$$\leq S_{s,\tau}^{-\frac{\tau}{2}}(1+1/\beta)^{(2-1)\frac{\tau}{2}}\xi_\infty^{\frac{\tau}{2}}.$$

Letting $\beta\to\infty$, we get $\nu_\infty\leq S_{s,\tau}^{-\frac{\tau}{2}}\xi_\infty{}^{\frac{\tau}{2}}$. Therefore, the proof is complete. □

6 Proof of Theorem 1.5

We need the following Lemma.

Lemma 6.1. *If $\{u_n\}_n$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$, then, passing to a subsequence if necessary, we may assume that $u_n\rightharpoonup 0$ in $X_0(\mathbb{R}^N)$. Then, for every $2_s^*<p<q=2^*$, we have*

$$c\geq\max\left\{\left(\frac{\lambda}{2}-\frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{p}{p-2}},\ \left(\frac{\mu}{2}-\frac{\mu}{2^*}\right)\left(\frac{S_{s,2^*}}{\lambda+\mu}\right)^{\frac{2^*}{2^*-2}}\right\}\quad or\quad c=0.$$

Proof. Let $\{u_n\}_n$ be a $(PS)_c$ sequence. By Lemma 2.3, $\{u_n\}_n$ is bounded in $X_0(\mathbb{R}^N)$. Up to a subsequence,

$$\begin{cases}u_n\rightharpoonup 0 & \text{in } X_0(\mathbb{R}^N),\\ |\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy\rightharpoonup\mu & \text{weakly in } \mathcal{M}(\mathbb{R}^N),\\ |u_n|^p\rightharpoonup\tilde{\nu} & \text{weakly in } \mathcal{M}(\mathbb{R}^N),\\ |u_n|^{2^*}\rightharpoonup\nu & \text{weakly in } \mathcal{M}(\mathbb{R}^N),\end{cases}\tag{6.1}$$

where μ, $\tilde{\nu}$, and ν are nonnegative bounded measures on $\mathbb{R}^N$. Then, by the concentration-compactness principle (Theorem 5.1), there exists an at most countable index set Λ such that

$$\begin{aligned}&\mu=|\nabla u|^2+\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy+\sum_{j\in\Lambda}\mu_j\delta_{x_j}+\tilde{\mu},\\&\nu=|u|^{2^*}+\sum_{j\in\Lambda}\nu_j\delta_{x_j},\quad\tilde{\nu}=|u|^p+\sum_{j\in\Lambda}\tilde{\nu}_j\delta_{x_j},\\&\nu_j\leq S_{s,2^*}^{-2^*/2}\mu_j^{2^*/2},\quad\tilde{\nu}_j\leq S_{s,p}^{-p/2}\mu_j^{p/2},\end{aligned}\tag{6.2}$$

where δ_{x_j} is the Dirac measure concentrated at $x_j\in\mathbb{R}^N$. First, suppose that $\Lambda\neq\emptyset$. For fixed $j\in\Lambda$ and $\varepsilon>0$, choose $\varphi_{\varepsilon,j}\in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_{\varepsilon,j}=1\text{ for }|x-x_j|\leq\varepsilon;\quad\varphi_{\varepsilon,j}=0\text{ for }|x-x_j|\geq 2\varepsilon,$$

and $|\nabla\varphi_{\varepsilon,j}|\leq 2/\varepsilon$. Clearly, $\varphi_{\varepsilon,j}u_n\in X_0(\mathbb{R}^N)$. Since $\langle J'_{\lambda,\mu}(u_n),\varphi_{\varepsilon,j}u_n\rangle\to 0$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^2\varphi_{\varepsilon,j}dx+\iint_{\mathbb{R}^{2N}}\mathcal{A}u_n(x,y)(u_n(x)-u_n(y))\varphi_{\varepsilon,j}(x)dxdy+o(1)\\&=-\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\varphi_{\varepsilon,j}dx-\iint_{\mathbb{R}^{2N}}\mathcal{A}u_n(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(x)dxdy\end{aligned}$$

$$-\lambda\int_{\mathbb{R}^N}|u_n|^p\varphi_{\varepsilon,j}(x)dx-\mu\int_{\mathbb{R}^N}|u_n|^{2^*}\varphi_{\varepsilon,j}(x)dx. \tag{6.3}$$

Using Hölder's inequality and Lemma 5.1, we have

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\varphi_{\varepsilon,j}dx\le C\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,j}|^2|u_n(x)|^2=0, \tag{6.4}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\mathcal{A}u_n(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(x)dxdy\\&\le C\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(x)|^2}{|x-y|^{N+2s}}\right)^{1/2}=0.\end{aligned} \tag{6.5}$$

On the other hand, from (6.2) we get

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\varphi_{\varepsilon,j}dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2\varphi_{\varepsilon,j}}{|x-y|^{N+2s}}dxdy=\mu_j, \tag{6.6}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\varphi_{\varepsilon,j}dx=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|u|^{2^*}\varphi_{\varepsilon,j}dx+\nu_j=\nu_j, \tag{6.7}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\varphi_{\varepsilon,j}dx=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|u|^p\varphi_{\varepsilon,j}dx+\tilde{\nu}_j=\tilde{\nu}_j. \tag{6.8}$$

Then from (6.3)–(6.8) we deduce that

$$\mu\nu_j+\lambda\tilde{\nu}_j=\mu_j. \tag{6.9}$$

Combining this equality with (6.2), we obtain

$$(\lambda+\mu)\max\{\nu_j,\tilde{\nu}_j\}\ge\mu\nu_j+\lambda\tilde{\nu}_j=\mu_j.$$

We now consider two cases.

Case 1. If $\nu_j>\tilde{\nu}_j$, then

$$(\lambda+\mu)\nu_j\ge\mu_j\ge S_{s,2^*}\nu_j^{\frac{2}{2^*}}. \tag{6.10}$$

From (6.10) it follows that

$$\nu_j\ge\left(\frac{S_{s,2^*}}{\lambda+\mu}\right)^{\frac{2^*}{2^*-2}}. \tag{6.11}$$

Case 2. If $\tilde{\nu}_j>\nu_j$, then

$$(\lambda+\mu)\tilde{\nu}_j\ge\mu_j\ge S_{s,p}\tilde{\nu}_j^{\frac{2}{p}}. \tag{6.12}$$

Inequality (6.12) implies

$$\tilde{\nu}_j\ge\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{p}{p-2}}. \tag{6.13}$$

On the other hand, by (6.2) we have

$$c=\lim_{n\to\infty}\left(J_{\lambda,\mu}(u_n)-\frac{1}{2}\langle J'_{\lambda,\mu}(u_n),u_n\rangle\right)$$

$$\geq \left(\frac{\lambda}{2}-\frac{\lambda}{p}\right)\tilde{\nu}_j+\left(\frac{\mu}{2}-\frac{\mu}{2^*}\right)\nu_j$$
$$\geq \max\left\{\left(\frac{\lambda}{2}-\frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{p}{p-2}},\ \left(\frac{\mu}{2}-\frac{\mu}{2^*}\right)\left(\frac{S_{s,2^*}}{\lambda+\mu}\right)^{\frac{2^*}{2^*-2}}\right\}. \tag{6.14}$$

This completes the proof.

Now consider the case $\Lambda=\emptyset$. For $R>0$, define

$$\xi_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\left(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dy\right)dx,$$

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}|u_n(x)|^{2^*}dx,\quad \tilde{\nu}_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}|u_n(x)|^p dx.$$

By Theorem 5.2, the quantities ξ_∞, ν_∞, and $\tilde{\nu}_\infty$ are well defined, and we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dy\right)dx=\xi(\mathbb{R}^N)+\xi_\infty, \tag{6.15}$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)|^{2^*}dx=\nu(\mathbb{R}^N)+\nu_\infty,\quad \limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)|^p dx=\tilde{\nu}(\mathbb{R}^N)+\tilde{\nu}_\infty. \tag{6.16}$$

Now suppose $\nu_\infty\neq 0$ or $\tilde{\nu}_\infty\neq 0$. Let $\chi_R\in C^\infty(\mathbb{R}^N)$ satisfy $\chi_R\in[0,1]$, $\chi_R(x)=0$ for $|x|<R$, $\chi_R(x)=1$ for $|x|>2R$, and $|\nabla\chi_R|<2/R$. By an argument similar to the proof of Theorem 5.2, we obtain

$$\xi_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\chi_R(x)\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\chi_R(x)\,dy\,dx, \tag{6.17}$$

and

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)\chi_R(x)|^{2^*}dx,\quad \tilde{\nu}_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)\chi_R(x)|^p dx. \tag{6.18}$$

Moreover, we have

$$S_{s,2^*}\nu_\infty^{\frac{2}{2^*}}\leq\xi_\infty,\quad S_{s,p}\tilde{\nu}_\infty^{\frac{2}{p}}\leq\xi_\infty. \tag{6.19}$$

Since $\|u_n\|^2$, $\int_{\mathbb{R}^N}|u_n(x)|^p dx$, and $\int_{\mathbb{R}^N}|u_n(x)|^{2^*}dx$ are bounded, we may assume, up to a subsequence, that they converge. Then by (6.15) and (6.16) we obtain

$$\lim_{n\to\infty}\|u_n\|^2=\int_{\mathbb{R}^N}d\xi+\xi_\infty, \tag{6.20}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)|^p dx=\int_{\mathbb{R}^N}d\tilde{\nu}+\tilde{\nu}_\infty,\quad \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)|^{2^*}dx=\int_{\mathbb{R}^N}d\nu+\nu_\infty. \tag{6.21}$$

From $\langle J'_{\lambda,\mu}(u_n),\chi_R u_n\rangle\to 0$ as $n\to\infty$, we have

$$\int_{\mathbb{R}^N}\nabla u_n\nabla(\chi_R u_n)\,dx+\iint_{\mathbb{R}^{2N}}\mathcal{A}u_n(x,y)(\chi_R(x)u_n(x)-\chi_R(y)u_n(y))\,dxdy$$
$$=\lambda\int_{\mathbb{R}^N}|u_n(x)|^{p-2}u_n(x)u_n(x)\chi_R(x)\,dx+\mu\int_{\mathbb{R}^N}|u_n(x)|^{2^*-2}u_n(x)u_n(x)\chi_R(x)\,dx+o(1). \tag{6.22}$$

By (5.18), (5.19), and Hölder's inequality, we obtain

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla u_n\nabla\chi_R\, u_n\,dx=0, \tag{6.23}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+2s}}\,dxdy=0. \tag{6.24}$$

Hence, from (6.17)–(6.24) it follows that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\chi_R\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2\chi_R(x)}{|x-y|^{N+2s}}\,dxdy\right)\\
&=\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\{x\in\mathbb{R}^N:|x|>R\}}|\nabla u_n|^2\,dx+\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dydx\right)\\
&=\xi_\infty.
\end{aligned}\tag{6.25}$$

Therefore, combining (6.22)–(6.25) and (6.18), we conclude that

$$\xi_\infty=\lambda\tilde{\nu}_\infty+\mu\nu_\infty.$$

Similarly, we obtain

$$\begin{aligned}
c&=\lim_{n\to\infty}\left(J_{\lambda,\mu}(u_n)-\frac{1}{2}\langle J'_{\lambda,\mu}(u_n),u_n\rangle\right)\\
&\geq\left(\frac{\lambda}{2}-\frac{\lambda}{p}\right)\tilde{\nu}_\infty+\left(\frac{\mu}{2}-\frac{\mu}{q}\right)\nu_\infty\\
&\geq\max\left\{\left(\frac{\lambda}{2}-\frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{p}{p-2}},\ \left(\frac{\mu}{2}-\frac{\mu}{2^*}\right)\left(\frac{S_{s,2^*}}{\lambda+\mu}\right)^{\frac{2^*}{2^*-2}}\right\}.
\end{aligned}\tag{6.26}$$

This completes the proof.

Now suppose $\Lambda=\emptyset$ and $\nu_\infty=0$. Then

$$\int_{\mathbb{R}^N}|u_n(x)|^p dx\to 0,\qquad \int_{\mathbb{R}^N}|u_n(x)|^{2^*}dx\to 0 \tag{6.27}$$

as $n\to\infty$, and hence $u_n\to 0$ in $L^p(\mathbb{R}^N)\cap L^{2^*}(\mathbb{R}^N)$. We now show that $u_n\to u$ in $X_0(\mathbb{R}^N)$.

Since $\langle J'_{\lambda,\mu}(u_n)-J'_{\lambda,\mu}(u),u_n-u\rangle\to 0$, we have

$$\begin{aligned}
\langle u_n,u_n-u\rangle-\langle u,u_n-u\rangle&=\lambda\int_{\mathbb{R}^N}\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\,dx\\
&\quad+\mu\int_{\mathbb{R}^N}\left(|u_n|^{2^*-2}u_n-|u|^{2^*-2}u\right)(u_n-u)\,dx+o(1),
\end{aligned}\tag{6.28}$$

where

$$\langle u_n,u_n-u\rangle:=\int_{\mathbb{R}^N}\nabla u_n\nabla(u_n-u)\,dx+\iint_{\mathbb{R}^{2N}}\mathcal{A}u_n(x,y)\left((u_n-u)(x)-(u_n-u)(y)\right)dxdy.$$

By Hölder's inequality, we obtain

$$\int_{\mathbb{R}^N} \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n-u)\,dx \le C\int_{\mathbb{R}^N} |u_n-u|^p dx, \tag{6.29}$$

$$\int_{\mathbb{R}^N} \left(|u_n|^{2^*-2}u_n - |u|^{2^*-2}u\right)(u_n-u)\,dx \le C\int_{\mathbb{R}^N} |u_n-u|^{2^*} dx, \tag{6.30}$$

and

$$\lim_{n\to\infty} \left(\langle u_n, u_n-u\rangle - \langle u, u_n-u\rangle\right) = 0. \tag{6.31}$$

From (6.29)–(6.31), it follows that $u_n \to u$ in $X_0(\mathbb{R}^N)$, and hence $J_{\lambda,\mu}(u_n) \to J_{\lambda,\mu}(u) = 0$. This concludes the proof. □

Proof of Theorem 1.5. Firstly, we assume that the sum $\lambda+\mu$ is sufficiently small. We now verify the assumptions of the mountain pass theorem. By Theorem 1.2, there exists $\tilde{v}_1 \in X_0(\mathbb{R}^N)$ such that $\|\tilde{v}_1\|^2 = S_{s,p}$ and $\|\tilde{v}_1\|_p = 1$. For any $t \ge 0$, we have

$$J_{\lambda,\mu}(t\tilde{v}_1) = \frac{1}{2}t^2\|\tilde{v}_1\|^2 - \frac{\lambda}{p}t^p\int_{\mathbb{R}^N}|\tilde{v}_1|^p dx - \frac{\mu}{2^*}t^{2^*}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx.$$

Since $2 < 2_s^* < p < 2^*$, there exists $\tilde{e}_1 := t_1\tilde{v}_1$ such that $\|\tilde{e}_1\| > \rho$ and $J_{\lambda,\mu}(\tilde{e}_1) \le 0$. Note that

$$\begin{aligned} J_{\lambda,\mu}(t\tilde{v}_1) &= \frac{1}{2}t^2\|\tilde{v}_1\|^2 - \frac{\lambda}{p}t^p\int_{\mathbb{R}^N}|\tilde{v}_1|^p dx - \frac{\mu}{2^*}t^{2^*}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx \\ &= \frac{t^2}{2}S_{s,p} - \frac{\lambda}{p}t^p - \frac{\mu}{2^*}t^{2^*}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx := g(t). \end{aligned}$$

To proceed, we study the maximum points of the function g. We claim that

$$g(t) < \left(\frac{\lambda}{2} - \frac{\lambda}{p}\right)\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{p}{p-2}}.$$

To prove this claim, first observe that

$$g'(t) = tS_{s,p} - \lambda t^{p-1} - \mu t^{2^*-1}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx.$$

Define

$$\tilde{g}(t) = S_{s,p} - \lambda t^{p-2} - \mu t^{2^*-2}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx. \tag{6.32}$$

Since $\tilde{g}(t)$ is monotonic, there exists $\tilde{t}_1 > 0$ such that $\tilde{g}(\tilde{t}_1) = 0$ and $J_{\lambda,\mu}(t\tilde{v}_1) \le J_{\lambda,\mu}(\tilde{t}_1\tilde{v}_1)$ for all $t \ge 0$. Since

$$\tilde{g}(t) = S_{s,p} - (\lambda+\mu)\,t^{\theta-2} + (\lambda+\mu)\,t^{\theta-2} - \lambda t^{p-2} - \mu t^{2^*-2}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx,$$

where $\theta \in (p, 2^*)$. We get that

$$\tilde{g}\left(\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{1}{\theta-2}}\right) = (\lambda+\mu)\frac{S_{s,p}}{\lambda+\mu} - \lambda\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{p-2}{\theta-2}} - \mu\left(\frac{S_{s,p}}{\lambda+\mu}\right)^{\frac{2^*-2}{\theta-2}}\int_{\mathbb{R}^N}|\tilde{v}_1|^{2^*}dx$$

$$= \lambda \left(\frac{S_{s,p}}{\lambda+\mu} - \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{p-2}{\theta-2}} \right) + \mu \left(\frac{S_{s,p}}{\lambda+\mu} - \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{2^*-2}{\theta-2}} \int_{\mathbb{R}^N} |\tilde{v}_1|^{2^*} dx \right). \quad (6.33)$$

Therefore, for any given $\lambda > 0$, there exists $\mu^{**} > 0$ such that for all $\mu \in (0, \mu^{**})$, $\tilde{g}\left(\left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{1}{\theta-2}} \right) > 0$ and $\left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{1}{\theta-2}} \leq \tilde{t}_1$. Hence, by the mean value theorem,

$$\begin{aligned} g(t) &\leq \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left(\frac{S_{s,p}}{\lambda} \right)^{\frac{p}{p-2}} - \frac{\mu}{2^*} \tilde{t}_1^{2^*} \int_{\mathbb{R}^N} |\tilde{v}_1|^{2^*} dx \\ &\leq \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left[\left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{p}{p-2}} + \left(\frac{S_{s,p}}{\lambda} \right)^{\frac{2}{p-2}} \frac{pS_{s,p}\mu}{(p-2)\lambda(\lambda+\mu)} \right] - \frac{\mu}{2^*} \tilde{t}_1^{2^*} \int_{\mathbb{R}^N} |\tilde{v}_1|^{2^*} dx \\ &\leq \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{p}{p-2}} + \frac{\mu}{2} \left(\frac{S_{s,p}}{\lambda} \right)^{\frac{p}{p-2}} - \frac{\mu}{2^*} \tilde{t}_1^{2^*} \int_{\mathbb{R}^N} |\tilde{v}_1|^{2^*} dx. \end{aligned} \quad (6.34)$$

Note that $\left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{1}{\theta-2}} \leq \tilde{t}_1$ for all $\mu \in (0, \mu^{**})$. If we can verify

$$\frac{\mu}{2} \left(\frac{S_{s,p}}{\lambda} \right)^{\frac{p}{p-2}} - \frac{\mu}{2^*} \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{2^*}{\theta-2}} \int_{\mathbb{R}^N} |\tilde{v}_1|^{2^*} dx \leq 0,$$

then the proof is complete. In fact, we can choose θ such that $\frac{p}{p-2} < \frac{2^*}{\theta-2}$. It is easy to verify that there exists $\mu^{**} > 0$ such that for all $\mu \in (0, \mu^{**})$,

$$\frac{\mu}{2} \left(\frac{S_{s,p}}{\lambda} \right)^{\frac{p}{p-2}} - \frac{\mu}{2^*} \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{2^*}{\theta-2}} \int_{\mathbb{R}^N} |\tilde{v}_1|^{2^*} dx \leq 0,$$

Hence, $g(t) < \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{p}{p-2}}$.

Define

$$\tilde{c}_{\lambda,\mu} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X_0(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = \tilde{e}_1\}$.

By Lemma 4.2, we have

$$\vartheta \leq \tilde{c}_{\lambda,\mu} < \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left(\frac{S_{s,p}}{\lambda} \right)^{\frac{p}{p-2}}, \ \tilde{c}_{\lambda,\mu} < \max_{t \geq 0} g(t) < \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{p}{p-2}}$$

. Therefore, we obtain a $(PS)_c$ sequence $\{\tilde{u}_n\}_n$ satisfying $\tilde{u}_n \rightharpoonup \tilde{u}$ with $\tilde{u} \neq 0$, and the level c satisfies $c < \left(\frac{\lambda}{2} - \frac{\lambda}{p} \right) \left(\frac{S_{s,p}}{\lambda+\mu} \right)^{\frac{p}{p-2}}$. One can verify that $J'_{\lambda,\mu}(\tilde{u}) = 0$ and $J_{\lambda,\mu}(\tilde{u}) = \tilde{c}_{\lambda,\mu}$. Thus, $\tilde{u}$ is a nontrivial solution to (1.1). □

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