Infinitely many normalized solutions for the Kirchhoff equation with localized nonlinearities on noncompact metric graphs

Abstract

In this paper, we study the following Kirchhoff problem

$$\begin{cases} -(a+b\int_{\mathcal{G}} |u'|^2\,dx)u'' + \lambda u = \kappa(x)|u|^{p-2}u & x \in \mathcal{G}, \\ \int_{\mathcal{G}} |u|^2\,dx = \mu, \end{cases}$$

where $p > 10$, $a, b > 0$, $\mu > 0$ is a prescribed mass, $\mathcal{G}$ is a noncompact metric graph, κ is the characteristic function of the compact core $\mathcal{K}$ of $\mathcal{G}$ and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. By using minimax principle, we prove the existence of infinitely many normalized solutions for any prescribed mass in the L^2-supercritical case.

Keywords: Kirchhoff equation; L^2-supercritical; infinitely many solutions; noncompact metric graph.

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1 Introduction and main results

In this paper, we consider the following Kirchhoff equation with localized nonlinearities on metric graphs

$$-(a+b\int_{\mathcal{G}} |u'|^2\,dx)u'' + \lambda u = \kappa(x)|u|^{p-2}u, \quad x \in \mathcal{G}, \tag{1.1}$$

coupled with the Kirchhoff conditions at the vertices, where $p > 10$, $a, b > 0$, $\lambda \in \mathbb{R}$, and κ is the characteristic function of the compact core $\mathcal{K}$, which is a subgraph of $\mathcal{G}$ consisting of all the bounded edges.

The equation (1.1) is an extension of the clsaaical D'Alembert's wave equations for free vibration of elastic strings which is proposed by Kirchhoff. The specific mathematical model is given by

$$\varrho \frac{\partial^2 \Phi}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial \Phi}{\partial x}\right|^2 dx \right) \frac{\partial^2 \Phi}{\partial x^2} = 0, \tag{1.2}$$

where Φ is the horizontal displacement, ϱ is the mass density, h is the cross-section area, L is the length, E is the Young's modulus and T_0 is the initial tension. For more physical background, see [10] and references therein. Notice that equation (1.1) has the term $-(a+b\int_{\mathcal{G}}|u'|^2\,dx)u''$, which is nonlocal, such that the Kirchhoff type equation is not a pointwise identify. It casues some difficulties when dealing with this kind of issue.

Throughout the paper we consider that $\mathcal{G}=(\mathcal{E},\mathcal{V})$ is a connected noncompact metric graph which satisfies:

- $\mathcal{G}$ has a finite number of edges and vertices;
- $\mathcal{G}$ has a nontrivial compact core $\mathcal{K}$;
- $\mathcal{G}$ has at least one half-line (that is, $\mathcal{G}$ has at least an unbounded edge).

Moreover, each edge $e\in\mathcal{E}$ (bounded or unbounded) is identified with an interval I_e($[0,l_e]$ or $[0,+\infty)$), where $l_e:=|e|$ is the length of edge $e\in\mathcal{E}$. And there exists a natural metric which is defined by the shortest path between points. For more details of metric graph, see [3, 11].

A function u defined on $\mathcal{G}$ can be viewed as a vector of functions $\{u_e\}_{e\in\mathcal{E}}$, where each u_e is defined on the edge e, that is the restriction of u on each edge e. We can define the Lebesgue space $L^p(\mathcal{G})$,

$$L^p(\mathcal{G}):=\left\{u=\{u_e\}_{e\in\mathcal{E}}: u \text{ is measurable and } p\text{-th integrable on each edge } e\right\},$$

equipped with norm

$$\|u\|^p_{L^p(\mathcal{G})}=\sum_{e\in\mathcal{E}}\|u_e\|^p_{L^p(e)},$$

the corresponding inner product denoted by $(\cdot,\cdot)$. And the Sobolev space $H^1(\mathcal{G})$ defined by

$$H^1(\mathcal{G}):=\left\{u=\{u_e\}_{e\in\mathcal{E}}: u \text{ is continuous on } \mathcal{G}, \text{ and } u_e\in H^1(e) \text{ for all } e\in\mathcal{E}\right\},$$

equipped with norm

$$\|u\|^2_{H^1(\mathcal{G})}=\sum_{e\in\mathcal{E}}\left(\|u'_e\|^2_{L^2(e)}+\|u_e\|^2_{L^2(e)}\right),$$

the corresponding inner product denoted by $\langle\cdot,\cdot\rangle$. From [3], we say that u satisfies the Kirchhoff condition at vertex, namely

$$\sum_{e\succ v}\frac{du_e}{dx_e}(v)=0 \tag{1.3}$$

holds for any vertex $v\in\mathcal{V}$. Here $e\succ v$ means that the edge e is incident at vertex v and the notation $\frac{du_e}{dx_e}(v)$ stands for $u'_e(0)$ or $-u'_e(l_e)$, according to whether the vertex v is identified with 0 or l_e.

To get solutions of (1.1), a direct way is to find the critical points for the functional $\tilde{E}(\cdot,\mathcal{G}):H^1(\mathcal{G})\to\mathbb{R}$ defined by

$$\tilde{E}(u,\mathcal{G}):=\frac{a}{2}\int_{\mathcal{G}}|u'|^2\,dx+\frac{b}{4}\left(\int_{\mathcal{G}}|u'|^2\,dx\right)^2+\frac{\lambda}{2}\int_{\mathcal{G}}|u|^2\,dx-\frac{1}{p}\int_{\mathcal{K}}|u|^p\,dx. \tag{1.4}$$

From a physical perspective, an alternative approach is to prescribe the L^2 mass of u, i.e., to impose the mass constraint

$$\int_{\mathcal{G}}|u|^2\,dx=\mu>0. \tag{1.5}$$

In this time, the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. This type of solutions often called normalized solutions, and can be obtained by searching for the critical points of functional $E(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$

$$E(u, \mathcal{G}) := \frac{a}{2} \int_{\mathcal{G}} |u'|^2 \, dx + \frac{b}{4} \left(\int_{\mathcal{G}} |u'|^2 \, dx \right)^2 - \frac{1}{p} \int_{\mathcal{K}} |u|^p \, dx \tag{1.6}$$

constrained on the L^2-sphere

$$H^1_\mu(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \;\middle|\; \|u\|^2_{L^2(\mathcal{G})} = \mu \right\}.$$

It is easy to check that $E(\cdot, \mathcal{G})$ is class of C^2 on $H^1(\mathcal{G})$, and is an even functional.

When we consider the normalized solutions of (1.1), it is well known that the exponent $p = 10$ plays a special role, which is called L^2-critical exponent. This constant can be derived form the Gagliardo-Nirenberg inequality on noncompact metric graph in [2]: for every $u \in H^1(\mathcal{G})$ and every noncompact metric graph $\mathcal{G}$, there exists $C_p > 0$ such that

$$\|u\|^p_{L^p(\mathcal{G})} \leqslant C_p \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}.$$

Recently, Qi [13] considered the following problem

$$\begin{cases} -(a + b \int_{\mathcal{G}} |u'|^2 \, dx) u'' + \lambda u = |u|^{p-2} u, \quad x \in \mathcal{G}, \\ \int_{\mathcal{G}} |u|^2 \, dx = \mu, \end{cases} \tag{1.7}$$

with the Kirchhoff conditions at the vertices, where $2 < p < 10$, $a, b > 0$ and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. In this case, by Gagliardo-Nirenberg inequality, it is easy to see that the functional $E(\cdot, \mathcal{G})$ is bounded from below and coercive. Thus, the ground state solutions can be expected. However, when $p > 10$, the functional $E(\cdot, \mathcal{G})$ is always unbounded from below. And due to the fact that graphs are not invariant under scaling, the classic scaling techniques to deal with the mass constraint problem in [15, 16] do not work.

If $a = 1$ and $b = 0$, we can see that the equation (1.1) will reduce to the nonlinear Schrödinger equation on metric graph. In the last few decades, this problem has been extensively studied and has obtained a variety of results. The reader can consult [1, 2, 4, 6–9, 12, 14] and references therein. In particular, Carrillo et al. [6] studied the following problem

$$\begin{cases} -u'' + \lambda u = \kappa(x) |u|^{p-2} u, \quad x \in \mathcal{G}, \\ \int_{\mathcal{G}} |u|^2 \, dx = \mu, \end{cases} \tag{1.8}$$

with the Kirchhoff conditions at the vertices, and proved that (1.8) has infinitely many normalized solutions on the L^2-supercritical case (namely $p > 6$).

Motivated by the results already mentioned above, in this paper, we consider the existence of infinitely many normalized solutions for the equation (1.1) on the L^2-supercritical case for any mass $\mu > 0$.

Our main result can be stated as follows:

Theorem 1.1. *Let $\mathcal{G}$ be a noncompact metric graph and $p > 10$. Then, for any $\mu > 0$, there exist infinitely many solutions to equation (1.1) with the mass constraint (1.5). These solutions correspond to the critical points of functional $E(\cdot, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$ whose energy levels go to $+\infty$.*

It is easy to check that, for any $\mu > 0$, the functional $E(\cdot, \mathcal{G})$ has a mountain pass geometry in $H^1_\mu(\mathcal{G})$. But it is difficult to obtain the boundedness of Palais-Smale sequences for $E(\cdot, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$. In order to overcome this issue, similar to [4, 7], we consider the family of functionals $E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ given by

$$E_\rho(u, \mathcal{G}) := \frac{a}{2} \int_{\mathcal{G}} |u'|^2 \, dx + \frac{b}{4} \left(\int_{\mathcal{G}} |u'|^2 \, dx \right)^2 - \frac{\rho}{p} \int_{\mathcal{K}} |u|^p \, dx, \quad \forall u \in H^1(\mathcal{G}),\ \forall \rho \in [\frac{1}{2}, 1]. \tag{1.9}$$

Clearly a critical point of $E_\rho(u, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$ is a solution to

$$\begin{cases} -(a + b\int_{\mathcal{G}} |u'|^2 \, dx)u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u & \text{on every edge } e \in \mathcal{E}, \\ \sum\limits_{e \succ v} \frac{du_e}{dx_e}(v) = 0 & \text{at every vertex } v \in \mathcal{V}, \end{cases} \tag{1.10}$$

where λ is associated Lagrange multiplier. Denote by $m(u)$ the Morse index of the solution $u \in H^1_\mu(\mathcal{G})$ to (1.10).

This paper is organized as follows. Section 2 contains some Preliminaries. In Section 3, we show that functional $E_\rho(\cdot, \mathcal{G})$ has infinitely many distinct critical levels. In Section 4, we provide two auxiliary results. The first result is showing that the solutions to (1.10) with $\lambda = 0$ have a L^2 norm going to infinity when their values of $E_\rho(\cdot, \mathcal{G})$ goes to infinity (see Proposition 4.1). The second result is Theorem 4.1, which is used to derive Theorem 1.1 with $\rho_n \to 1^-$. Finally, in Section 5, we prove Theorem 1.1.

2 Preliminaries

In this section, we present some existing results that will be used in the rest of the paper. Firstly, we give some definitions.

Definition 2.1. *Let E be a infinite-dimensional Hilbert space, $\varphi : E \to \mathbb{R}$ be a C^2-functional and $\alpha \in (0, 1]$. If for any $R > 0$, one can find $M = M(R)$ such that, for any $u_1, u_2 \in B(0, R)$,*

$$\|\varphi'(u_1) - \varphi'(u_2)\|_* \leq M\|u_1 - u_2\|^\alpha$$

and

$$\|\varphi''(u_2) - \varphi''(u_2)\|_{**} \leq M\|u_1 - u_2\|^\alpha$$

then we say that $\varphi'(u)$ and $\varphi''(u)$ are α-Hölder continuous on bounded sets, where $\|\cdot\|_*$ and $\|\cdot\|_{**}$ show the operator norm of $\mathcal{L}(E, \mathbb{R})$ and of $\mathcal{L}(E, \mathcal{L}(E, \mathbb{R}))$ respectively.

Definition 2.2. *Let $\mathcal{G}$ be a noncompact metric graph, $p > 10$, $\mu > 0$ and $\rho \in [\frac{1}{2}, 1]$. For every $u \in H^1_\mu(\mathcal{G})$ and $\theta \geqslant 0$, we define the approximate Morse index of u with respect to θ by*

$$\tilde{m}_\theta(u) = \sup\left\{ dim L \middle| L\, is\, a\, subspace\, of\, T_u H^1_\mu(\mathcal{G})\, such\, that\, D^2E_\rho(u, \mathcal{G})[w, w] < -\theta\|w\|^2_{H^1(\mathcal{G})}, \forall w \in L\backslash\{0\} \right\},$$

where $T_u H^1_\mu(\mathcal{G})$ is the tangent space to $H^1_\mu(\mathcal{G})$ at u and

$$D^2E_\rho(u, \mathcal{G})[w, w] := E''_\rho(u, \mathcal{G})[w, w] - \frac{E'_\rho(u, \mathcal{G})[u]}{\|u\|^2_{L^2(\mathcal{G})}}\|w\|^2_{L^2(\mathcal{G})}.$$

If u is a critical point for E_ρ constrained to $H^1_\mu(\mathcal{G})$ and $\theta = 0$, then $\tilde{m}_0(u)$ is the Morse index of u as a constrained critical point.

Theorem 2.1. *[5, Theorem 1.12] Let $\mathcal{G}$ be a noncompact metric graph, $p > 10$ and $\mu > 0$. Let $I \subset (0, +\infty)$ be an interval and denote the family of C^2 functionals $E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ by the following form*

$$E_\rho(u, \mathcal{G}) = A(u, \mathcal{G}) - \rho B(u, \mathcal{G}), \qquad \rho \in I,$$

where $B(u, \mathcal{G}) \geqslant 0$ for all $u \in H^1(\mathcal{G})$ and

$$A(u, \mathcal{G}) \to +\infty \ \text{ or } \ B(u, \mathcal{G}) \to +\infty \qquad \text{as } \ u \in H^1(\mathcal{G}) \ \text{ and } \ \|u\|_{H^1(\mathcal{G})} \to +\infty. \tag{2.1}$$

Suppose that, for every $\rho \in I$, $E_\rho(u, \mathcal{G}) \mid_{H^1_\mu(\mathcal{G})}$ is even and $E'_\rho(u, \mathcal{G})$ and $E''_\rho(u, \mathcal{G})$ are α-Hölder continuous on bounded sets in the sense of Definition 2.1 for some $\alpha \in (0, 1]$. Moreover, suppose that there exists an integer $N \geqslant 2$ and two odd functions $\gamma_{i,N} : \mathbb{S}^{N-2} \to H^1_\mu(\mathcal{G})$, where $i = 0, 1$, such that the set

$$\Gamma_N := \left\{\gamma \in C([0,1] \times \mathbb{S}^{N-2}, H^1_\mu(\mathcal{G})) \mid \forall t \in [0,1], \gamma(t, \cdot) \text{is odd}, \gamma(0, \cdot) = \gamma_{0,N}, \text{and } \gamma(1, \cdot) = \gamma_{1,N}\right\} \tag{2.2}$$

is non void and

$$c^N_\rho := \inf_{\gamma \in \Gamma_N} \max_{(t,s) \in [0,1] \times \mathbb{S}^{N-2}} E_\rho(\gamma(t,s), \mathcal{G}) > \max_{s \in \mathbb{S}^{N-2}} \max \left\{E_\rho(\gamma_{0,N}(s), \mathcal{G}), E_\rho(\gamma_{1,N}(s), \mathcal{G})\right\}, \ \forall \rho \in I. \tag{2.3}$$

Then, for almost every $\rho \in I$, there exist sequences $\left\{u^N_{\rho,n}\right\} \subset H^1_\mu(\mathcal{G})$ and $\zeta^N_{\rho,n} \to 0^+$ such that, as $n \to +\infty$,

- *(i)* $E_\rho(u^N_{\rho,n}, \mathcal{G}) \to c^N_\rho$;
- *(ii)* $E'_\rho(u^N_{\rho,n}, \mathcal{G}) - \frac{E'_\rho(u^N_{\rho,n}, \mathcal{G})[u^N_{\rho,n}]}{\mu} u^N_{\rho,n} \to 0$ *in the dual of* $H^1_\mu(\mathcal{G})$;
- *(iii)* $\left\{u^N_{\rho,n}\right\}$ *is bounded in* $H^1_\mu(\mathcal{G})$;
- *(iv)* $\tilde{m}_{\zeta_n}(u^N_{\rho,n}) \leqslant N$.

Remark 2.1. From [6, Remark 2.6], we know that if the sequence $\left\{u^N_{\rho,n}\right\} \subset H^1_\mu(\mathcal{G})$ provided by the previous Theorem converges to some $u^N_\rho \in H^1_\mu(\mathcal{G})$, then in view of points (i)-(ii), u^N_ρ is a critical point of $E_\rho(u, \mathcal{G}) \mid_{H^1_\mu(\mathcal{G})}$ at level c^N_ρ. And the Morse index of u^N_ρ, as a constrained critical point, satisfies $\tilde{m}_0(u^N_\rho) \leqslant N$.

Lemma 2.1. *[6, Lemma 2.7] Let $\mathcal{G}$ be a noncompact, $N \geqslant 2$ be an integer, $p > 10$, $\mu > 0$ and $\rho > 0$. Assume that $\left\{u^N_{\rho,n}\right\} \subset H^1_\mu(\mathcal{G})$, $\left\{\lambda^N_{\rho,n}\right\} \subset \mathbb{R}$ and $\left\{\zeta^N_{\rho,n}\right\} \subset \mathbb{R}^+$ with $\zeta^N_{\rho,n} \to 0^+$ as $n \to +\infty$, and for a given $M \in \mathbb{N}$, the following conditions hold:*
(i) if all subspace $W_n \subset H^1(\mathcal{G})$ satisfies

$$E''_\rho(u^N_{\rho,n}, \mathcal{G})[w, w] + \lambda^N_{\rho,n} \|w\|^2_{L^2(\mathcal{G})} < -\zeta^N_{\rho,n} \|w\|^2_{H^1(\mathcal{G})}, \qquad \forall w \in W_n \setminus \{0\}, \tag{2.4}$$

for large enough $n \in \mathbb{N}$, then $dim(W_n) \leqslant M$. Here $\lambda^N_{\rho,n}$ is called the almost Lagrange multipliers, which dedined by

$$\lambda^N_{\rho,n} := -\frac{1}{\mu} E'_\rho(u^N_{\rho,n}, \mathcal{G})[u^N_{\rho,n}].$$

(ii) there exist $\lambda \in \mathbb{R}$, a subspace $Y \subset H^1(\mathcal{G})$ with $dim(Y) \geqslant M + 1$ and $\zeta > 0$ such that, for large enough $n \in \mathbb{N}$,

$$E''_\rho(u^N_{\rho,n}, \mathcal{G})[w, w] + \lambda \|w\|^2_{L^2(\mathcal{G})} \leqslant -\zeta \|w\|^2_{H^1(\mathcal{G})}, \qquad \forall w \in Y. \tag{2.5}$$

Then $\lambda^N_{\rho,n} > \lambda$ for all large enough $n \in \mathbb{N}$. In particular, if (2.5) holds for any $\lambda < 0$, then we have $\liminf_{n \to \infty} \lambda^N_{\rho,n} \geqslant 0$.

3 Existence of infinitely many minimax values

The main purpose of this section is to prove the following proposition.

Proposition 3.1. *Let $\mathcal{G}$ be a noncompact metric graph. For any $\mu > 0$ and $p > 10$, there exists $N_0 \in \mathbb{N}$ such that if $N \geqslant N_0$, there exist functions $\gamma_{0,N}$ and $\gamma_{1,N}$ such that the family of functionals*

$$E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R} : u \mapsto \frac{a}{2}\int_{\mathcal{G}} |u'|^2\, dx + \frac{b}{4}\left(\int_{\mathcal{G}} |u'|^2\, dx\right)^2 - \frac{\rho}{p}\int_{\mathcal{K}} |u|^p\, dx, \qquad \rho \in [\frac{1}{2}, 1]$$

satisfies the assumptions of Theorem 2.1. In particular,

$$\Gamma_N = \left\{\gamma \in C([0,1] \times \mathbb{S}^{N-2}, H^1_\mu(\mathcal{G})) \mid \forall t \in [0,1], \gamma(t,\cdot) is\ odd, \gamma(0,\cdot) = \gamma_{0,N}, and \gamma(1,\cdot) = \gamma_{1,N}\right\} \quad (3.1)$$

is non void and

$$c^N_\rho = \underset{\gamma\in\Gamma_N}{inf} \underset{(t,s)\in[0,1]\times\mathbb{S}^{N-2}}{max} E_\rho(\gamma(t,s), \mathcal{G}) > \underset{s\in\mathbb{S}^{N-2}}{max}\, max\left\{E_\rho(\gamma_{0,N}(s), \mathcal{G}), E_\rho(\gamma_{1,N}(s), \mathcal{G})\right\}, \forall \rho \in [\frac{1}{2}, 1] \quad (3.2)$$

Furthermore, $c^N_\rho \to +\infty$ uniformly w.r.t. $\rho \in [\frac{1}{2}, 1]$, as $N \to +\infty$. In particular, there exist infinitely many distinct values of c^N_ρ.

Remark 3.1. Notice that the levels $c^N_\rho \in \mathbb{R}$ for every $N \geqslant N_0$ and every $\rho \in [\frac{1}{2}, 1]$, since they are defined by infima over nonempty sets(equivalent to $c^N_\rho < +\infty$) and the inequality (3.2) implies that $c^N_\rho > -\infty$.

Remark 3.2. Setting

$$A(u, \mathcal{G}) = \frac{a}{2}\int_{\mathcal{G}} |u'|^2\, dx + \frac{b}{4}\left(\int_{\mathcal{G}} |u'|^2\, dx\right)^2 \quad \text{and} \quad B(u, \mathcal{G}) = \frac{1}{p}\int_{\mathcal{K}} |u|^p\, dx,$$

it is easy to check the assumption (2.1) holds, since

$$u \in H^1_\mu(\mathcal{G}) \text{ and } \|u\|_{H^1(\mathcal{G})} \to +\infty \quad \Longrightarrow \quad A(u, \mathcal{G}) \to +\infty.$$

Let $E'_\rho(u, \mathcal{G})$ and $E''_\rho(u, \mathcal{G})$ denote respectively the free first and second Fréchet derivatives of $E_\rho(u, \mathcal{G})$. Clearly, $E'_\rho(u, \mathcal{G})$ and $E''_\rho(u, \mathcal{G})$ are both of class C^1, and hence locally Hölder continuous on $H^1_\mu(\mathcal{G})$, which implies $E'_\rho(u, \mathcal{G})$ and $E''_\rho(u, \mathcal{G})$ are α-Hölder continuous on bounded sets in the sense of Definition 2.1 for some $\alpha \in (0, 1]$. Then, it is only to show that (3.1) and (3.2) posed on Γ_N hold.

As this point, we shall rely on some results from [6], which are following lemmas.

Lemma 3.1. *Let $\{u_1, \cdots, u_{N-1}\} \subset H^1_\mu(\mathcal{G})$ and $\{v_1, \cdots, v_{N-1}\} \subset H^1_\mu(\mathcal{G})$ be orthogonal families for the inner product $(\cdot, \cdot)$. Setting the odd functions*

$$\gamma_{0,N} : \mathbb{S}^{N-2} \to H^1_\mu(\mathcal{G}) \ \ by \ \ \gamma_{0,N}(s_1, \cdots, s_{N-1}) = \sum_{i=1}^{N-1} s_i u_i$$

and

$$\gamma_{1,N} : \mathbb{S}^{N-2} \to H^1_\mu(\mathcal{G}) \ \ by \ \ \gamma_{1,N}(s_1, \cdots, s_{N-1}) = \sum_{i=1}^{N-1} s_i v_i,$$

the set Γ_N defined by (2.2) is non void.

Lemma 3.2. *Let $\mathcal{G}$ be a noncompact metric graph, $p > 10, \mu > 0, \rho > 0, N \geqslant 2$ be an integer, and $\gamma_{i,N} : \mathbb{S}^{N-2} \to H^1_\mu(\mathcal{G}), i = 1, 2,$ be two odd functions. Assume that the set*

$$\Gamma_N := \left\{\gamma \in C([0,1] \times \mathbb{S}^{N-2}, H^1_\mu(\mathcal{G})) \mid \forall t \in [0,1], \gamma(t,\cdot) is\ odd, \gamma(0,\cdot) = \gamma_{0,N}, and \gamma(1,\cdot) = \gamma_{1,N}\right\}$$

is not empty. Assume further that there exists a continuous even functional $J : H^1(\mathcal{G}) \to \mathbb{R}, \beta_N \in \mathbb{R}$, and a subspace $W \subset H^1(\mathcal{G})$ with $dim(W) \leqslant N - 2$ such that

(H1) $J(\gamma_{0,N}(s)) < \beta_N < J(\gamma_{1,N}(s))$, *for all* $s \in \mathbb{S}^{N-2}$;

(H2) $\max_{s\in\mathbb{S}^{N-2}} \max \{E_\rho(\gamma_{0,N}(s), \mathcal{G}),\ E_\rho(\gamma_{1,N}(s), \mathcal{G})\} < \inf_{u\in B_N} E_\rho(u, \mathcal{G}),$

where, $B_N = \left\{u \in H^1_\mu(\mathcal{G}) \cap W^\perp \mid J(u, \mathcal{G}) = \beta_N\right\}$. *Then*

$$c^N_\rho = \inf_{\gamma\in\Gamma_N} \max_{(t,s)\in[0,1]\times\mathbb{S}^{N-2}} E_\rho(\gamma(t, s), \mathcal{G}) \geqslant \inf_{u\in B_N} E_\rho(u, \mathcal{G}). \tag{3.3}$$

The following two lemmas will provide orthogonal families to be used in Lemma 3.1.

Lemma 3.3. *Let $\mathcal{G}$ be a noncompact metric graph, $p > 10$ and $\mu > 0$. For any $\xi > 0$, there exists a sequence of functions $\{\phi_1, \phi_2, \cdots\}$ such that for any $i, j \in \mathbb{N}^*$ and any $\rho \in [\frac{1}{2}, 1]$:*

(i) $\phi_i \in H^1_\mu(\mathcal{G})$; $\|\phi_i'\|_{L^2(\mathcal{G})} = \xi$; $E_\rho(\phi_i, \mathcal{G}) = \frac{a}{2}\xi^2 + \frac{b}{4}\xi^4$;

(ii) ϕ_i *has compact support and* $supp(\phi_i) \cap supp(\phi_j) = \varnothing$ *for* $i \neq j$;

(iii) for any $N \geqslant 2$ *and* $s \in \mathbb{S}^{N-2}$, $\left\|\left(\sum_{i=1}^{N-1} s_i\phi_i\right)'\right\|_{L^2(\mathcal{G})} = \xi$ *and* $E_\rho\left(\sum_{i=1}^{N-1} s_i\phi_i, \mathcal{G}\right) = \frac{a}{2}\xi^2 + \frac{b}{4}\xi^4$.

Proof. The proof is very similar to [6, Lemma 4.3], only need to replace the expression of $E_\rho(u, \mathcal{G})$. □

Lemma 3.4. *Let $\mathcal{G}$ be a noncompcat metric graph, $p > 10$ and $\mu > 0$. For any fixed integer $N \geqslant 2$ and any given values of $\overline{\xi} > 0$, $\overline{b} > 0$, there exist functions $\overline{\phi}_1, \cdots, \overline{\phi}_N$, compactly supported in $\mathcal{K}$, such that for all $i, j \in \{1, \cdots, N\}$ and all $\rho \in [\frac{1}{2}, 1]$,*

(i) $\overline{\phi}_i \in H^1_\mu(\mathcal{G})$; $\left\|\overline{\phi}_i'\right\|_{L^2(\mathcal{G})} \geqslant \overline{\xi}$;

(ii) $supp\left(\overline{\phi}_i\right) \cap supp\left(\overline{\phi}_i\right) = \varnothing$ *for* $i \neq j$;

(iii) if $s \in \mathbb{S}^{N-2}$ *then* $\left\|\left(\sum_{i=1}^{N-1} s_i\overline{\phi}_i\right)'\right\|_{L^2(\mathcal{G})} \geqslant \overline{\xi}$ *and* $E_\rho\left(\sum_{i=1}^{N-1} s_i\overline{\phi}_i, \mathcal{G}\right) \leqslant \overline{b}$.

Proof. Let e $= [0, l_e]$ be any bounded edge of $\mathcal{G}$. Let $\phi \in C^\infty_c\left((0, \frac{l_e}{N})\right)$ be any function such that $\|\phi\|_{L^2(\mathbb{R})} = \mu$. For $t \in \mathbb{R}^+$, define the function ϕ^t by

$$\phi^t(x) := t^{\frac{1}{2}}\phi(tx). \tag{3.4}$$

We can see that supp $(\phi^t) \subset \left(0, \frac{l_e}{N}\right)$ whenever $t \geqslant 1$. Define the functions

$$\overline{\phi}_i := \phi^t\left(x - \frac{(i-1)l_e}{N}\right), \quad i = 1, \cdots, N$$

where $t \geqslant 1$ will be chosen later. Note that

$$\text{supp }(\overline{\phi}_i) \subset \left(\frac{(i-1)l_e}{N}, \frac{il_e}{N}\right),$$

so the functions $\overline{\phi}_i$ have disjoint supports, then (ii) is satisfied. Viewing now $\overline{\phi}_i$ as functions in $H^1(\mathcal{G})$ supported in e, we may compute the energy of function $\sum\limits_{i=1}^{N-1} s_i\overline{\phi}_i$ with $s \in \mathbb{S}^{N-2}$ as follows

$$\begin{aligned} E_\rho\left(\sum_{i=1}^{N-1} s_i\overline{\phi}_i, \mathcal{G}\right) &= \frac{a}{2}\int_e \left|\sum_{i=1}^{N-1} s_i\overline{\phi}_i{}'\right|^2 dx + \frac{b}{4}\left(\int_e \left|\sum_{i=1}^{N-1} s_i\overline{\phi}_i{}'\right|^2 dx\right)^2 - \frac{\rho}{p}\int_e \left|\sum_{i=1}^{N-1} s_i\overline{\phi}_i\right|^p dx \\ &= \frac{at^2}{2}\sum_{i=1}^{N-1} s_i^2 \int_0^{\frac{l_e}{N}} |\phi'|^2\,dx + \frac{b}{4}\left(t^2\sum_{i=1}^{N-1} s_i^2 \int_0^{\frac{l_e}{N}} |\phi'|^2\,dx\right)^2 - \frac{\rho t^{\frac{p-2}{2}}}{p}\sum_{i=1}^{N-1} |s_i|^p \int_0^{\frac{l_e}{N}} |\phi|^p\,dx \\ &\leqslant \frac{at^2}{2}\|\phi'\|^2_{L^2(\mathcal{G})} + \frac{bt^4}{4}\|\phi'\|^4_{L^2(\mathcal{G})} - \frac{Ct^{\frac{p-2}{2}}}{2p}\|\phi\|^p_{L^p(\mathcal{K})}, \end{aligned}$$

where $C = \min\limits_{s\in\mathbb{S}^{N-2}} \sum\limits_{i=1}^{N-1} |s_i|^p$. It is straightforward to verify that the last expression tends to $-\infty$ as $t \to +\infty$. Thus, for all $\overline{b} \in \mathbb{R}$, there exists $t_0 > 0$ such that for all $t > t_0$ we have $E_\rho\left(\sum\limits_{i=1}^{N-1} s_i\overline{\phi}_i, \mathcal{G}\right) < \overline{b}$. As a result, if we chose

$$t := \max\left\{1, \frac{\overline{\xi}}{\|\phi'\|_{L^2(\mathcal{G})}}, t_0\right\},$$

the functions $\overline{\phi}_i$ satisfy all of the desired properties. Indeed, $\overline{\phi}_i \in H^1_\mu(\mathcal{G})$ and a direct calculation yields

$$\|\overline{\phi}_i{}'\|_{L^2(\mathcal{G})} = t\|\phi'\|_{L^2(\mathcal{G})} \geqslant \overline{\xi}, \tag{3.5}$$

which implies (i). Finally, the choice of t, (3.5) and $\left\|\left(\sum\limits_{i=1}^{N-1} s_i\overline{\phi}_i\right)'\right\|^2_{L^2(\mathcal{G})} = \sum\limits_{i=1}^{N-1} s_i^2 \left\|\overline{\phi}_i{}'\right\|^2_{L^2(\mathcal{G})}$ show (iii). $\square$

Now let $\{Y_N\}$ be a sequence of linear subspaces of $H^1(\mathcal{G})$ with dim $(Y_N) = N$, which is exhausting $H^1(\mathcal{G})$ in the sense that

$$\bigcup_{N\geqslant 1} Y_N$$

is dense in $H^1(\mathcal{G})$. We recall that for separable Hilbert spaces, such as $H^1(\mathcal{G})$, such sequence always exists.

Lemma 3.5. *[6, Lemma 4.5] For any* $p > 10$ *there holds:*

$$S_N := \mathop{\mathit{inf}}_{u\in Y_{N-2}^{\perp}} \frac{\int_{\mathcal{G}} |u'|^2 + |u|^2\,dx}{\left(\int_{\mathcal{K}} |u|^p\,dx\right)^{\frac{2}{p}}} \to \infty, \quad \textit{as } N \to \infty.$$

Now, we define

$$\beta_N := \left(\frac{S_N^{\frac{p}{2}}}{\aleph}\right)^{\frac{1}{p-2}}, \qquad \text{where } \aleph = \aleph(p) := \frac{3}{ap}\max_{x>0}\frac{(\mu+x^2)^{\frac{p}{2}}}{\mu+x^p}.$$

Using Lemma 3.5, we have that $\beta_N \to \infty$. Thus, setting

$$\tilde{c}_\rho^N := \underset{u\in B_N}{inf}\ E_\rho(u,\mathcal{G}), \qquad \text{where } B_N := \left\{u \in Y_{N-2}^{\perp} \cap H_\mu^1(\mathcal{G}) \mid \|u'\|_{L^2(\mathcal{G})} = \beta_N\right\},$$

we can obtain that

Lemma 3.6. *$\tilde{c}_\rho^N \to +\infty$ as $N \to +\infty$, uniformly in $\rho \in [\frac{1}{2}, 1]$.*

Proof. For every $u \in B_N$, using Lemma 3.5 we have

$$\begin{aligned}
E_\rho(u,\mathcal{G}) =& \frac{a}{2}\int_{\mathcal{G}}|u'|^2\,dx + \frac{b}{4}\left(\int_{\mathcal{G}}|u'|^2\,dx\right)^2 - \frac{\rho}{p}\int_{\mathcal{K}}|u|^p\,dx \\
\geqslant& \frac{a}{2}\int_{\mathcal{G}}|u'|^2\,dx + \frac{b}{4}\left(\int_{\mathcal{G}}|u'|^2\,dx\right)^2 - \frac{1}{p}\left(\frac{\mu+\int_{\mathcal{G}}|u'|^2\,dx}{S_N}\right)^{\frac{p}{2}} \\
\geqslant& \frac{a}{2}\int_{\mathcal{G}}|u'|^2\,dx + \frac{b}{4}\left(\int_{\mathcal{G}}|u'|^2\,dx\right)^2 - \frac{a\aleph}{3S_N^{\frac{p}{2}}}\left(\mu + \|u'\|_{L^2(\mathcal{G})}^p\right) \\
=& \frac{a}{2}\beta_N^2 + \frac{b}{4}\beta_N^4 - \frac{a}{3}\beta_N^{2-p}\left(\mu+\beta_N^p\right) \\
=& \frac{a}{6}\beta_N^2 + \frac{b}{4}\beta_N^4 + o(1).
\end{aligned}$$

The proof is completed by taking the infimum over B_N. ☐

Proof of Proposition 3.1. From Remark 3.2, we can konw that $E_\rho(u,\mathcal{G})$ satisfies the assumptions of Theorem 2.1. By Lemma 3.5 and Lemma 3.6, for each $N \geqslant 2$, both sequences $\{\beta_N\}$ and $\left\{\tilde{c}_\rho^N\right\}$ are diverge.

Then consider a sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ as given by Lemma 3.3 taking $\xi = 1$ and a set of N functions $\{\overline{\phi}_i\}_{i=1}^{N}$ given by Lemma 3.4 taking $\overline{\xi} = 2\beta_N$ and $\overline{b} = 1$. And, define the functions

$$\gamma_{0,N} : \mathbb{S}^{N-2} \to H_\mu^1(\mathcal{G}) : (s_1, \cdots, s_{N-1}) \mapsto \sum_{i=1}^{N-1} s_i\phi_i,$$

$$\gamma_{1,N} : \mathbb{S}^{N-2} \to H_\mu^1(\mathcal{G}) : (s_1, \cdots, s_{N-1}) \mapsto \sum_{i=1}^{N-1} s_i\overline{\phi}_i,$$

which satisfy, for each $N \geqslant 2$ and $s \in \mathbb{S}^{N-2}$,

$$\begin{cases} \|\gamma_{0,N}(s)'\|_{L^2(\mathcal{G})} = 1, \\ E_\rho(\gamma_{0,N}(s),\mathcal{G}) = \frac{a}{2} + \frac{b}{4}, \end{cases}$$

and

$$\begin{cases} \|\gamma_{1,N}(s)'\|_{L^2(\mathcal{G})} \geqslant 2\beta_N, \\ E_\rho(\gamma_{1,N}(s), \mathcal{G}) \leqslant 1. \end{cases}$$

From Lemma 3.1, we can get that the set

$$\Gamma_N = \left\{\gamma \in C([0,1] \times \mathbb{S}^{N-2}, H^1_\mu(\mathcal{G})) \mid \forall t \in [0,1], \gamma(t,\cdot) is\ odd, \gamma(0,\cdot) = \gamma_{0,N}, and \gamma(1,\cdot) = \gamma_{1,N}\right\}$$

is not empty, (3.1) holds.

Finally, we use Lemma 3.2 with the choice $J(u, \mathcal{G}) = \|u'\|_{L^2(\mathcal{G})}$ and $W = Y_{N-2}$. It is easy to check that assumptions $(H1)$ and $(H2)$ are satisfied for any N sufficiently large (uniformly *w.r.t.* ρ), thus (3.2) holds. And using $\tilde{c}^N_\rho \to +\infty$ as $N \to +\infty$ and (3.3), we get that $c^N_\rho \to +\infty$ uniformly *w.r.t.* $\rho \in [\frac{1}{2}, 1]$, as $N \to +\infty$. □

4 Two auxiliary results

4.1 The first auxiliary result

In this subsection, we focus on deriving properties of solutions to (1.10) when $\lambda = 0$, as stated in the following proposition.

Proposition 4.1. *Let $\mathcal{G}$ be a noncompact metric graph and $p > 10$. Let $\{u_{\rho_k}\} \subset H^1(\mathcal{G})$ and $\{\rho_k\} \subset [\frac{1}{2}, 1]$ be sequences such that u_{ρ_k} is a solution to (1.10) with $\rho = \rho_k$ and $\lambda = 0$. If $E_{\rho_k}(u_{\rho_k}, \mathcal{G}) \to +\infty$, then $\|u_{\rho_k}\|_{L^2(\mathcal{G})} \to \infty$.*

Remark 4.1. Note that if (λ, u) is a solution of

$$-(a + b\int_I |u'|^2\, dx)u'' + \lambda u = \rho|u|^{p-2}u,$$

with L^2 mass μ_0 on some interval $I \subseteq \mathbb{R}^+$. Set $z = \varpi^{\frac{1}{2-p}}u$, where $\varpi = a + b\int_I |u'|^2\, dx$. Then $\left(\frac{\lambda}{\varpi}, z\right)$ is a solution of

$$-z'' + \frac{\lambda}{\varpi}z = \rho|z|^{p-2}z,$$

with L^2 mass $\varpi^{\frac{2}{2-p}}\mu_0$ on some interval $I \subseteq \mathbb{R}^+$. Since a function on a metric graph is a family of functions defined along its edges and the fact that each edge of a metric graph can be identified with an interval $I \subseteq \mathbb{R}^+$. We can say that if (λ, u) is a solution of

$$-(a + b\int_{\mathcal{G}} |u'|^2\, dx)u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u,$$

with L^2 mass μ on metric graph $\mathcal{G}$, then $\left(\frac{\lambda}{\omega}, f\right)$ is a solution of

$$-f'' + \frac{\lambda}{\omega}f = \rho\kappa(x)|f|^{p-2}f, \tag{4.1}$$

with L^2 mass $\omega^{\frac{2}{2-p}}\mu$ on metric graph $\mathcal{G}$.

Then, for the following problem

$$\begin{cases} -f'' + \frac{\lambda}{\omega} f = \rho\kappa(x)|f|^{p-2}f & \text{on every edge } e \in \mathcal{E}, \\ \sum\limits_{e \succ v} \frac{df_e}{dx_e}(v) = 0 & \text{at every vertex } v \in \mathcal{V}, \end{cases} \tag{4.2}$$

we define the family of functionals $J_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ by

$$J_\rho(f, \mathcal{G}) := \frac{1}{2} \int_{\mathcal{G}} \left|f'\right|^2 dx - \frac{\rho}{p} \int_{\mathcal{K}} |f|^p \, dx,$$

constrained on the L^2-sphere

$$S_\mu := \left\{ f \in H^1(\mathcal{G}) \;\middle|\; \|f\|^2_{L^2(\mathcal{G})} = \omega^{\frac{2}{2-p}} \mu \right\}.$$

From [6], we can know that if f is a solution to $-f'' + \frac{\lambda}{\omega} f = \rho|f|^{p-2}f$ on some interval $I \subseteq \mathbb{R}^+$, set

$$H_f(x) := \frac{1}{2} \left(f'(x)\right)^2 + V_{\omega,\lambda}(f(x)) \quad \text{where} \quad V_{\omega,\lambda}(f) := \frac{\rho}{p} |f|^p - \frac{\lambda}{2\omega} f^2,$$

it is easy to see that $H_f(x)$ is a constant on I. Since $H_f'(x) := f'(x) \cdot (f''(x) + V'_{\omega,\lambda}(f(x))) = 0$. The H_f is called the *ODE energy* of the solution f on I.

The proofs of the following two lemmas follow closely those in [6, Proposition 3.1 and Lemma 3.2], with λ replaced by $\frac{\lambda}{\omega}$.

Lemma 4.1. *Let $\mathcal{G}$ be a noncompact metric graph. Let $p > 10$, $\frac{\lambda}{\omega} \in \mathbb{R}$, and f be a solution to (4.2). For each bounded edge e of $\mathcal{G}$, let the ODE energy of the solution f on e be given by*

$$H_f(e) := H_f(x) = \frac{1}{2} \left|f'(x)\right|^2 + \frac{\rho}{p} |f(x)|^p - \frac{\lambda}{2\omega} |f(x)|^2, \tag{4.3}$$

where x is an arbitrary point of e. Finally, define

$$P_\rho(f, \mathcal{G}) := \sum_{e \text{ is a bounded edge of } \mathcal{G}} l_e H_f(e), \tag{4.4}$$

where l_e is the length of the edge e. Then, one has

$$\frac{1}{2} \|f'\|^2_{L^2(\mathcal{G})} + \frac{\rho}{p} \|\kappa f\|^p_{L^p(\mathcal{G})} = \frac{\lambda}{2\omega} \|f\|^2_{L^2(\mathcal{G})} + P_\rho(f, \mathcal{G}).$$

Lemma 4.2. *Let $\mathcal{G}$ be a noncompact metric graph. Let $p > 10$ and $\frac{\lambda}{\omega} \in \mathbb{R}$. Let $f \in H^1(\mathcal{G})$ be a solution to (4.2). Then, one has*

$$J_\rho(f, \mathcal{G}) = \frac{(p-6)\lambda}{2\omega(p+2)} \|f\|^2_{L^2(\mathcal{G})} + \frac{p-2}{p+2} P_\rho(f, \mathcal{G}),$$

where $P_\rho(f, \mathcal{G})$ is defined by (4.3)-(4.4).

The following lemma is crucial for proving Proposition 4.1, which is an already mentioned result from [6, Corollary 3.5].

Lemma 4.3. *Let $l > 0, 0 < \underline{\alpha} < \overline{\alpha} < \infty$, and $p > 10$. For every $\underline{\mu} > 0$, there exists $\underline{H} > 0$ such that if $f : [0, l] \to \mathbb{R}$ is a solution to*

$$-f'' = \alpha\,|f|^{p-2}\,f, \qquad \text{with } \alpha \in [\underline{\alpha}, \overline{\alpha}] \text{ and } H_f \geqslant \underline{H},$$

then

$$\int_0^l |f(x)|^2\, dx \geqslant \underline{\mu}.$$

Proof of Proposition 4.1. From Remark 4.1, we have $f_{\rho_k} = \omega^{\frac{1}{2-p}} u_{\rho_k}$ is a solution to (4.2) with $\rho = \rho_k$ and $\lambda = 0$. Since $E_{\rho_k}(u_{\rho_k}, \mathcal{G}) \to +\infty$, we can know that $J_{\rho_k}(f_{\rho_k}, \mathcal{G}) \to +\infty$, then Lemma 4.2 implies that $P_{\rho_k}(f_{\rho_k}, \mathcal{G}) \to +\infty$. Let $e_0 \in \mathcal{E}$ be a bounded edge such that $l_{e_0} H_{f_{\rho_k}}(e_0) \geqslant l_e H_{f_{\rho_k}}(e)$ for all bounded edges $e \in \mathcal{E}$. Since $\mathcal{E}$ is finite, then we can get that

$$P_{\rho_k}(f_{\rho_k}, \mathcal{G}) \leqslant \#\{\mathcal{E}\} l_{e_0} H_{f_{\rho_k}}(e_0),$$

so $H_{f_{\rho_k}}(e_0) \to +\infty$. Finally, using Lemma 4.3 with $f = f_{\rho_k}, \alpha = \rho_k$ and $[0, l] = e_0$, we deduce that $\|f_{\rho_k}\|_{L^2(e_0)} \to \infty$ and thus $\|f_{\rho_k}\|_{L^2(\mathcal{G})} \to \infty$. Also have $\|u_{\rho_k}\|_{L^2(\mathcal{G})} \to \infty$, since $\|f_{\rho_k}\|_{L^2(\mathcal{G})} = \omega^{\frac{2}{2-p}} \|u_{\rho_k}\|_{L^2(\mathcal{G})}$. ☐

4.2 The second auxiliary result

In this subsection, we present the main result to prove Theorem 1.1.

Theorem 4.1. *Let $\mathcal{G}$ be a noncompact metric graph and $p > 10$. For any $\mu > 0$, there exists $N_0 \in \mathbb{N}$ such that for almost every $\rho \in [\frac{1}{2}, 1]$, there exist sequences of Lagrange multipliers $\{\lambda_\rho^N\}_{N=N_0}^{\infty} \subset \mathbb{R}^+$ and solutions $\{u_\rho^N\}_{N=N_0}^{\infty} \subset H_\mu^1(\mathcal{G})$ to*

$$\begin{cases} -(a + b\int_{\mathcal{G}} \left|(u_\rho^N)'\right|^2 dx)(u_\rho^N)'' + \lambda_\rho^N u_\rho^N = \rho\kappa(x)\left|u_\rho^N\right|^{p-2} u_\rho^N & \text{on every edge } e \in \mathcal{E}, \\ \sum\limits_{e \succ v} \frac{du_{\rho,e}^N}{dx_e}(v) = 0 & \text{at every vertex } v \in \mathcal{V}. \end{cases} \tag{4.5}$$

In addition, $c_\rho^N = E_\rho(u_\rho^N, \mathcal{G}) \to +\infty$ uniformly w.r.t. $\rho \in [\frac{1}{2}, 1]$, as $N \to +\infty$ and $m(u_\rho^N) \leqslant N + 1$.

As a consequence of Proposition 3.1, we can see that the all assumptions of Theorem 2.1 are satisfied; then for all $N \in \mathbb{N}$ large enough and for almost every $\rho \in [\frac{1}{2}, 1]$, we deduce the existence of a bounded sequence $\{u_{\rho,n}^N\}_{n=1}^{\infty} \subset H_\mu^1(\mathcal{G})$, such that

$$E_\rho(u_{\rho,n}^N, \mathcal{G}) \to c_\rho^N \tag{4.6}$$

and

$$E'_\rho(u_{\rho,n}^N, \mathcal{G}) + \lambda_{\rho,n}^N(u_{\rho,n}^N, \cdot) \to 0 \quad \text{in the dual of } H_\mu^1(\mathcal{G}) \tag{4.7}$$

where

$$\lambda_{\rho,n}^N = -\frac{1}{\mu} E'_\rho(u_{\rho,n}^N, \mathcal{G})[u_{\rho,n}^N]. \tag{4.8}$$

Finally, there exists a sequence $\{\zeta_{\rho,n}^N\} \subset \mathbb{R}^+$ with $\zeta_{\rho,n}^N \to 0^+$ such that, if the inequality

$$a\int_{\mathcal{G}} \left|w'\right|^2 dx + b\int_{\mathcal{G}} \left|(u_{\rho,n}^N)'\right|^2 dx \int_{\mathcal{G}} \left|w'\right|^2 dx + 2b\left(\int_{\mathcal{G}} (u_{\rho,n}^N)' w'\, dx\right)^2$$

$$\begin{aligned}&-(p-1)\rho\int_{\mathcal{K}}\left|u_{\rho,n}^N\right|^{p-2}|w|^2\,dx+\lambda_{\rho,n}^N\int_{\mathcal{G}}|w|^2\,dx\\ =&E_\rho''(u_{\rho,n}^N,\mathcal{G})[w,w]+\lambda_{\rho,n}^N\|w\|_{L^2(\mathcal{G})}^2<-\zeta_{\rho,n}^N\|w\|_{H^1(\mathcal{G})}^2,\end{aligned}\tag{4.9}$$

holds for any $w\in W_n\setminus\{0\}$ in a subspace W_n of $T_{u_{\rho,n}^N}H_\mu^1(\mathcal{G})$, then dim $(W_n)\leqslant N$.

Since $\left\{u_{\rho,n}^N\right\}\subset H^1(\mathcal{G})$ is bounded, and therefore up to a subsequence, we can assume that there exists some $u_\rho^N\in H^1(\mathcal{G})$ such that

$$u_{\rho,n}^N\rightharpoonup u_\rho^N\quad\text{in } H^1(\mathcal{G}),\tag{4.10}$$

$$u_{\rho,n}^N\to u_\rho^N\quad\text{in } L_{loc}^p(\mathcal{G}).\tag{4.11}$$

In particular, since $\left\{u_{\rho,n}^N\right\}\subset H^1(\mathcal{G})$ is a bounded sequence, it follows from (4.8) that $\left\{\lambda_{\rho,n}^N\right\}\subset\mathbb{R}$ is bounded. Up to a subsequence, there exists $\lambda_\rho^N\in\mathbb{R}$ such that $\lim\limits_{n\to+\infty}\lambda_{\rho,n}^N=\lambda_\rho^N$.

We claim that the limit $u_\rho^N\in H^1(\mathcal{G})$ solves (4.5). Using (4.7) and $\lim\limits_{n\to+\infty}\lambda_{\rho,n}^N=\lambda_\rho^N$, we have

$$\begin{aligned}0&=\lim_{n\to\infty}\left[E_\rho'(u_{\rho,n}^N,\mathcal{G})+\lambda_{\rho,n}^N(u_{\rho,n}^N,\cdot)\right][\eta]\\ &=\lim_{n\to\infty}\left[a\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx+b\int_{\mathcal{G}}|(u_{\rho,n}^N)'|^2\,dx\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx-\rho\int_{\mathcal{K}}|u_{\rho,n}^N|^{p-2}u_{\rho,n}^N\eta\,dx+\lambda_{\rho,n}^N\int_{\mathcal{G}}u_{\rho,n}^N\eta\,dx\right]\\ &=a\int_{\mathcal{G}}(u_\rho^N)'\eta'\,dx+b\int_{\mathcal{G}}|(u_\rho^N)'|^2\,dx\int_{\mathcal{G}}(u_\rho^N)'\eta'\,dx-\rho\int_{\mathcal{K}}|u_\rho^N|^{p-2}u_\rho^N\eta\,dx+\lambda_\rho^N\int_{\mathcal{G}}u_\rho^N\eta\,dx\end{aligned}\tag{4.12}$$

for any $\eta\in H^1(\mathcal{G})$. So the claim is true.

Next, we focus on proving the strong convergence of the sequence $\left\{u_{\rho,n}^N\right\}\subset H^1(\mathcal{G})$. We have the following lemma.

Lemma 4.4. *If $\lambda_\rho^N\geqslant 0$, the following convergence holds:*

$$\int_{\mathcal{G}}|(u_{\rho,n}^N-u_\rho^N)'|^2\,dx+\lambda_\rho^N\int_{\mathcal{G}}|u_{\rho,n}^N-u_\rho^N|^2\,dx\to 0,\quad \textit{as } n\to+\infty.$$

In particular, if $\lambda_\rho^N>0$, then sequence $\left\{u_{\rho,n}^N\right\}$ converges strongly in $H^1(\mathcal{G})$.

Proof. From (4.7), we have

$$\begin{aligned}o(1)\|\eta\|_{H^1(\mathcal{G})}=&a\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx+b\int_{\mathcal{G}}|(u_{\rho,n}^N)'|^2\,dx\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx\\ &-\rho\int_{\mathcal{K}}|u_{\rho,n}^N|^{p-2}u_{\rho,n}^N\eta\,dx+\lambda_{\rho,n}^N\int_{\mathcal{G}}u_{\rho,n}^N\eta\,dx\\ =&a\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx+b\int_{\mathcal{G}}|(u_{\rho,n}^N)'|^2\,dx\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx\\ &-\rho\int_{\mathcal{K}}|u_{\rho,n}^N|^{p-2}u_{\rho,n}^N\eta\,dx+\lambda_\rho^N\int_{\mathcal{G}}u_{\rho,n}^N\eta\,dx+(\lambda_{\rho,n}^N-\lambda_\rho^N)\int_{\mathcal{G}}u_{\rho,n}^N\eta\,dx,\end{aligned}$$

equivalent to

$$o(1)\|\eta\|_{H^1(\mathcal{G})}=a\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx+b\int_{\mathcal{G}}|(u_{\rho,n}^N)'|^2\,dx\int_{\mathcal{G}}(u_{\rho,n}^N)'\eta'\,dx$$

$$- \rho \int_{\mathcal{K}} |u_{\rho,n}^N|^{p-2} u_{\rho,n}^N \eta \, dx + \lambda_\rho^N \int_{\mathcal{G}} u_{\rho,n}^N \eta \, dx. \tag{4.13}$$

Then, combining (4.12) and (4.13), choosing $\eta = \eta_{\rho,n}^N = u_{\rho,n}^N - u_\rho^N$ and using (4.11) and also $\{\eta_{\rho,n}^N\}$ is bounded, we obtain

$$\begin{aligned} o(1) =& a \int_{\mathcal{G}} ((u_{\rho,n}^N)' - (u_\rho^N)')(\eta_{\rho,n}^N)' \, dx + b \int_{\mathcal{G}} |(u_{\rho,n}^N)'|^2 \, dx \int_{\mathcal{G}} (u_{\rho,n}^N)'(\eta_{\rho,n}^N)' \, dx \\ & - b \int_{\mathcal{G}} |(u_\rho^N)'|^2 \, dx \int_{\mathcal{G}} (u_\rho^N)'(\eta_{\rho,n}^N)' \, dx - \rho \int_{\mathcal{K}} (|u_{\rho,n}^N|^{p-2} u_{\rho,n}^N - |u_\rho^N|^{p-2} u_\rho^N) \eta_{\rho,n}^N \, dx \\ & + \lambda_\rho^N \int_{\mathcal{G}} (u_{\rho,n}^N - u_\rho^N) \eta_{\rho,n}^N \, dx \\ =& a \int_{\mathcal{G}} ((u_{\rho,n}^N)' - (u_\rho^N)')(\eta_{\rho,n}^N)' \, dx + \lambda_\rho^N \int_{\mathcal{G}} (u_{\rho,n}^N - u_\rho^N) \eta_{\rho,n}^N \, dx \\ & + b \int_{\mathcal{G}} |(u_{\rho,n}^N)'|^2 \, dx \left(\int_{\mathcal{G}} (u_{\rho,n}^N)'(\eta_{\rho,n}^N)' \, dx - \int_{\mathcal{G}} (u_\rho^N)'(\eta_{\rho,n}^N)' \, dx \right) + o(1) \\ =& a \int_{\mathcal{G}} ((u_{\rho,n}^N)' - (u_\rho^N)')(\eta_{\rho,n}^N)' \, dx + \lambda_\rho^N \int_{\mathcal{G}} (u_{\rho,n}^N - u_\rho^N) \eta_{\rho,n}^N \, dx \\ & + b \int_{\mathcal{G}} |(u_{\rho,n}^N)'|^2 \, dx \int_{\mathcal{G}} |{\eta_{\rho,n}^N}'|^2 \, dx + o(1) \\ \geqslant& a \int_{\mathcal{G}} |(u_{\rho,n}^N - u_\rho^N)'|^2 \, dx + \lambda_\rho^N \int_{\mathcal{G}} |u_{\rho,n}^N - u_\rho^N|^2 \, dx + o(1). \end{aligned}$$

It is easy to see $\int_{\mathcal{G}} |(u_{\rho,n}^N - u_\rho^N)'|^2 \, dx + \lambda_\rho^N \int_{\mathcal{G}} |u_{\rho,n}^N - u_\rho^N|^2 \, dx \to 0$, *as* $n \to +\infty$. □

Then, to obtain the strong convergence of the sequence $\{u_{\rho,n}^N\} \subset H^1(\mathcal{G})$, it suffices to show that $\lambda_\rho^N > 0$. Firstly, we show that $\lambda_\rho^N < 0$ is impossible by using Lemma 2.1. We begin with the following lemma.

Lemma 4.5. *For any $\lambda < 0$ and $d \in \mathbb{N}$, there exists a subspace Y of $H^1(\mathcal{G})$ with $dim(Y) = d$ such that*

$$\begin{aligned} E_\rho''(u_{\rho,n}^N, \mathcal{G})[w,w] + \lambda \|w\|_{L^2(\mathcal{G})}^2 =& a \int_{\mathcal{G}} \left|w'\right|^2 \, dx + b \int_{\mathcal{G}} \left|(u_{\rho,n}^N)'\right|^2 \, dx \int_{\mathcal{G}} \left|w'\right|^2 \, dx \\ & + 2b \left(\int_{\mathcal{G}} (u_{\rho,n}^N)' w' \, dx \right)^2 + \lambda \int_{\mathcal{G}} |w|^2 \, dx \leqslant \frac{\lambda}{2} \|w\|_{H^1(\mathcal{G})}^2, \quad \forall w \in Y. \end{aligned}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ with supp $\phi \subset (0,1)$ and such that $\int_0^{+\infty} |\phi|^2 \, dx = 1$. Viewing ϕ as a function in $H^1(\mathcal{G})$ whose support is contained in a half-line which we identify with $[0, +\infty)$. Using the notation of (3.4), we can define

$$\phi_1 := \phi^\tau,$$

where $\tau > 0$ is taken small enough so that

$$(a + 3b\delta)\tau^2 \|\phi'\|_{L^2(\mathbb{R})}^2 + \lambda \leqslant \frac{\lambda}{2}(\tau^2 \|\phi'\|_{L^2(\mathbb{R})}^2 + 1), \tag{4.14}$$

where $\delta = \int_{\mathcal{G}} |(u_{\rho,n}^N)'|^2 \, dx$. By a direct calculation yields that

$$\|\phi_1\|_{L^2(\mathcal{G})}^2 = 1, \quad \|\phi_1'\|_{L^2(\mathcal{G})}^2 = \tau^2 \|\phi'\|_{L^2(\mathcal{G})}^2$$

Next, for $i \geqslant 2$, define

$$\phi_i(x) := \phi_1\left(x - \frac{i-1}{\tau}\right).$$

Notice that supp $(\phi_i) \subset \left(\frac{i-1}{\tau}, \frac{i}{\tau}\right)$, and all the ϕ_i have disjoint supports. Let $Y \subset H^1(\mathcal{G})$ be the subspace generated by $\phi_1, \cdots, \phi_d$. Any element $w \in Y$ writes as

$$w := \sum_{i=1}^{d} \theta_i \phi_i, \quad i = 1, \cdots, d,$$

where $\theta_i \in \mathbb{R}$. By a direct calculation and Hölder inequality, we have

$$\begin{aligned} & a\int_{\mathcal{G}} \left|w'\right|^2 dx + b\int_{\mathcal{G}} \left|(u_{\rho,n}^N)'\right|^2 dx \int_{\mathcal{G}} \left|w'\right|^2 dx + 2b\left(\int_{\mathcal{G}} (u_{\rho,n}^N)' w'\, dx\right)^2 + \lambda \int_{\mathcal{G}} |w|^2\, dx \\ \leqslant & a\tau^2\left(\sum_{i=1}^{d} \theta_i^2 \|\phi'\|_{L^2(\mathbb{R})}^2\right) + b\delta\tau^2\left(\sum_{i=1}^{d} \theta_i^2 \|\phi'\|_{L^2(\mathbb{R})}^2\right) + 2b\delta\tau^2\left(\sum_{i=1}^{d} \theta_i^2 \|\phi'\|_{L^2(\mathbb{R})}^2\right) + \lambda\left(\sum_{i=1}^{d} \theta_i^2\right) \\ = & \left((a+3b\delta)\tau^2 \|\phi'\|_{L^2(\mathbb{R})}^2 + \lambda\right) \sum_{i=1}^{d} \theta_i^2. \end{aligned}$$

And also, $\|w\|_{H^1(\mathcal{G})}^2 = \left(\tau^2 \|\phi'\|_{L^2(\mathbb{R})}^2 + 1\right) \sum\limits_{i=1}^{d} \theta_i^2$. Thus (4.14) implies that

$$a\int_{\mathcal{G}} \left|w'\right|^2 dx + b\int_{\mathcal{G}} \left|(u_{\rho,n}^N)'\right|^2 dx \int_{\mathcal{G}} \left|w'\right|^2 dx + 2b\left(\int_{\mathcal{G}} (u_{\rho,n}^N)' w'\, dx\right)^2 + \lambda \int_{\mathcal{G}} |w|^2\, dx \leqslant \frac{\lambda}{2} \|w\|_{H^1(\mathcal{G})}^2.$$

Finally, using the fact that w vanishes outside the half-line justifies the equality stated in the claim, which completes the proof. □

Since the codimension of $T_{u_{\rho,n}^N} H_\mu^1(\mathcal{G})$ in $H^1(\mathcal{G})$ is one. Thus, if (4.9) holds for any $w \in W_n \setminus \{0\}$ in a subspace W_n of $H^1(\mathcal{G})$, then dim $(W_n) \leqslant N+1$. Let $\lambda < 0$ and Y be the space of dimension $d = N+2$ provide by Lemma 4.5. Apply Lemma 2.1, we can obtain that

$$\lambda_\rho^N \geqslant 0. \tag{4.15}$$

Thus, from Lemma 4.4 and (4.15), we have

$$\int_{\mathcal{G}} |(u_{\rho,n}^N - u_\rho^N)'|^2\, dx \to 0.$$

Using (4.11) and the fact that nonlinearity acts only on the compact core $\mathcal{K}$, we get that

$$E_\rho(u_{\rho,n}^N, \mathcal{G}) \to E_\rho(u_\rho^N, \mathcal{G}).$$

In particular, from (4.6), it follows that

$$E_\rho(u_\rho^N, \mathcal{G}) = c_\rho^N. \tag{4.16}$$

Then, with the following result, we complete the proof that $\lambda_\rho^N > 0$, thereby establishing the strong convergence of the sequence $\left\{u_{\rho,n}^N\right\} \subset H^1(\mathcal{G})$.

Lemma 4.6. *It cannot be* $\lambda_\rho^N = 0$.

Proof. Assuming that $N \in \mathbb{N}$ is large enough uniformly in $\rho \in [\frac{1}{2}, 1]$ and by contradiction that there exists a subsequence $\{u_{\rho_k}^{N_k}\}_{k=1}^{\infty}$, with $N_k \to +\infty$ and $\rho_k \in [\frac{1}{2}, 1]$ for all k, such that $u_{\rho_k}^{N_k} \in H_\mu^1(\mathcal{G})$ have an associated $\lambda_{\rho_k}^{N_k} = 0$. By Proposition 3.1, $c_{\rho_k}^{N_k} \to +\infty$ uniformly *w.r.t.* $\rho_k \in [\frac{1}{2}, 1]$, as $N_k \to +\infty$. Then, from (4.16), we have $E_{\rho_k}(u_{\rho_k}^{N_k}, \mathcal{G}) \to +\infty$ as $k \to \infty$. It is impossible, as a consequence of Proposition 4.1, we can know that $\|u_{\rho_k}^{N_k}\|_{L^2(\mathcal{G})} \to \infty$, but $\{u_{\rho_k}^{N_k}\}_{k=1}^{\infty} \subset H_\mu^1(\mathcal{G})$, a contradiction. □

In the end, we will show that the Morse index $m(u_\rho^N)$ of u_ρ^N as a solution to (4.5) satisfies $m(u_\rho^N) \leqslant N + 1$. Let us recall the following definition.

Definition 4.1. *For any metric graph* $\mathcal{G}$, *and any solution* $u \in H^1(\mathcal{G})$ *of* (1.10), *we consider*

$$\begin{aligned} Q(w; u, \mathcal{G}) :=& a \int_{\mathcal{G}} |w'|^2 \, dx + b \int_{\mathcal{G}} |u'|^2 \, dx \int_{\mathcal{G}} |w'|^2 \, dx + 2b \left(\int_{\mathcal{G}} u'w' \, dx \right)^2 \\ & - (p-1)\rho \int_{\mathcal{K}} |u|^{p-2} |w|^2 \, dx + \lambda \int_{\mathcal{G}} |w|^2 \, dx, \end{aligned}$$

Then, the Morse index of u *is the maximal dimension of a subspace* $W \subset H^1(\mathcal{G})$ *such that* $Q(w; u, \mathcal{G}) < 0$ *for all* $w \in W \setminus \{0\}$.

Observe that the relationship between the Morse index of a solution to (1.10) and the Morse index as a constrained critical point (see Definition 2.2) via the equality

$$\begin{aligned} D^2 E_\rho(u_\rho^N, \mathcal{G})[w, w] :=& E_\rho''(u_\rho^N, \mathcal{G})[w, w] + \lambda(w, w) \\ =& a \int_{\mathcal{G}} |w'|^2 \, dx + b \int_{\mathcal{G}} |(u_\rho^N)'|^2 \, dx \int_{\mathcal{G}} |w'|^2 \, dx + 2b \left(\int_{\mathcal{G}} (u_\rho^N)'w' \, dx \right)^2 \\ & - (p-1)\rho \int_{\mathcal{K}} |u_\rho^N|^{p-2} |w|^2 \, dx + \lambda_\rho^N \int_{\mathcal{G}} |w|^2 \, dx, \quad \text{for all } w \in H^1(\mathcal{G}). \end{aligned} \tag{4.17}$$

Since $u_{\rho,n}^N \to u_\rho^N$ as $n \to \infty$, from Remark 2.1, we can know that the Morse index of $u_\rho^N \in H_\mu^1(\mathcal{G})$ as a constrained critical point is less than N. In view of (4.17) and the fact that the codimension of $H_\mu^1(\mathcal{G})$ in $H^1(\mathcal{G})$ is one. So we can deduce that

$$m(u_\rho^N) \leqslant N + 1. \tag{4.18}$$

Proof of Theorem 4.1. Integrating the previous content, we immediately have, for any $\mu > 0$ and $N \in \mathbb{N}$ sufficiently large, and for almost every $\rho \in [\frac{1}{2}, 1]$. There exist sequences of Lagrange multipliers $\{\lambda_\rho^N\}_{N=N_0}^{\infty} \subset \mathbb{R}^+$ and solutions $\{u_\rho^N\}_{N=N_0}^{\infty} \subset H_\mu^1(\mathcal{G})$ to (4.5). We also have by (4.16) that $E_\rho(u_\rho^N, \mathcal{G}) = c_\rho^N \to +\infty$ as $N \to +\infty$. Finally, the estimate (4.18) completes the proof. □

5 Proof of Theorem 1.1

Let $\mu > 0$ and $N \in \mathbb{N}$ be sufficiently large. By Theorem 4.1, it is possible to choose a sequence $\rho_n \to 1^-$, and a corresponding sequence of critical points $u_{\rho_n}^N \in H_\mu^1(\mathcal{G})$ of $E_{\rho_n}(u, \mathcal{G})$ constrained to $H_\mu^1(\mathcal{G})$, at the level $c_{\rho_n}^N$ and having a Morse index $m(u_{\rho_n}^N) \leqslant N + 1$. Additionally, the Lagrange multipliers satisfy $\lambda_{\rho_n}^N > 0$.

Thus, in order to prove the Theorem 1.1, it is only to show that $\left\{u_{\rho_n}^N\right\} \subset H_\mu^1(\mathcal{G})$ converges. For this the key point is to show that $\left\{u_{\rho_n}^N\right\} \subset H_\mu^1(\mathcal{G})$ is bounded. Observe the monotonicity of c_ρ^N, as a function of $\rho \in [\frac{1}{2}, 1]$ implies that $\left\{c_{\rho_n}^N\right\}$ is bounded as it belongs to $[c_1^N, c_{\frac{1}{2}}^N]$ with $c_1^N, c_{\frac{1}{2}}^N \in \mathbb{R}$ (see Remark 3.1). In addition, due to the Kirchhoff boundary condition

$$a \int_{\mathcal{G}} \left|(u_{\rho_n}^N)'\right|^2 dx + b\left(\int_{\mathcal{G}} \left|(u_{\rho_n}^N)'\right|^2 dx\right)^2 + \lambda_{\rho_n}^N \int_{\mathcal{G}} \left|u_{\rho_n}^N\right|^2 dx = \rho_n \int_{\mathcal{K}} \left|u_{\rho_n}^N\right|^p dx,$$

it follows that

$$c_{\rho_n}^N = E_{\rho_n}(u_{\rho_n}^N, \mathcal{G}) = \left(\frac{a}{2} - \frac{a}{p}\right) \int_{\mathcal{G}} \left|(u_{\rho_n}^N)'\right|^2 dx + \left(\frac{b}{4} - \frac{b}{p}\right) \left(\int_{\mathcal{G}} \left|(u_{\rho_n}^N)'\right|^2 dx\right)^2 - \frac{\lambda_{\rho_n}^N \mu}{p},$$

equivalent to

$$\left(\frac{a}{2} - \frac{a}{p}\right) \int_{\mathcal{G}} \left|(u_{\rho_n}^N)'\right|^2 dx + \left(\frac{b}{4} - \frac{b}{p}\right) \left(\int_{\mathcal{G}} \left|(u_{\rho_n}^N)'\right|^2 dx\right)^2 = c_{\rho_n}^N + \frac{\lambda_{\rho_n}^N \mu}{p}.$$

Therefore, if $\left\{\lambda_{\rho_n}^N\right\} \subset (0, +\infty)$ is bounded, then $\left\{u_{\rho_n}^N\right\} \subset H_\mu^1(\mathcal{G})$ is bounded as well. So using the following result, we can conclude the proof of Theorem 1.1.

Lemma 5.1. *Let $\mathcal{G}$ be a noncompact metric graph and $p > 10$. Assume that $\{\rho_n\} \subseteq [\frac{1}{2}, 1]$ is a sequence converging to 1. Let $\{(\lambda_{\rho_n}, u_{\rho_n})\} \subseteq \mathbb{R} \times H^1(\mathcal{G})$ be a sequence of solutions with $\rho = \rho_n$ to*

$$\begin{cases} -(a + b\int_{\mathcal{G}} |u'|^2 dx)u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u & \textit{on every edge } e \in \mathcal{E}, \\ \sum\limits_{e \succ v} \frac{du_e}{dx_e}(v) = 0 & \textit{at every vertex } v \in \mathcal{V}. \end{cases}$$

and satisfy additionally, for some $\mu > 0$,

$$\int_{\mathcal{G}} |u_{\rho_n}|^2 dx = \mu, \quad \textit{for all } n \in \mathbb{N},$$

and whose Morse indices $m(u_{\rho_n})$ are bounded. Then, the sequence $\{\lambda_{\rho_n}\} \subset \mathbb{R}$ is bounded from above.

Proof. From Remark 4.1, we know that $\left\{\left(\frac{\lambda_{\rho_n}}{\omega}, f_{\rho_n}\right)\right\} \subseteq \mathbb{R} \times H^1(\mathcal{G})$ is a sequence of solutions with $\rho = \rho_n$ to

$$\begin{cases} -f'' + \frac{\lambda}{\omega} f = \rho\kappa(x)|f|^{p-2}f & \text{on every edge } e \in \mathcal{E}, \\ \sum\limits_{e \succ v} \frac{df_e}{dx_e}(v) = 0 & \text{at every vertex } v \in \mathcal{V}, \end{cases}$$

with mass $\int_{\mathcal{G}} |f_{\rho_n}|^2 dx = \omega^{\frac{2}{2-p}} \mu$. Using the [6, Lemma 6.1], we have that the sequence $\left\{\frac{\lambda_{\rho_n}}{\omega}\right\} \subset \mathbb{R}$ is bounded from above, namely the sequence $\{\lambda_{\rho_n}\} \subset \mathbb{R}$ is bounded. □

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Data Availability:

Data sharing is not applicable to this article as no new data were created analyzed in this study.

References

[1] R. Adami, E. Serra and P. Tilli, NLS ground states on graphs. *Calc. Var. Partial Differential Equations*, 54(1):743–761, 2015.

[2] R. Adami, E. Serra and P. Tilli, Threshold phenomena and existence results for NLS ground states on metric graphs. *J. Funct. Anal.*, 271(1):201–223, 2016.

[3] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, volume 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Province, RI, 2013.

[4] J. Borthwick, X. Chang, L. Jeanjean and N. Soave, Normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities. *Nonlinearity*, 36(7):3776–3795, 2023.

[5] J. Borthwick, X. Chang, L. Jeanjean and N. Soave, Bounded Palais-Smale sequences with Morse type information for some constrained functionals. *Trans. Amer. Math. Soc.*, 377(6):4481–4517, 2024.

[6] P. Carrillo, D. Galant, L. Jeanjean and C. Troestler, Infinitely many normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities. *arXiv:2403.10959*, 2024.

[7] X. Chang, L. Jeanjean and N. Soave, Normalized solutions of L^2-supercritical NLS equations on compact metric graphs. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 41(4):933–959, 2024.

[8] S. Dovetta, Existence of infinitely many stationary solutions of the L^2-subcritical and critical NLSE on compact metric graphs. *J. Differential Equations*, 264(7):4806–4821, 2018.

[9] S. Dovetta and L. Tentarelli, L^2-critical NLS on noncompact metric graphs with localized nonlinearity: topological and metric features. *Calc. Var. Partial Differential Equations*, 58(3):Paper No. 108, 26, 2019.

[10] G. Kirchhoff. Mechanik, Teubner, Leipzig, (1883).

[11] A. Kairzhan, D. Noja and D. E. Pelinovsky, Standing waves on quantum graphs. *J. Phys. A*, 55(24):Paper No. 243001, 51, 2022.

[12] D. Noja, Nonlinear schrödinger equation on graphs: recent results and open problems. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2007):20130002, 2014.

[13] S. Qi, Normalized solutions for the Kirchhoff equation on noncompact metric graphs. *Nonlinearity*, 34(10):6963–7004, 2021.

[14] E. Serra and L. Tentarelli, Bound states of the NLS equation on metric graphs with localized nonlinearities. *J. Differential Equations*, 260(7):5627–5644, 2016.

[15] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities. *J. Differential Equations*, 269(9):6941–6987, 2020.

[16] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case. *J. Funct. Anal.*, 279(6):108610, 43, 2020.