EXTENSION OF HERMITE-HADAMARD INEQUALITY TYPE WITH K-FRACTIONAL COMPANIONS OF RIEMANN-LIOUVILLE INTEGRALS

Abstract. This present study deals with the derivation of some new companions of Riemann-Liouville k-fractional integrals type by means of generalized convex functions, through relative semi-convex, relative h-convex and relative g-convex functions.

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1. Introduction

The study of convexity and its generalizations continues to serve as a foundation tools in the analysis of inequalities and their applications in fractional calculus. In recent years, attention was shifted toward exploring more generalized form of convex functions such as h-convex and s-convex to broaden the scope of classical inequality like Hardy inequality, Opial inequality and Hermite-Hadamard inequality.

A convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, induces the following classical inequality

$$f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_q^q f(x)dx \leq \frac{f(p)+f(q)}{2} \tag{1}$$

This inequality was discovered for the first time by Hermite see, Mitrinovic and Lackovic (1985). However, this result was no where mentioned in the mathematical literature before Mitrinovic and Lackovic (1985) and was not widely known as Hermite's result see Pecaric, Proschan and Tong (1992). Over the years, this inequality has been modified, extended and generalized for different classes of functions as can be found in many research papers and books devoted to the field. For more details see [Latif and Alomari (2009), Mubeen and Habibullah (2012), Iqbal, Bahtti and Muddassar (2011), and Noor, Noor and Awan (2014)].

The classical Hardy inequality was first introduced by G.H. Hardy in 1920 in the context of divergent series and inequalities. The integral form that commonly used today is:

Let $1 < p < \infty$ and let $f \geq 0$ be measurable on $(0, \infty)$. Then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p\, dx. \tag{2}$$

The constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Opial inequality is also a fundamental integral inequality relating a function and its derivative. It plays a significant role in analysis of differential equations and boundary value problems.The Opial inequality states:
let $x(t)$ be absolutely continuous in $[0, a]$ and $x(0) = 0$, and let p be a positive integer. Then, the following inequality hold:

$$\int_0^b |u(x)^p u'(x)|\, dx \leq \frac{b^p}{p+1} \int_0^b |u'(x)|^{p+1}\, dx. \tag{3}$$

The Riemann-Liouville fractional integral has been extensively applied in the development of Hermite-Hadamard type inequalities for convex functions. In particular, fractional versions of the classical Hermite-Hadamard inequality were established using the Riemann-Liouville operator, providing important bounds in fractional analysis Sarikaya *et al.* (2013). Riemann-Liouville Integrals states:

Let $f \in [p, q]$, the Riemann Liouville integrals $J^\gamma_{p+} f$ and $J^\gamma_{q-} f$ of order $\gamma > 0$ with $p \geq 0$, be defined by:

$$J^\gamma_{p+} f(x) = \frac{1}{\Gamma(\gamma)} \int_p^x (x - r)^{\gamma - 1} f(r) dr, \qquad x > p \tag{4}$$

and

$$J^\gamma_{q-} f(x) = \frac{1}{\Gamma(\gamma)} \int_x^q (r - x)^\gamma - 1 f(r) dr, \qquad q > x \tag{5}$$

where $\Gamma(\gamma) = \int_0^\infty r^{\gamma - 1} e^{-r} dr$.

Budak, Ali and Tarhanac (2020) contributed to this field by exploring quantum Hermite-Hadamard-type inequalities for coordinated convex functions, which are functions that are convex in each variable when the other is fixed. Using tools from quantum calculus, the authors derived new integral inequalities in a quantum setting where traditional derivatives are replaced by quantum derivatives. One of their main results established that if $f : [a, b] \times [c, d] \to \mathbb{R}$ is coordinated convex, then the double quantum integral average f is bounded between the function's corner values and midpoints, generalizing the classical two-variable Hermite-Hadamard framework.

In a subsequent work, Budak, Erden and Ali (2021) extended these ideas further by formulating Simpson and Newton-type inequalities using newly defined quantum integrals. This study builds on the premise that convex functions when integrated in a quantum sense yield sharper or more flexible bounds than those obtainable through classical methods.

Ali, Budak and Murtaza (2021) also published a paper on Hermite-Hadamard type inequalities for h-convex functions utilizing generalized fractional integrals. The study contributes to the field by offering a more comprehensive understanding of the interplay between convexity and fractional integration. The authors constructed novel Hermite-Hadamard type inequalities under the assumption that the function in question satisfies a generalized h-convexity condition. This work represents a significant advancement in the study of convex functions and fractional integrals, providing researchers with new tools to explore and apply Hermite-Hadamard type inequalities in more generalized settings.

Definition 1.1 (Ali *et al.* (2021))**.** : Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $K_1, K_2 \in I$, with $K_1 < K_2$. If $F' \in L[K_1, K_2]$, then we have the following identity that holds for generalized fractional integrals:

$$F\left(\tfrac{K_1+K_2}{2}\right) - \tfrac{1}{2(1)}\left[(\tfrac{K_1+K_2}{2})^{+} I_{\phi} F(K_2) + (\tfrac{K_1+K_2}{2})^{-} I_{\phi} F(K_1)\right] =$$

$$\frac{K_2 - K_1}{4(1)} \int_0^1 \Lambda(\psi)\Big[F'\Big(\psi K_2 + (1-\psi)\frac{K_1+K_2}{2} - F'\Big(\psi K_1 + (1-\psi)\frac{K_1+K_2}{2}\Big)\Big]d\psi \tag{6}$$

Definition 1.2 (Mubeen and Habibullah (2012))**.** : Let $f \in [p, q]$, the k-fractional Riemann Liouville integrals ${}_kJ^{\gamma}_{p^+}f$ and ${}_kJ^{\gamma}_{p^-}f$ of order $\gamma > 0$ with $p \geq 0, k > 0$ be defined by:

$$ {}_kJ^{\gamma}_{p^+}f(x) = \frac{1}{k\Gamma k(\gamma)} \int_p^x (x-r)^{\frac{\gamma}{k}-1} f(r)dr, \qquad x > p \tag{7}$$

and

$$ {}_kJ^{\gamma}_{q^-}f(x) = \frac{1}{k\Gamma k(\gamma)} \int_x^q (r-x)^{\frac{\gamma}{k}-1} f(r)dr, \qquad q > x \tag{8}$$

where $\Gamma_k(\gamma)$ is the k-gamma function given by $\Gamma_k(\gamma) = \int_0^{\infty} r^{\gamma-1} e^{\frac{-r^k}{k}} dr$.

Definition 1.3. Let $h : (0,1) \subseteq \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be h-convex or that f is said to belong to the class $SX(h, I)$, if f is non-negative and $\forall\ x, y \in I$ and $\gamma \in (0,1)$, we have

$$f(rx + (1-r)y) \leq h(r)f(x) + h(1-r)f(y) \tag{9}$$

Definition 1.4 (Hussain *et al.* (2016))**.** : Let $f : [p, q] \to \mathbb{R}$ be a positive function with $0 \leq p < q$ and $f \in L_1[p, q]$. If f is a h-convex function on $[p, q]$, then the following inequalities for k-fractional Riemann-Liouville integrals hold:

$$f\Big(\frac{p+q}{2}\Big) \leq h\Big(\frac{1}{2}\Big)\frac{\Gamma_k(\gamma+k)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q) + {}_kJ^{\gamma}_{q^-}f(p)]$$

$$\leq h\Big(\frac{1}{2}\Big)\Gamma_k(\gamma+k)[f(p) + f(q)][{}_kJ^{\gamma}_{1^-}h(0) + {}_kJ^{\gamma}_{0^+}h(1)] \tag{10}$$

Definition 1.5 (Hussain *et al.* (2016))**.** : Let $f : [p, q] \to \mathbb{R}$ be a positive function with $0 \leq p < q$ and $f \in L_1[p, q]$. If f is s-convex function in the second sense on $[p,q]$, then the following inequalities for k-fractional Riemann-Liouville integrals hold:

$$f\left(\frac{p+q}{2}\right) \leq \frac{\Gamma_k(\gamma+k)}{2^s(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) + {}_kJ^{\gamma}_{q-}f(p)] \leq \left(\frac{\frac{\gamma}{k}}{2^s}\right)[f(p)+f(q)]$$

$$\left[\frac{1}{\frac{\gamma}{k}+s} + \frac{s(s-1)(s-2)...(s-n)}{\frac{\gamma}{k}(\frac{\gamma}{k}+1)(\frac{\gamma}{k}+2)...(\frac{\gamma}{k}+n)}\int_0^1 r^{\frac{\gamma}{k}+n}(1-r)^{s-(n+1)}\mathrm{d}r\right] \tag{11}$$

Definition 1.6 (Hussain *et al.* (2016))**.** : Let $f : \triangle \subseteq \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated h-convex on $\triangle := [p,q]\times[m,n]$ in $\mathbb{R}^2$ with $0 \leq p < q$, $0 \leq m < n$ and $f \in L_1(\triangle)$. Then, following inequalities hold:

$$f\Big(\frac{p+q}{2}, \frac{m+n}{2}\Big) \leq \left[h\Big(\frac{1}{2}\Big)\right]^2 \frac{\Gamma_k(\gamma+k)\Gamma_k(\delta+k)}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}$$

$$\left[{}_kJ^{\gamma,\delta}_{p+,m+}f(q,n) + {}_kJ^{\gamma,\delta}_{p+,n-}f(q,m) + {}_kJ^{\gamma,\delta}_{q-,m+}f(p,n) + {}_kJ^{\gamma,\delta}_{q-,n-}f(p,m)\right]$$

$$\leq \left[h\left(\frac{1}{2}\right)\right]^2 \left[{}_kJ^{\gamma}_{1-}h(0) + {}_kJ^{\gamma}_{0+}h(1)\right]\left[{}_kJ^{\delta}_{1-}h(0) + {}_kJ^{\delta}_{0+}h(1)\right]$$

$$\Gamma_k(\gamma+k)\Gamma_k(\delta+k)[f(p,m)+f(p,n)+f(q,m)+f(q,n)] \tag{12}$$

Definition 1.7. A function f is said to be a relative semi-convex function, if and only if there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(rg(x)+(1-r)g(y)) \leq rf(x)+(1-r)f(y) \tag{13}$$

where $x, y \in M$, $r \in [0,1]$

Definition 1.8. A function $f : [p,q] \to \mathbb{R}$ is said to be relative h-convex function with two functions $h : [0,1] \to (0,+\infty)$ and $g : \mathbb{R} \to \mathbb{R}$ such that $[p,q]$ is relative convex set, if;

$$f(rx+(1-r)g(y)) \leq h(r)f(x)+h(1-r)f(g(y)) \tag{14}$$

$\forall x, y \in \mathbb{R} : x, g(y) \in [p,q], r \in [0,1]$.

Definition 1.9. A function f is said to be relative g-convex function on a relative g-convex set I_g if and only if, there exist a function g :$\mathbb{R} \to \mathbb{R}$ such that,

$$f(rg(x)+(1-r)g(y)) \leq rf(g(x))+(1-r)f(g(y)) \tag{15}$$

$\forall\ x, y \in \mathbb{R} :\ g(x), g(y) \in I_g$, $r \in [0,1]$.

2. Results

The following results are the main results.

Theorem 2.1. *Let function $f:\mathbb{R}\to\mathbb{R}$ be relative semi-convex on a relative convex set $M\subseteq\mathbb{R}$, then*

$$f\big(\frac{p+q}{2}\big)\leq\frac{\Gamma k(\gamma+k)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q-}f(p)]\leq f(p)+f(q)\tag{16}$$

Proof. Since f is relative semi-convex, from (13)in definition (1.7) we have

$$f\big(rg(x)+(1-r)g(y)\big)\leq rf(x)+(1-r)f(y)$$

Let $g(x)=rg(p)+(1-r)g(q)$ and
$g(y)=(1-r)g(p)+rg(q)$
then,

$$f\big(\frac{p+q}{2}\big)\leq f\big(rg(p)+(1-r)g(q)\big)+f\big((1-r)g(p)+rg(q)\big)\tag{17}$$

Multiply (17)through by $r^{\frac{\gamma}{k}-1}$ and then integrate with respect to r over $[0,1]$

$$\int_0^1 r^{\frac{\gamma}{k}-1}f\Big(\frac{p+q}{2}\Big)dr\leq\int_0^1 r^{\frac{\gamma}{k}-1}f\big(rg(p)+(1-r)g(q)\big)dr+\int_0^1 r^{\frac{\gamma}{k}-1}f\big((1-r)g(p)+rg(q)\big)dr$$

hence,

$$\frac{k}{\gamma}f\big(\frac{p+q}{2}\big)\leq\frac{k\Gamma k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q-}f(p)]\tag{18}$$

divide (18) through by k then,

$$\frac{1}{\gamma}f\big(\frac{p+q}{2}\big)\leq\frac{\Gamma k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q-}f(p)]\tag{19}$$

By replacing $x=p$ and $y=q$ in (13) yields:

$$f\big(rg(p)+(1-r)g(q)\big)\leq rf(p)+(1-r)f(q)\tag{20}$$

and

$$f\big((1-r)g(p)+rg(q)\big)\leq(1-r)f(p)+rf(q)\tag{21}$$

Adding (20) and (21), we get

$$f\big(rg(p)+(1-r)g(q)\big)+f\big((1-r)g(p)+rg(q)\big)\leq rf(p)+(1-r)f(q)+(1-r)f(p)+rf(q)$$

$$f\big(rg(p)+(1-r)g(q)\big)+f\big((1-r)g(p)+rg(q)\big)\leq[f(p)+f(q)][r+(1-r)]\tag{22}$$

Multiply both sides of (22) by $r^{\frac{\gamma}{k}-1}$ and integrate with respect to r over $[0,1]$, we have

$$\int_0^1 r^{\frac{\gamma}{k}-1}f\big(rg(p)+(1-r)g(q)\big)dr+\int_0^1 r^{\frac{\gamma}{k}-1}f\big((1-r)g(p)+rg(q)\big)dr$$
$$\leq[f(p)+f(q)]\int_0^1 r^{\frac{\gamma}{k}-1}\big(r+(1-r)\big)dr$$

$$\frac{k\Gamma k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)]\leq[f(p)+f(q)]\int_0^1 r^{\frac{\gamma}{k}-1}dr$$

$$\frac{\Gamma k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)]\leq\frac{1}{\gamma}[f(p)+f(q)] \tag{23}$$

using (19) and (23), we have

$$\frac{1}{\gamma}f\big(\frac{p+q}{2}\big)\leq\frac{\Gamma k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)]\leq\frac{1}{\gamma}[f(p)+f(q)] \tag{24}$$

multiply (24) by γ, we obtain the required result in (16).

$$f\big(\frac{p+q}{2}\big)\leq\frac{\Gamma k(\gamma+k)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)]\leq f(p)+f(q)$$

□

Theorem 2.2. *Let $f:[p,q]\to\mathbb{R}$ be a relative h-convex function, such that $h(\frac{1}{2})\neq 0$, then*

$$f\Big(\frac{p+g(q)}{2}\Big)\leq h\Big(\frac{1}{2}\Big)\frac{\Gamma k(\gamma+k)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)]$$

$$\leq h\Big(\frac{1}{2}\Big)\Gamma k(\gamma+k)[f(p)+f(g(q))][{}_kJ^{\gamma}_{1^-}h(0)+_k J^{\gamma}_{0^+}h(1)] \tag{25}$$

Proof. Since f is a relative h-convex function, from (14) in definition (1.8) we have

$$f(rx+(1-r)g(y))\leq h(r)f(x)+h(1-r)f(g(y))$$

Let $x=r_1p+(1-r_1)g(q)$, $g(y)=(1-r_1)p+r_1g(q)$ and $r=\frac{1}{2}$, then

$$f\Big(\frac{p+g(q)}{2}\Big)\leq h\Big(\frac{1}{2}\Big)\Big[f(r_1p+(1-r_1)g(q))+f((1-r_1)p+r_1g(q))\Big] \tag{26}$$

Multiply (26) through by $r_1^{\frac{\gamma}{k}-1}$ and integrate with respect to r_1 over $[0,1]$

$$\int_0^1 r_1^{\frac{\gamma}{k}-1}f\Big(\frac{p+g(q)}{2}\Big)dr_1\leq h\Big(\frac{1}{2}\Big)\Big[\int_0^1 r_1^{\frac{\gamma}{k}-1}f(r_1p+(1-r_1)g(q))dr_1 + \int_0^1 r_1^{\frac{\gamma}{k}-1}f((1-r_1)p+r_1g(q))dr_1\Big]$$

$$\frac{k}{\gamma h\big(\frac{1}{2}\big)}f\Big(\frac{p+g(q)}{2}\Big)\leq\frac{k\Gamma k(\gamma)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)] \tag{27}$$

by dividing (27) through by k then,

$$\frac{1}{\gamma h\big(\frac{1}{2}\big)}f\Big(\frac{p+g(q)}{2}\Big)\leq\frac{\Gamma k(\gamma)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q)+_k J^{\gamma}_{q^-}f(p)] \tag{28}$$

By replacing $x=p$ and $y=q$ in (14) yields:

$$f(rp+(1-r)g(q))\leq h(r)f(p)+h(1-r)f(g(q)) \tag{29}$$

and

$$f((1-r)p+rg(q)) \leq h(1-r)f(p)+h(r)f(g(q)) \tag{30}$$

Adding (29) and (30), we get

$$f(rp+(1-r)g(q))+f((1-r)p+rg(q)) \leq h(r)f(p)+h(1-r)f(g(q)) +h(1-r)f(p)+h(r)f(g(q))$$

$$f(rp+(1-r)g(q))+f((1-r)p+rg(q)) \leq [f(p)+f(g(q)][h(r)+h(1-r)] \tag{31}$$

Multiplying both sides of (31) by $r^{\frac{\gamma}{k}-1}$ and integrate with respect to r over $[0,1]$

$$\int_0^1 r^{\frac{\gamma}{k}-1} f(rp+(1-r)g(q))dr + \int_0^1 r^{\frac{\gamma}{k}-1} f((1-r)p+rg(q))dr \leq [f(p)+f(g(q)] \int_0^1 r^{\frac{\gamma}{k}-1}[h(r)+h(1-r)]dr$$

$$\frac{k\Gamma k(\gamma)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)] \leq [f(p)+f(g(q)] \int_0^1 r^{\frac{\gamma}{k}-1}[h(r)+h(1-r)]dr \tag{32}$$

from (7) and (8) in definition (1.2), we have

$$\int_0^1 r^{\frac{\gamma}{k}-1} h(r)dr = k\Gamma k(\gamma)_k J^{\gamma}_{1-}h(0)$$

and

$$\int_0^1 r^{\frac{\gamma}{k}-1} h(r-1)dr = k\Gamma k(\gamma)_k J^{\gamma}_{0+}h(1)$$

Using the above results in (32), we get

$$\frac{\Gamma k(\gamma)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)] \leq \Gamma k(\gamma)[f(p)+f(g(q)][{}_kJ^{\gamma}_{1-}h(0) +_k J^{\gamma}_{0+}h(1)] \tag{33}$$

from (28) and (33), we have

$$\frac{1}{\gamma h\left(\frac{1}{2}\right)} f\left(\frac{p+g(q)}{2}\right) \leq \frac{\Gamma k(\gamma)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)]$$

$$\leq \Gamma k(\gamma)[f(p)+f(g(q)][{}_kJ^{\gamma}_{1-}h(0) +_k J^{\gamma}_{0+}h(1)] \tag{34}$$

By multiplying (34) by $\gamma h(\frac{1}{2})$, we get the required result in (25).

$$f\left(\frac{p+g(q)}{2}\right) \leq h\left(\frac{1}{2}\right)\frac{\Gamma k(\gamma+k)}{(g(q)-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)]$$

$$\leq h\left(\tfrac{1}{2}\right)\Gamma k(\gamma+k)[f(p)+f(g(q))][{}_kJ^{\gamma}_{1-}h(0) +_k J^{\gamma}_{0+}h(1)]$$

□

Theorem 2.3. *If a function $f : \mathbb{R} \to \mathbb{R}$ is a relative g-convex function on a relative g- convex set $I_g \subseteq \mathbb{R}$, then*

$$f\Big(\frac{g(p)+g(q)}{2}\Big) \le \frac{\Gamma k(\gamma+k)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)] \le f(g(p)) + f(g(q)) \tag{35}$$

Proof. Since f is a relative g-convex function, from (15) in definition (1.9) we have

$$f(rg(x) + (1-r)g(y)) \le rf(g(x)) + (1-r)f(g(y))$$

Let $g(x) = rg(p) + (1-r)g(q)$ and $g(y) = (1-r)g(p) + rg(q)$ then,

$$f\Big(\frac{g(p)+g(q)}{2}\Big) \le \Big[f(rg(p) + (1-r)g(q)) + f((1-r)g(p) + rg(q))\Big] \tag{36}$$

Multiply (36) through by $r^{\frac{\gamma}{k}-1}$ and integrate with respect to r over $[0, 1]$

$$\int_0^1 r^{\frac{\gamma}{k}-1} f\Big(\frac{g(p)+g(q)}{2}\Big)dr \le \Big[\int_0^1 r^{\frac{\gamma}{k}-1} f(rg(p) + (1-r)g(q))dr + \int_0^1 r^{\frac{\gamma}{k}-1} f((1-r)g(p) + rg(q))dr\Big]$$

we obtain,

$$\frac{k}{\gamma} f\Big(\frac{g(p)+g(q)}{2}\Big) \le \frac{k\Gamma k(\gamma)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)] \tag{37}$$

by dividing (37) through by k then,

$$\frac{1}{\gamma} f\Big(\frac{g(p)+g(q)}{2}\Big) \le \frac{\Gamma k(\gamma)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) +_k J^{\gamma}_{q-}f(p)] \tag{38}$$

By replacing $x = p$ and $y = q$ in (15) yields:

$$f(rg(p) + (1-r)g(q)) \le rf(g(p)) + (1-r)f(g(q)) \tag{39}$$

and

$$f((1-r)g(p) + rg(q)) \le (1-r)f(g(p)) + (r)f(g(q)) \tag{40}$$

Adding (39) and (40), we get

$$f(rg(p) + (1-r)g(q)) + f((1-r)g(p) + rg(q)) \le rf(g(p)) + (1-r)f(g(q)) + (1-r)f(g(p)) + rf(g(q))$$

hence,

$$f(rg(p) + (1-r)g(q)) + f((1-r)g(p) + rg(q)) \le [f(g(p)) + f(g(q)][r + (1-r)] \tag{41}$$

Multiplying both sides of (41) by $r^{\frac{\gamma}{k}-1}$ and integrate with respect to r over $[0,1]$, we obtain

$$\int_0^1 r^{\frac{\gamma}{k}-1} f(rg(p)+(1-r)g(q))dr + \int_0^1 r^{\frac{\gamma}{k}-1} f((1-r)g(p)+rg(q))dr \leq [f(g(p))+f(g(q))] \int_0^1 r^{\frac{\gamma}{k}-1}[(r+(1-r)]dr$$

then,

$$\frac{k\Gamma k(\gamma)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q) +_k J^{\gamma}_{q^-}f(p)] \leq [f(g(p))+f(g(q))]\int_0^1 r^{\frac{\gamma}{k}-1}dr$$

and

$$\frac{k\Gamma k(\gamma)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q) +_k J^{\gamma}_{q^-}f(p)] \leq [f(g(p))+f(g(q))]\frac{k}{\gamma} \tag{42}$$

by dividing (42) through by k hence,

$$\frac{\Gamma k(\gamma)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q) +_k J^{\gamma}_{q^-}f(p)] \leq [f(g(p))+f(g(q))]\frac{1}{\gamma} \tag{43}$$

Using (38) and (43), we obtain

$$\frac{1}{\gamma}f\Big(\frac{g(p)+g(q)}{2}\Big) \leq \frac{\Gamma k(\gamma)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q) +_k J^{\gamma}_{q^-}f(p)] \leq [f(g(p))+f(g(q))]\frac{1}{\gamma} \tag{44}$$

By multiplying the (44) by γ, we get the required result in (35).

$$f\Big(\frac{g(p)+g(q)}{2}\Big) \leq \frac{\Gamma k(\gamma+k)}{(g(q)-g(p))^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p^+}f(q) +_k J^{\gamma}_{q^-}f(p)] \leq f(g(p))+f(g(q)).$$

□

Conclusion

This study focuses on developing new results related to Riemann–Liouville k-fractional integrals by exploring the role of generalized convex functions. In particular, the work makes use of concepts such as relative semi-convexity, relative, relative h-convex and relative g-convex functions.

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