A Unified Framework for Generalized Hardy-type Inequalities on Time Scales

Abstract: This paper presents weighted Hardy-type inequalities established within the framework of time scale calculus. By employing Hölder's inequality as an analytical tool, we derive refined inequalities that extend classical Hardy results to a more general and unified setting. The approach relies on suitable generalizations of Opial-type inequalities on time scales, allowing the treatment of continuous and discrete cases under a single structure. Furthermore, the inequalities may serve as useful tools in the qualitative analysis of dynamic equations, particularly in studying boundedness, oscillation and stability properties of solutions on arbitrary time scales.

Keywords: Hardy–type inequalities, Continuous functions, Weighted functions, Hölder′s inequality.

1 INTRODUCTION

Result in [9] states that if $f(x)$ is a non–negative p–integrable function defined on $(0,\infty)$, and $\rho > 1$. Then f integrable over the interval $(0,x)$ for each x and the following inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^\rho dx \le \left(\frac{\rho}{\rho-1}\right)^\rho \int_0^\infty f^\rho(x)dx \tag{1.1}$$

holds, where $\left(\frac{\rho}{\rho-1}\right)^\rho$ is the best possible constant. Inequality (1.1) is called the continuous Hardy inequality.

In [8], the discrete version of (1.1) states that: if $1 < \rho < \infty$, $A_n = \sum_{k=1}^{n} a_k$, $a_n = \{a_k\}$ is a sequence of non–negative real numbers and $C_\rho > 0$, then

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} A_n \right|^{\rho} \le C_\rho \sum_{n=1}^{\infty} |a_n|^{\rho} . \tag{1.2}$$

The summary of Hardy inequalities with weighted functions could be sourced from ([1], [2], [11], [14] and [16]) and the references therein.

(1.1) was extended to the form

$$\int_{\alpha}^{\beta} \left(\left(\int_{\alpha}^{x} f(t)dt \right)^{\gamma} \mu(x)dx \right)^{\frac{1}{\gamma}} \le C \left(\int_{\alpha}^{\beta} f^{\rho}(x)\nu(x)dx \right)^{\frac{1}{\gamma}}, \tag{1.3}$$

with α, β real numbers, satisfying $\alpha < \beta$, and μ, ν are positive and measurable functions in the interval (α, β), ρ, γ are parameters satisfying $0 < \gamma < \infty$ and $1 \le \rho < \infty$.

[3] proved some Hardy–type inequalities of the form

$$\int_{\alpha}^{\beta} \phi(x) \left(\int_{0}^{x} f(t)dt \right)^{\rho} dx \le \int_{\alpha}^{\beta} \psi(x) f^{\rho}(x)dx, \tag{1.4}$$

where ψ and ϕ satisfy the differential equation

$$\frac{d}{dx}(\psi(x)(y^{'})^{\rho-1}) + \phi(x)y^{\rho-1}(x) = 0. \tag{1.5}$$

In time scales analysis, [6 & 7] summarized and organized the time scale calculus.

[14] pioneered the time scales version of Hardy inequality in an attempt to unify and extend the classical Hardy integral inequality and the discrete Hardy inequality by means of the theory of time scales and obtained the following result:

$$\int_{a}^{\infty} \left(\frac{F^{\sigma}(x)}{\sigma(x) - a} \right)^{p} \Delta x \le \left(\frac{p}{p-1} \right)^{p} \int_{a}^{\infty} f^{p}(x)\Delta x, \tag{1.6}$$

where $p > 1$, $F(x) := \int_0^x f(t)\Delta t$ and f is a non–negative function.

Time scale calculus was initiated by [10] in order to create a theory that can unify discrete and continuous analysis. A *time scale* is an arbitrary non–empty closed subset of the real numbers. The three most popular examples of time scale calculus are differential calculus, difference calculus and quantum calculus, that is $\mathbb{T} = \mathbb{R}$, $\quad \mathbb{T} = \mathbb{N}$, $\quad \mathbb{T} = q^{\mathbb{N}_0} =$

$\{q^t : t \in \mathbb{N}_0\}$, where $q > 1$

Delta derivative f^{Δ} for a function f defined on $\mathbb{T}$ as:

(i)$f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$; and

(ii)$f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

We shall assume throughout this article that time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

In this paper our aim is to establish some inequalities with weighted functions of Hardy-type which are proved using Hölder′s inequality on time scales via some generalizations of Opial inequalities.

2 MAIN RESULTS

The main results will be proved by Hölder′s inequality on time scale as expressed in [4] states that

$$\int_{\alpha}^{\beta} |f(x)g(x)|\,\Delta x \leq \left[\int_{\alpha}^{\beta} |f(x)|^{\rho}\,\Delta x\right]^{\frac{1}{\rho}} \left[\int_{\alpha}^{\beta} |g(x)|^{\gamma}\,\Delta t\right]^{\frac{1}{\gamma}}, \tag{2.1}$$

where $\alpha, \beta \in \mathbb{T}$ and $f, g \in C_{rd}(\mathbb{I}, \mathbb{R})$, $\rho > 1$ and $1/\rho + 1/\gamma = 1$. Also, by [15] which states that if f is absolutely continuous on $[\alpha, \beta]$ and $f(\alpha) = f(\beta) = 0$, then

$$\int_{\alpha}^{\beta} |f(x)| \left|f^{'}(x)\right| dx \leq \frac{\beta - \alpha}{4} \int_{\alpha}^{\beta} \left|f^{'}(x)\right|^{2} dx, \tag{2.2}$$

with a best constant 1/4. Opial inequality had already been simplified by [12], it is proved that if f is real and absolutely continuous on (α, β) and with $f(a) = 0$ and $\mathbb{T} = \mathbb{R}$, then

$$\int_{\alpha}^{\beta} |f(x)| \left|f^{'}(x)\right| dx \leq \frac{\beta}{2} \int_{\alpha}^{\beta} \left|f^{'}(x)\right|^{2} dx \tag{2.3}$$

In this paper, all functions are assumed to be positive and measurable and all integrals exist and finite.

Theorem 2.1. *Let $\mathbb{T}$ be time scale with $\alpha, \beta \in \mathbb{T}$. Assume that ψ, ϕ be positive rd–continuous functions on $(\alpha, \beta)_{\mathbb{T}}$. If $f : [\alpha, \beta] \cap \mathbb{T} \to \mathbb{R}^{+}$ with $f(\alpha) = f(\beta) = 0$.* Then

$$\left(\int_{\alpha}^{\beta} \psi(x) \left(\int_{\alpha}^{x} f(t)\Delta t\right) \Delta x\right)^{2} \leq \int_{\alpha}^{\beta} \frac{\Psi^{2}(x, \beta)}{\phi(x)} \Delta x \left(\int_{\alpha}^{\beta} \phi(x)(f(x))^{2}\Delta\right), \tag{2.4}$$

for all functions $f \geq 0$*, where*

$$\Psi(x,\beta) = \int_x^\beta \psi(x)\Delta x. \tag{2.5}$$

Proof: Let $F(x) = \int_\alpha^x f(t)\Delta t$. Since $F(\alpha) = 0$ and $F^{'}(x) = f(x)$, we have

$$\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)\Delta x = \int_x^\beta \psi(x)F(x)\Delta x. \tag{2.6}$$

Integrating by parts the R.H.S., we have

$$\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)\Delta x = -\Psi(x,\beta)F(x)|_\alpha^\beta + \int_\alpha^\beta \Psi(x,\beta)F^{'}(x)\Delta x. \tag{2.7}$$

By the assumptions $\Psi(\beta,\beta) = 0$ and $F(\alpha)$, we have

$$\begin{aligned}\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)\Delta x &= \int_\alpha^\beta \Psi(x,\beta)F^{'}(x)\Delta x.\\ &= \int_\alpha^\beta \frac{\Psi^2(x,\beta)}{\sqrt{\phi(x)}}\sqrt{\phi(x)}F^{'}(x)\Delta x.\end{aligned} \tag{2.8}$$

Applying the inequality (2.1) with $\rho = \gamma = 2$, we have

$$\begin{aligned}\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)\Delta x &\leq \left(\int_\alpha^\beta \frac{\Psi^2(x,\beta)}{\phi(x)}\Delta x\right)^{\frac{1}{2}}\left(\int_\alpha^\beta \phi(x)(F^{'}(x))^2\Delta x\right)^{\frac{1}{2}}\\ &= \left(\int_\alpha^\beta \frac{\Psi^2(x,\beta)}{\phi(x)}\Delta x\right)^{\frac{1}{2}}\left(\int_\alpha^\beta \phi(x)(f(x))^2\Delta x\right)^{\frac{1}{2}}.\end{aligned} \tag{2.9}$$

Hence, the proof is complete.

Theorem 2.2. *Let* $\mathbb{T}$ *be time scale with* $\alpha, \beta \in \mathbb{T}$*. Assume that* ψ *be positive rd–continuous functions on* $(\alpha,\beta)_\mathbb{T}$*. Then*

$$\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)^2\Delta x \leq (\beta-\alpha)\sup_{\alpha<x<\beta}\left(\int_x^\beta \psi(x)\Delta x\right)\int_\alpha^\beta (f(x))^2\Delta x, \tag{2.10}$$

for all functions $f \geq 0$*, where*

$$\Psi(x,\beta) = \int_x^\beta \psi(x)\Delta x. \tag{2.11}$$

Proof: Let $F(x) = \int_\alpha^x f(t)\Delta t$. Since $F(\alpha) = 0$ and $F^{'}(x) = f(x)$, we have

$$\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)^2 \Delta x = \int_\alpha^\beta \psi(x)F^2(x)\Delta x. \tag{2.12}$$

Integrating by parts the R.H.S., we have

$$\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)^2 \Delta x = -\Psi(x,\beta)F^2(x)|_\alpha^\beta + 2\int_\alpha^\beta \Psi(x,\beta)F(x)F^{'}(x)\Delta x. \tag{2.13}$$

By the assumptions $\Psi(\beta,\beta) = 0$ and $F(\alpha)$, we have

$$\begin{aligned}\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)^2 \Delta x =& 2\int_\alpha^\beta \Psi(x,\beta)F(x)F^{'}(x)\Delta x.\\ &\leq 2\sup_{\alpha<x<\beta}\Psi(x,\beta)\int_\alpha^\beta)F(x)F^{'}(x)\Delta x.\end{aligned} \tag{2.14}$$

Applying the inequality (2.3) with $F(\alpha) = 0$, we have

$$\begin{aligned}\int_\alpha^\beta \psi(x)\left(\int_\alpha^x f(t)\Delta t\right)^2 \Delta x &\leq (\beta-\alpha)\sup_{\alpha<x<\beta}\Psi(x,\beta)\int_\alpha^\beta \left(F^{'}(x)\right)^2 \Delta x.\\ &= (\beta-\alpha)\sup_{\alpha<x<\beta}\Psi(x,\beta)\left(\int_\alpha^\beta (f(x))^2\Delta x\right).\end{aligned} \tag{2.15}$$

Hence, the proof is complete.

Remark 2.1. Applying a generalization of inequality (2.3) due to [4] to prove a new inequality of Hardy′s type. The inequality due Beesack states that:

Theorem 2.3. *Let* $\mathbb{T}$ *be time scale with* $\alpha, \beta \in \mathbb{T}$. *If* f *is an absolutely continuous functions on* $[\alpha, \beta]$ *with* $f(\alpha) = f(\beta) = 0$ *then*

$$\int_\alpha^\beta |f(t)|\left|f^{'}(t)\right|\Delta t \leq \frac{1}{2}\int_\alpha^\beta \frac{1}{\psi(t)}\int_\alpha^\beta \psi(t)\left|f^{'}(t)\right|^2 \Delta t, \tag{2.16}$$

where $\psi(t)$ is positive and continuous with $\int_\alpha^\beta \Delta t/r(t) < \infty$.

Proof: Using this inequality on the term

$$2\int_\alpha^\beta \Psi(x,\beta)F(x)F^{'}(x)\Delta x \leq 2\sup_{\alpha<x<\beta}\Psi(x,\beta)\int_\alpha^\beta)F(x)F^{'}(x)\Delta x, \tag{2.17}$$

we obtain

$$\begin{aligned}2\int_{\alpha}^{\beta}\Psi(x,\beta)F(x)F^{'}(x)\Delta x &\leq 2\sup_{\alpha<x<\beta}\Psi(x,\beta)\frac{1}{2}\left(\int_{\alpha}^{\beta}\frac{\Delta x}{\phi(x)}\right)\int_{\alpha}^{\beta}\phi(x)\left(F^{'}(x)\right)^{2}\Delta x.\\ &= \sup_{\alpha<x<\beta}\Psi(x,\beta)\left(\int_{\alpha}^{\beta}\frac{\Delta x}{\phi(x)}\right)\int_{\alpha}^{\beta}\phi(x)\left(f(x)\right)^{2}\Delta x.\end{aligned} \tag{2.18}$$

Remark 2.2. Using this inequality (2.18) and as in the proof of Theorem 2.2 we have

$$\int_{\alpha}^{\beta}\psi(x)\left(\int_{\alpha}^{x}f(t)\Delta t\right)^{2}\Delta x \leq \sup_{\alpha<x<\beta}\Psi(x,\beta)\left(\int_{\alpha}^{\beta}\frac{\Delta x}{\phi(x)}\right)\int_{\alpha}^{\beta}\phi(x)\left(f(x)\right)^{2}\Delta x. \tag{2.19}$$

Hence, the proof is complete.

Theorem 2.4. *Let $\mathbb{T}$ be time scale with $\alpha,\beta\in\mathbb{T}$. Assume that ψ,ϕ be positive rd–continuous functions on $(\alpha,\beta)_{\mathbb{T}}$. If $f:[\alpha,\beta]\cap\mathbb{T}\to\mathbb{R}^{+}$ with $f(\alpha)=f(\beta)=0$.* Then

$$\int_{\alpha}^{\beta}\psi(x)\left(\int_{\alpha}^{x}f(t)\Delta t\right)^{2}\Delta x \leq \sup_{\alpha<x<\beta}\Psi(\alpha,x)\left(\int_{\alpha}^{\beta}\frac{\Delta x}{\phi(x)}\right)\int_{\alpha}^{\beta}\phi(x)\left(f(x)\right)^{2}\Delta x. \tag{2.20}$$

for all functions $f\geq 0$.

If we apply [12] inequality, which states that:

Theorem 2.5. *Let $\mathbb{T}$ be time scale with $\alpha,\beta\in\mathbb{T}$. Let f be absolutely continuous function on $[\alpha,\beta]$ with $f(\alpha)=f(\beta)=0$, then*

$$\int_{\alpha}^{\beta}|f(t)|\left|f^{'}(t)\right|\Delta t \leq \frac{1}{2}\left(\int_{\alpha}^{\beta}\left(\frac{1}{\psi(t)}\right)^{\rho-1}\Delta t\right)^{\frac{2}{\rho}}\left(\int_{\alpha}^{\beta}\psi(t)\left|f^{'}(t)\right|^{\gamma}\Delta t\right)^{\frac{2}{\gamma}}, \tag{2.21}$$

where $\int_{\alpha}^{\beta}(1/\psi(t))^{\rho-1}$, $\rho\geq 1$ and $\frac{1}{\rho}+\frac{1}{\gamma}=1$.

Proof: Using inequality (2.21) on the term $\sup_{\alpha<x<\beta}\Psi(x,\beta)\int_{\alpha}^{\beta}F(x)F^{'}(x)\Delta x$, we have

$$\begin{aligned}&2\int_{\alpha}^{\beta}\Psi(x,\beta)F(x)F^{'}(x)\Delta x \leq 2\sup_{\alpha<x<\beta}\Psi(x,\beta)\int_{\alpha}^{\beta}F(x)F^{'}(x)\Delta x\\ &\leq \sup_{\alpha<x\beta}\Psi(x,\beta)\left(\int_{\alpha}^{\beta}\left(\frac{1}{\phi(x)}\right)^{\rho-1}\Delta x\right)^{\frac{2}{\rho}}\left(\int_{\alpha}^{\beta}\phi(x)\left(F^{'}(x)\right)^{\gamma}\Delta x\right)^{\frac{2}{\gamma}} \qquad (2.22)\\ &= \sup_{\alpha<x<\beta}\Psi(x,\beta)\left(\int_{\alpha}^{\beta}\left(\frac{1}{\phi(x)}\right)^{\rho-1}\Delta x\right)^{\frac{2}{\rho}}\left(\int_{\alpha}^{\beta}\phi(x)(f(x))^{\gamma}\Delta x\right)^{\frac{2}{\gamma}}.\end{aligned}$$

Remark 2.3. Using the inequality (2.22) and proceed as in the proof of Theorem 2.2, we have the following results:

Theorem 2.6. *Let $\mathbb{T}$ be time scale with $\alpha, \beta \in \mathbb{T}$. Let f be absolutely continuous function on $[\alpha, \beta]$ and $\rho > 1$ such that $\frac{1}{\rho} + \frac{1}{\gamma} = 1$. Then*

$$\begin{aligned}\int_{\alpha}^{\beta} \psi(x) \left(\int_{\alpha}^{x} f(t) \Delta t \right)^{2} \Delta x \leq \sup_{\alpha < x < \beta} \Psi(x, \beta) \left(\int_{\alpha}^{\beta} \left(\frac{1}{\phi(x)} \right)^{\rho - 1} \Delta x \right)^{\frac{2}{\rho}} \\ \times \left(\int_{\alpha}^{\beta} \phi(x) (f(x))^{\gamma} \Delta x \right)^{\frac{2}{\gamma}}\end{aligned} \tag{2.23}$$

for all functions $f \geq 0$.

Theorem 2.7. *Let $\mathbb{T}$ be time scale with $\alpha, \beta \in \mathbb{T}$. Let ψ be positive and continuous function on $[\alpha, \beta]$ and $\rho \geq 1$ such that where $\int_{\alpha}^{\beta} (1/\Psi(\alpha, x))^{\rho - 1} < \infty$, $\rho \geq 1$ and $\frac{1}{\rho} + \frac{1}{\gamma} = 1$. Then*

$$\begin{aligned}\int_{\alpha}^{\beta} \psi(x) \left(\int_{\alpha}^{x} f(t) \Delta t \right)^{2} \Delta x \leq \sup_{\alpha < x < \beta} \Psi(\alpha, x) \left(\int_{\alpha}^{\beta} \left(\frac{1}{\phi(x)} \right)^{\rho - 1} \Delta x \right)^{\frac{2}{\rho}} \\ \times \left(\int_{\alpha}^{\beta} \phi(x) (f(x))^{\gamma} \Delta x \right)^{\frac{2}{\gamma}}\end{aligned} \tag{2.24}$$

for all functions $f \geq 0$.

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