LCD code construction from Hankel matrix

Abstract

Shi et al. [Finite Fields their Appl., 75(2021), pp. 101892] and Cheng [Finite Fields their Appl., 95(2024), pp. 102380] constructed LCD double Toeplitz codes from tridiagonal symmetric Toeplitz matrix and skew-symmetric tridiagonal Toeplitz matrices by using the factorization of Dickson polynomials. We extend their results to the anti-tridiagonal Hankel matrices with 0 on the sub-diagonal.

KeyWords: LCD codes, Toeplitz matrices, Hankel matrices

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1 Introduction

We say that C is a linear complementary dual (LCD) code if its dual code $C^{\perp}$ satisfies $C \cap C^{\perp} = \{0\}$. It was introduced by Massey[4] in 1992 and was proved to be asymptotically good by Sendrier[5], who used it in relation with the equivalence testing of linear codes. There were not people who had paid attention to the research on LCD codes between 2000 and 2015. Until 2016, Carlet and Guilley[2] found the important role of LCD codes in Side Channel Attack (SCA) and Fault Injection Attack (FIA), which had led to widespread research on LCD codes. Cyclic LCD codes are very important for studying LCD codes, and double-cycle LCD codes are an important class of cyclic codes. An important generalization of them is the double Toeplitz (DT) code. In general, a code is double Toeplitz (DT) if its generator matrix is of the form (I,T) with I as the identity matrix and T as a Toeplitz matrix of the same order. A $n \times n$ matrix $T = [t_{ij}]$ is a Toeplitz matrix if it satisfies $t_{ij} = t_{j-i}, i, j = 1, 2, \cdots, n$. What conditions need to be met for a DT code to be LCD code? Extend the issue, What conditions need to be met for a code from a Toeplitz matrix to be an LCD code?

Based on this idea, Shi et al.[6] constructed LCD double Toeplitz codes from tridiagonal symmetric Toeplitz matrix in 2021. This was the first paper using the factorization of Dickson polynomials for the construction of LCD codes. In 2023, Li and Shi[7] constructed LCD codes from tridiagonal Toeplitz matrices of index t, which were selfdual (FSD) codes when t is 2. And they also constructed many ELCD codes and HLCD codes. In 2024, Li and Shi et al.[8] introduced Toeplitz codes of index t which is the promotion of quasi-cyclic codes of index t. Their generator matrices were $G = (I, A_1, \cdots, A_{t-1})$ with A_i, $1 \leq i \leq t-1$ were Toeplitz matrices of the same order, and they satisfied the asymptotic Gilbert-Varshamov boundary. In addition, they also constructed many Toeplitz LCD codes and optimal LCD codes. In the same year, Cheng[3] constructed LCD double Toeplitz codes from skew-symmetric tridiagonal Toeplitz matrices by using the factorization of Dickson polynomials, and constructed LCD codes with arbitrary minimum distance by using concatenation.

Inspired by the above work, It's easy to wonder if there are other classes of DT codes being LCD. This paper investigates the remaining classes of DT codes whose generating matrices are of the form $(I, T_{2n+1}(b, 0, c))$, I is a $(2n+1) \times (2n+1)$ identity matrix, $T_{2n+1}(b, 0, c)$ is a tridiagonal Toeplitz matrix

of the same order with 0 on the main diagonal, which can be denoted by

$$T_{2n+1}(b,0,c)=\begin{pmatrix} 0 & c & \cdots & 0 & 0 & 0 \\ b & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & c \\ 0 & 0 & \cdots & 0 & b & 0 \end{pmatrix}, n \geq 1.$$

We often call a matrices is Hankel matrix if the elements on each subdiagonal are equal. In fact, Hankel matrices can be regarded as the upside down form of Toeplitz matrices. So in this paper, we also study the linear code that generates matrices of the form $(I, H_{2n+1}(b,0,c))$, H_{2n+1} is the $(2n+1)\times(2n+1)$ anti-tridiagonal Hankel matrix with 0 on the subdiagonal, which can be denoted by

$$H_{2n+1}(b,0,c)=\begin{pmatrix} 0 & 0 & \cdots & 0 & b & 0 \\ 0 & 0 & \cdots & b & 0 & c \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & \cdots & 0 & 0 & 0 \end{pmatrix}, n \geq 1.$$

Since it is too complicated to construct the LCD code from $T_{2n+1}(b,0,c)$, this paper first constructs the LCD code from $H_{2n+1}(b,0,c)$. And the key tools used in this paper are the factorization of Dickson polynomials and the symmetric tridiagonal 2-Toeplitz matrix with 0 on the main diagonal. We usually say that a matrix is a 2-Toeplitz matrix if it satisfies $t_{i+2,j+2}=t_{i,j}, i,j=1,2,\cdots,n-r$.

When p = 3, we can also construct the LCD codes with the generation matrix of the form $(I, T_{2n+1}(b,1,-b))$ by this method. $T_{2n+1}(b,1,-b)$ is a $(2n+1)\times(2n+1)$ Toeplitz matrices with 1 on the main diagonal, which can be denoted by

$$T_{2n+1}(b,1,-b)=\begin{pmatrix} 1 & -b & \cdots & 0 & 0 & 0 \\ b & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & -b \\ 0 & 0 & \cdots & 0 & b & 1 \end{pmatrix}, n \geq 1.$$

Given some sufficiently necessary conditions for a DT code to be an LCD code, we can construct LCD codes over a finite field. Since there is a limit on the minimum distance of a DT code constructed in this way, an isometric mapping is applied. In addition, a construction method is introduced to make the final result optimal.

This paper is organized as follows. In section 2, we obtain a mathematical formula to describe the relation between a symmetric tridiagonal 2-Toeplitz matrix with 0 on the main diagonal and an anti-tridiagonal Hankel matrix with 0 on the subdiagonal and study the factorization of characteristic polynomial of the 2-Toeplitz matrix. These results will be used in Section 3 to construct the LCD code from Hankel matrix.

2 The relation between $T_{2n+1}^{(2)'}$ and $H_{2n+1}(b,0,c)$

We can let $T_{2n+1}^{(2)'}(b,0,c)$ be a $(2n+1)\times(2n+1)$ symmetric tridiagonal 2-Toeplitz matrix with 0 on the main diagonal, which can be denoted by

$$T_{2n+1}^{(2)'}(b,0,c)=\begin{pmatrix} 0 & b & 0 & \cdots & \cdots & \cdots \\ b & 0 & c & \cdots & \cdots & \cdots \\ 0 & c & 0 & b & \cdots & \cdots \\ \cdots & \cdots & b & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, n\geq 1.$$

Usually we abbreviate $T_{2n+1}^{(2)'}(b,0,c)$ as $T_{2n+1}^{(2)'}$.
And we can set the permutation

$$\sigma=\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n+1 & \cdots & 2n-2 & 2n-1 & 2n & 2n+1 \\ 1 & 2n & 3 & 2n-2 & \cdots & n+1 & \cdots & 4 & 2n-1 & 2 & 2n+1 \end{pmatrix}.$$

It is clear to know that the permutation matrix P_σ is an involution and $P_\sigma H_{2n+1}(b,0,c)P_\sigma = T_{2n+1}^{(2)'}$ by recalling [1].

This shows that $H_{2n+1}(b,0,c)$ and $T_{2n+1}^{(2)'}$ can be interconverted to each other by P_σ. Using the fact that P_σ is an involution, we have $P_\sigma H_{2n+1}(b,0,c)P_\sigma = T_{2n+1}^{(2)'}$ and $P_\sigma T_{2n+1}^{(2)'}P_\sigma = H_{2n+1}(b,0,c)$.

We know that the study of the eigenvalues of the Hankel matrix facilitates the construction of LCD codes generated from the Hankel matrix, so we should study the eigenvalues of the Hankel matrix. But it is too complicated to study the eigenvalues of the Hankel matrix, we can take an intermediate variable to transform the problem. Based on the previous relation, if the eigenvalues of $T_{2n+1}^{(2)'}$ are known, then the eigenvalues of $H_{2n+1}(b,0,c)$ can be determined. Therefore, we can study the eigenvalues of $T_{2n+1}^{(2)'}$.

3 LCD code construction from Hankel matrix

3.1 LCD code construction when $\mathrm{char}(\mathbb{F}_q)$ is even

In the previous section we obtained the factorization of the characteristic polynomials of $T_{2n+1}^{(2)'}$ and used a mathematical formula to express the relation between $T_{2n+1}^{(2)'}$ and $H_{2n+1}(b,0,c)$, so it is natural to obtain the factorization of characteristic polynomials of $H_{2n+1}(b,0,c)$. In this subsection the LCD code will be constructed by factorization of the characteristic polynomial of $H_{2n+1}(b,0,c)$.

For simplicity, we first define the LCD code from $H_{2n+1}(b,0,c)$ as $C_{2n+1}(b,0,c)$.

Definition 1. *Let n be a positive integer, define $C_{2n+1}(b,0,c)$ to be a $\mathbb{F}_q$-linear code of length 2(2n+1) and dimension 2n+1 with a generator matrix $G=[I_{2n+1},H_{2n+1}(b,0,c)], n\geq 1$.*

Next, we can get that $C_{2n+1}(b,0,c)$ is a sufficiently necessary condition for LCD code from the previous two relations.

Theorem 1. *For $b,c\in\mathbb{F}_q$, let $b\neq 0$, $n\in\mathbb{N}^+$. Assume that $\mathrm{char}\mathbb{F}_q$ is even and $\gcd(p,n+1)=1$. Then $C_{2n+1}(b,0,c)$ is LCD if and only if $b\notin\{\frac{1-c^2}{b}-c(\theta^i+\theta^{-i}):1\leq i\leq\frac{n}{2}\}$, where θ is a primitive $(n+1)$-th root of 1.*

Proof. It is well known that $C_{2n+1}(b,0,c)$ is LCD if and only if GG^T is invertible[**?**]. Note that

$$(I_{2n+1}, H_{2n+1}(b,0,c)) \begin{pmatrix} I_{2n+1} \\ H_{2n+1}(b,0,c)^T \end{pmatrix} = I_{2n+1} + H_{2n+1}(b,0,c)H_{2n+1}(b,0,c)^T$$

is invertible, I_{2n+1} is $(2n+1)\times(2n+1)$ identity matrix.

H_n is symmetric from $H_{2n+1}(b,0,c) = H_{2n+1}(b,0,c)^T$, so $I_{2n+1} + H_{2n+1}(b,0,c)H_{2n+1}(b,0,c)^T$ is invertible if and only if $I_{2n+1} + H_{2n+1}(b,0,c)^2$ is invertible.

We can recall the result due to lemma 2.1 in [6]: for $n \geq 1$ let A be an $n \times n$ matrix over $\mathbb{F}_q$. If $\mathrm{char}(\mathbb{F}_q)$ is even, then -1 is an eigenvalue of A^2 if and only if -1 is an eigenvalue of A.

Thus -1 is not an eigenvalue of $H_{2n+1}(b,0,c)^2$ if and only if -1 is not an eigenvalue of $H_{2n+1}(b,0,c)$, we can get $\det(I_{2n+1} + H_{2n+1}(b,0,c)) \neq 0$.

Because P_σ is an involution, we can have

$$\begin{aligned} \det(I_{2n+1} + H_{2n+1}(b,0,c)) &= \det(P_\sigma P_\sigma + P_\sigma P_\sigma H_{2n+1}(b,0,c) P_\sigma P_\sigma) \\ &= \det(I_{2n+1} + T^{(2)'}_{2n+1}) \\ &= b^{2n} \prod_{i=1}^{\frac{n}{2}} (v - (1+d^2) - d(\theta^i + \theta^{-i})) \neq 0, \end{aligned}$$

and $v = \frac{\lambda^2}{b^2} = \frac{1}{b^2}$, $d = \frac{c}{b}$. θ is a primitive $(n+1)$-th root of 1. Thus $C_{2n+1}(b,0,c)$ is LCD codes if and only if $b \notin \{\frac{1-c^2}{b} - c(\theta^i + \theta^{-i}) : 1 \leq i \leq \frac{n}{2}\}, \theta$ is a primitive $(n+1)$-th root of 1. □

We have extension of the results for $gcd(p, n+1) \neq 1$:

Theorem 2. *For $b, c \in \mathbb{F}_q$, let $b \neq 0$, $n \in \mathbb{N}^+$. Assume* $\mathrm{char}\mathbb{F}_q$ *is even, $r \in \mathbb{N}^+$, $m \in \mathbb{N}^+$, $p^r || (n+1)$, $n+1 = p^r(m+1)$. Then $C_{2n+1}(b,0,c)$ is LCD if and only if $b \notin \{\sqrt{1-c^2}\} \cup \{\frac{1-c^2}{b} - c(\theta^i + \theta^{-i}) : 1 \leq i \leq \frac{m}{2}\}$, where θ is a primitive $(m+1)$-th root of 1.*

Proof. $C_{2n+1}(b,0,c)$ is LCD $\Leftrightarrow -1$ is not an eigenvalue of $H_{2n+1}(b,0,c) \Leftrightarrow \det(I_{2n+1} + H_{2n+1}(b,0,c)) = \det(I_{2n+1} + T^{(2)'}_{2n+1}) = b^{2n}[\prod_{i=1}^{\frac{m}{2}}(v - 1 - d^2 - d(\theta^i + \theta^{-i}))^2]^{2^r}(v - (1+d^2))^{2^r - 1} \neq 0$, $v = \frac{1}{b^2}$, $d = \frac{c}{b}$, θ is a primitive $(n+1)$-th root of 1.

Thus $C_{2n+1}(b,0,c)$ is LCD if and only if $b \notin \{\sqrt{1-c^2}\} \cup \{\frac{1-c^2}{b} - c(\theta^i + \theta^{-i}) : 1 \leq i \leq \frac{m}{2}\}$, where θ is a primitive $(m+1)$-th root of 1. □

3.2 LCD code construction when $\mathrm{char}(\mathbb{F}_q)$ is odd

We have discussed the LCD code construction when $\mathrm{char}(\mathbb{F}_q)$ is even, now let's discuss the LCD code construction when $\mathrm{char}(\mathbb{F}_q)$ is odd.

Theorem 3. *For $b, c \in \mathbb{F}_q$, let $b \neq 0$, $n \in \mathbb{N}^+$. Assume* $\mathrm{char}\mathbb{F}_q$ *is odd and* $\gcd(p, n+1) = 1$. *Then $C_{2n+1}(b,0,c)$ is LCD if and only if $b \notin \{\frac{-1-c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq n\}$, where ξ is a primitive $2(n+1)$-th root of 1.*

Proof. $C_{2n+1}(b,0,c)$ is LCD if and only if GG^T is invertible, Note that

$$(I_{2n+1}, H_{2n+1}(b,0,c)) \begin{pmatrix} I_{2n+1} \\ H_{2n+1}(b,0,c)^T \end{pmatrix} = I_{2n+1} + H_{2n+1}(b,0,c)H_{2n+1}(b,0,c)^T$$

is invertible, I_{2n+1} is $(2n+1)\times(2n+1)$ identity matrix.

$H_{2n+1}(b,0,c)$ is symmetric from $H_{2n+1}(b,0,c) = H_{2n+1}(b,0,c)^T$, so $I_{2n+1}+H_{2n+1}(b,0,c)H_{2n+1}(b,0,c)^T$ is invertible if and only if $I_{2n+1} + H_{2n+1}(b,0,c)^2$ is invertible.

We can recall the result due to lemma 2.1 in [6]: for $n \geq 1$ let A be an $n \times n$ matrix over $\mathbb{F}_q$. If $\mathrm{char}(\mathbb{F}_q)$ is odd, then -1 is an eigenvalue of A^2 if and only if $-\mu$ or μ is an eigenvalue of A, $\mu \in \mathbb{F}_{q^2}, \mu^2 = -1$.

Thus -1 is not an eigenvalue of $H_{2n+1}(b,0,c)^2$ if and only if $-\mu$ or μ is not an eigenvalue of $H_{2n+1}(b,0,c)$, we can get $\det(-\mu I_{2n+1} + H_{2n+1}(b,0,c)) \neq 0$ or $\det(\mu I_{2n+1} + H_{2n+1}(b,0,c)) \neq 0$, $\mu \in \mathbb{F}_{q^2}, \mu^2 = -1$. Now let's discuss them in categories:

(1) When $\det(-\mu I_{2n+1} + H_{2n+1}(b,0,c)) \neq 0$.

Because P_σ is an involution, we can have

$$\det(-\mu I_{2n+1} + H_{2n+1}(b,0,c)) = \det(-\mu P_\sigma P_\sigma + P_\sigma P_\sigma H_{2n+1}(b,0,c) P_\sigma P_\sigma)$$

$$= \det(-\mu I_{2n+1} + T^{(2)'}_{2n+1}) = -\mu b^{2n} \prod_{i=1}^{n} (v - (1+d^2) - d(\xi^i + \xi^{-i})) \neq 0.$$

Where $v = \frac{\lambda^2}{b^2} = \frac{\mu^2}{b^2}$, $d = \frac{c}{b}$, ξ is a primitive $2(n+1)$-th root of 1.

Thus $C_{2n+1}(b,0,c)$ is LCD codes if and only if $b \notin \{\frac{\mu^2 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq n\}$, where ξ is a primitive $2(n+1)$-th root of 1.

(2) When $\det(\mu I_{2n+1} + H_{2n+1}(b,0,c)) \neq 0$.

Because P_σ is an involution, we can have

$$\det(\mu I_{2n+1} + H_{2n+1}(b,0,c)) = \det(\mu P_\sigma P_\sigma + P_\sigma P_\sigma H_{2n+1}(b,0,c) P_\sigma P_\sigma)$$

$$= \det(\mu I_{2n+1} + T^{(2)'}_{2n+1}) = \mu b^{2n} \prod_{i=1}^{n} (v - (1+d^2) - d(\xi^i + \xi^{-i})) \neq 0.$$

Where $v = \frac{\lambda^2}{b^2} = \frac{\mu^2}{b^2}, d = \frac{c}{b}$, and ξ is a primitive $2(n+1)$-th root of 1.

Thus $C_{2n+1}(b,0,c)$ is LCD codes if and only if

$$b \notin \{\frac{\mu^2 - c^2}{b} - c(\xi^i + \xi^{-i})\},$$

where ξ is a primitive $2(n+1)$-th root of 1.

In conclusion, $C_{2n+1}(b,0,c)$ is LCD codes if and only if

$$b \notin \{\frac{\mu^2 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq n\},$$

where ξ is a primitive $2(n+1)$-th root of 1.

$C_{2n+1}(b,0,c)$ is LCD codes if and only if

$$b \notin \{\frac{-1 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq n\},$$

where ξ is a primitive $2(n+1)$-th root of 1 from $\mu \in \mathbb{F}_{q^2}$, $\mu^2 = -1$, □

We have extension of the results for $gcd(p, n+1) \neq 1$:

Theorem 4. *For $b, c \in \mathbb{F}_q$, let $b \neq 0$, $n \in \mathbb{N}^+$. Assume* $\mathrm{char}\mathbb{F}_q$ *is odd, $r \in \mathbb{N}^+$, $m \in \mathbb{N}^+$, $p^r || (n+1)$, $n+1 = p^r(m+1)$. Then $C_{2n+1}(b,0,c)$ is LCD if and only if $b \notin \{\frac{-1-c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq m\} \cup \{\frac{-1-c^2}{b} - 2c\} \cup \{\frac{-1-c^2}{b} + 2c\}$, where ξ is a primitive $2(m+1)$-th root of 1.*

Proof. $C_{2n+1}(b,0,c)$ is LCD $\Leftrightarrow -\mu$ or μ is not an eigenvalue of $H_{2n+1}(b,0,c)$, $\mu \in \mathbb{F}_{q^2}, \mu^2 = -1$.

(1) When μ is not an eigenvalue of $H_{2n+1}(b,0,c)$:

$$\det(-\mu I_{2n+1} + H_{2n+1}(b,0,c)) = \det(-\mu I_{2n+1} + T^{(2)'}_{2n+1})$$
$$= -\mu b^{2n} \prod_{i=1}^{m} [v - 1 - d^2 - d(\xi^i + \xi^{-i})]^{p^r} [(v-1-d^2) - 2d]^{\frac{p^r-1}{2}} [(v-1-d^2) + 2d]^{\frac{p^r-1}{2}} \neq 0$$

where $v = \frac{\mu^2}{b^2}$, $d = \frac{c}{b}$, and ξ is a primitive $2(m+1)$-th root of 1.

So $C_{2n+1}(b,0,c)$ is LCD if and only if

$$b \notin \{\frac{\mu^2 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq m\} \cup \{\frac{\mu^2 - c^2}{b} - 2c\} \cup \{\frac{\mu^2 - c^2}{b} + 2c\}.$$

(2) When $-\mu$ is not an eigenvalue of $H_{2n+1}(b,0,c)$:

$$\det(\mu I_{2n+1} + H_{2n+1}(b,0,c)) = \det(\mu I_{2n+1} + T^{(2)'}_{2n+1})$$
$$= \mu b^{2n} \prod_{i=1}^{m} [v - 1 - d^2 - d(\xi^i + \xi^{-i})]^{p^r} [(v-1-d^2) - 2d]^{\frac{p^r-1}{2}} [(v-1-d^2) + 2d]^{\frac{p^r-1}{2}},$$

where $v = \frac{\mu^2}{b^2}$, $d = \frac{c}{b}$, and ξ is a primitive $2(m+1)$-th root of 1.

So $C_{2n+1}(b,0,c)$ is LCD if and only if

$$b \notin \{\frac{\mu^2 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq m\} \cup \{\frac{\mu^2 - c^2}{b} - 2c\} \cup \{\frac{\mu^2 - c^2}{b} + 2c\}.$$

Thus $C_{2n+1}(b,0,c)$ is LCD if and only if

$$b \notin \{\frac{\mu^2 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq m\} \cup \{\frac{\mu^2 - c^2}{b} - 2c\} \cup \{\frac{\mu^2 - c^2}{b} + 2c\},$$

where $\mu \in \mathbb{F}_{q^2}, \mu^2 = -1$.

In conclusion, $C_{2n+1}(b,0,c)$ is LCD if and only if

$$b \notin \{\frac{-1 - c^2}{b} - c(\xi^i + \xi^{-i}) : 1 \leq i \leq m\} \cup \{\frac{-1 - c^2}{b} - 2c\} \cup \{\frac{-1 - c^2}{b} + 2c\}.$$

□

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Competing Interests

Author has declared that no competing interests exist.

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