A note on the theory of topological spaces and topological variety

Abstract

In this paper, we study some concepts of algebraic topology. The basic idea is to transform a topological problem into an algebraic problem. The method consists of describing the topological structure of a space, associating with it an algebraic object. At each topological space X we associate a ring or an R-module and to the continuous functions we associate a homomorphism of rings or an R-homomorphism, where R is a commutative ring with identity not null.

Keywords: topological space; topological variety; algebraic topology; homeomorphism.

1 Introduction

This paper provide us a result of the algebraic topology, and with this result we provide an appropriate definition.

There are some central problems in topology, such as determining when two topological spaces are homeomorphic or not. If we want to show that two topological spaces are not homeomorphic we must verify that there is no continuous bijection $f : (X, \tau) \to (Y, \tau^{'})$ with continuous inverse, which is not always simple. Remembering that a homeomorphism preserves the topological structure of the spaces involved, that is, if (X, τ) is homeomorph to $(Y, \tau^{'})$, any property of X that can be expressed in terms of the topology τ over X is transmitted automatically to $(Y, \tau^{'})$ via the existing homeomorphism. These properties are called topological invariants. In this way, the most efficient method to show that two spaces are not homeomorphic is to find an invariant property by homeomorphism that is valid for one of the spaces but is not valid for the other.

For example, the unit disk

$$\mathrm{D}^2 = \left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y) \leq 1\|\right\},$$

with the induced subspace topology of $(\mathbb{R}^2, \tau_d)$, where τ_d is the metric topology usual.

Then, $(\mathrm{D}^2, \tau_{\mathrm{D}^2})$ is compact however $(\mathbb{R}^2, \tau_d)$ not is compact. In this way, these spaces are not homeomorphic. For example, $(\mathbb{R}, \tau_d)$ and $(\mathbb{R}^2, \tau_d)$ not are homeomorphic: in fact, if there exists $f : (\mathbb{R}, \tau_d) \to (\mathbb{R}^2, \tau_d)$ a homeomorphism then the your restriction $f_{|\mathbb{R}\setminus\{0\}} : \mathbb{R} \setminus \{0\} \to \mathbb{R}^2 \setminus \{f(0)\}$ also is a homeomorphism; however, since $\mathbb{R}^2 \setminus \{f(0)\}$ is connected (since is connected by paths) we should have $\mathbb{R} \setminus \{0\}$ connected, which does not occur.

Note that in these previous examples, compactness and connectivity were sufficient to distinguish two given topological spaces. However, in general, these invariants are not enough. So it is often necessary to introduce more sophisticated topological invariants.

2 Preliminary definitions

The purpose here is to put some preliminary definitions that we will use in the article.

Definition 2.1. We say that a topological space X is locally Euclidean of dimension n, if for every point $x \in X$ there exists an open $U \subset X$ containing the point x such that U is homeomorphic to an open subset of the $\mathbb{R}^n$. A open U in this conditions is called an Euclidean open neighborhood of the point x in question.

Lemma 2.2 ([1, Lemma 2.3])**.** *A topological space X is locally Euclidian of dimension n if and only if X satisfies one of the following properties:*

(1) For every point $x \in X$ there exists $U \subset X$ an open with $x \in U$ such that U is homeomorphic to an open ball in $\mathbb{R}^n$. In particular we can to consider open balls of ray one in $\mathbb{R}^n$.

(2) For every point $x \in X$ there exists $U \subset X$ with $x \in U$ such that U is homeomorphic to $\mathbb{R}^n$.

Remark 2.3. Let X be a topological space locally Euclidian of dimension n, and let $f : X \longrightarrow Y$ be a function over (with Y topological space) such that f is a local homeomorphism (hence is an open function). In this conditions we have that Y is locally Euclidian of dimension n.

Definition 2.4. A topological variety of dimension n is a space of Hausdorff X enumerable second, which is a space locally Euclidian of dimension n.

Given a topological variety of dimension n X, for each $x \in X$ the open set $U \subset X$ containing the point x with U homeomorph to an unitary open ball B^n of the $\mathbb{R}^n$ is called coordinated neighborhood of the point x and the homeomorphism $\phi : U \longrightarrow B^n$ is called a letter about U, for $n \geq 1$.

Example 2.5. (1) $\mathbb{R}^n$ is an n-variety (see [1, Lemma 2.16]).

(2) S^n is an n-variety, since that $S^n \setminus \{N\}$ is homeomorph to $\mathbb{R}^n$ in the stereographic projection.

Remark 2.6. We have that if M and N are topological varieties of dimension m and n, respectively, then $M \times N$ is an $(m+n)$-topological variety. More generally, this result is true for a finite product of topological varieties:

$$M_1 \times M_2 \times \ldots \times M_k.$$

Remark 2.7. Suppose that $p : E \rightarrow B$ is a recoating application. Thus, if E is an n-variety and if B is a topological space Hausdorff then B is an n-topological variety.

Since all n-variety X is a space locally Euclidean of dimension n it follows that X has the same local properties of the $\mathbb{R}^n$.

Theorem 2.8 (Theorem of the Excision)**.** *Let X be a topological space. Consider that $U \subset A \subset X$ and is such that $\bar{U} \subset \mathrm{int}(A)$. Then, for all $p \geq 0$, we have that*

$$\mathrm{H}_p(X, A, R) \cong \mathrm{H}_p(X \setminus U, A \setminus U, R),$$

where the isomorphism is given by the function i_*, with

$$i : (X \setminus U, A \setminus U) \hookrightarrow (X, A),$$

the inclusion and R a commutative ring and with identity not null.

Remark 2.9. Note that we have the exact sequence of the pair (X, A) of the following form:

$$\dots \mathrm{H}_p(A, R) \xrightarrow{i_*} \mathrm{H}_p(X, R) \xrightarrow{j_*} \mathrm{H}_p(X, A, R) \xrightarrow{\partial_p} \mathrm{H}_{p-1}(A, R) \xrightarrow{i_*} \mathrm{H}_{p-1}(X, R) \to \dots,$$

where $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ are inclusions.

Theorem 2.10 ([1])**.** *If X is a topological space connected by paths and if $A \subset X$ is a subset not empty, then we have that:*

$$\mathrm{H}_0(X, A, R) \cong \{0\}.$$

Let's now recall some definitions and notations that will be used.

We remember that $\left\{\mathrm{S}_p(X, A, R), \bar{\partial}_p\right\}$ is a complex of chains, where we have that $\mathrm{S}_p(X, A, R) := \frac{\mathrm{S}_p(X,R)}{\mathrm{S}_p(A,R)}$ is the free module over R whose base is formed by elements:

$$\sigma + \mathrm{S}_p(A, R),$$

with $\sigma : \Delta_p \to X$ an p-singular simplex of X whose image is not completely contained in A. Moreover, we have that

$$\bar{\partial}_p : \mathrm{S}_p(X, A, R) \to \mathrm{S}_{p-1}(X, A, R,)$$

is such that $c_p + \mathrm{S}_p(A, R) \mapsto \partial_p\left(c_p + \mathrm{S}_p(A, R)\right) = \partial_p(c_p) + \mathrm{S}_{p-1}(A, R)$, with $c_p \in \mathrm{S}_p(X, R)$, and thus,

$$c_p := \sum_{i=1}^{r} \alpha_i \cdot \sigma_i,$$

with $\alpha_i \in R$. Moreover, we have that

$$\mathrm{Ker}(\bar{\partial}_p) := \left\{c + \mathrm{S}_p(A, R) \mid \bar{\partial}_p(c + \mathrm{S}_p(A, R)) = \partial_p(c) + \mathrm{S}_{p-1}(A, R) = \mathrm{S}_{p-1}(A, R)\right\},$$

and therefore it follows that:

$$\mathrm{Ker}(\bar{\partial}_p) = \{c + \mathrm{S}_p(A, R) \mid c \in \mathrm{S}_p(X, R) \text{ and } \partial_p(c) \in \mathrm{S}_{p-1}(A, R)\}.$$

Remark 2.11. (1.) Note that:

$$\mathrm{H}_p(\mathrm{S}_*(X, A, R)) = \mathrm{H}_p(X, A, R) = \frac{\mathrm{Ker}(\bar{\partial}_p)}{\mathrm{Im}(\bar{\partial}_{p+1})} = \frac{\mathrm{Z}_p(X, A, R)}{B_p(X, A, R)}.$$

(2.) We have that $c + \mathrm{S}_p(A, R) \in \mathrm{S}_p(X, A, R)$ represents a relative homology class in $\mathrm{H}_p(X, A, R) \Leftrightarrow \partial_p(c) \in \mathrm{S}_{p-1}(A, R)$.

3 The result

From now on, X will denote an n-topological variety and R is a commutative ring with identity not null 1_R. When we do not specify otherwise, the homologies will have coefficients in R.

Theorem 3.1. *Let X be an n-variety. For all $x \in X$, we have that*

$$\mathrm{H}_n\left(X, X \setminus \{x\}, R\right) \cong R,$$

for all $n \geq 1$.

Proof. Given $x \in X$ and given U a coordinated neighborhood of X, we consider:

(a) $V = X \setminus U$, which is a set closed in X and

(b) the closure of V, $\bar{V} = V := X \setminus U \subset X \setminus \{x\} = \mathrm{int}(X \setminus \{x\})$ (the last equality it is because $X \setminus \{x\}$ is a set open).

Thus, it follows of the Theorem of the Excision that we have:

$$\mathrm{H}_n(X, X \setminus \{x\}, R) \cong \mathrm{H}_n(X \setminus V = U, (X \setminus \{x\}) \setminus V = U \setminus \{x\}, R),$$

and so it follows that

$$\mathrm{H}_n(X, X \setminus \{x\}, R) \cong \mathrm{H}_n(U, U \setminus \{x\}, R)).$$

We consider, now, the exact sequence of the pair: $(U, U \setminus \{x\})$ in two cases.

Case 1: We consider here that $n > 1$ (in this case, $n - 1 \geq 1$)

$$\rightarrow \ldots \mathrm{H}_n(U \setminus \{x\}, R) \rightarrow \mathrm{H}_n(U, R) \rightarrow \mathrm{H}_n(U, U \setminus \{x\}, R) \overset{\cong}{\rightarrow} \mathrm{H}_{n-1}(U \setminus \{x\}, R) \rightarrow \mathrm{H}_{n-1}(U, R) \rightarrow \ldots,$$

where we have $\mathrm{H}_n(U, R) \cong \{0\}$ and $\mathrm{H}_{n-1}(U, R) \cong \{0)$ (since U is contractile and $n > 1$, we have that $\mathrm{H}_n(U, R) \cong \{0\}$ and $\mathrm{H}_{n-1}(U, R) \cong \{0\}$).

Moreover, we have that $U \setminus \{x\} \cong S^{n-1}$. Then, in the exact sequence of the pair $(U, U \setminus \{x\})$ we have an isomorphism:

$$\mathrm{H}_n(U, U \setminus \{x\}, R) \cong \mathrm{H}_{n-1}(U \setminus \{x\}, R) \cong \mathrm{H}_{n-1}(S^{n-1}, R) \cong R.$$

Case 2: We consider here that $n = 1$.

In this case, we have that:

$$U \setminus \{x\} \cong S^0 = \{-1, 1\},$$

and hence, $\mathrm{H}_0(U \setminus \{x\}, R) \cong \mathrm{H}_0(S^0, R) \cong R \oplus R$.

Thus, in the exact sequence we have:

$$\mathrm{H}_1(U, R) \cong 0 \to \mathrm{H}_1(U, U \setminus \{x\}, R) \to \mathrm{H}_0(U \setminus \{x\}, R) \cong R \oplus R \to R \cong \mathrm{H}_0(U, R) \to 0 \cong \mathrm{H}_0(U, U \setminus \{x\}, R).$$

Since R is an R-free module we have that the sequence cinde and

$$\mathrm{H}_1(U, U \setminus \{x\}, R) \oplus R \cong R \oplus R \Longrightarrow \mathrm{H}_1(U, U \setminus \{x\}, R) \cong R.$$

Of the cases 1 and 2, it follows the proof of the result. □

Remark 3.2 (Geometric idea of the Theorem 3.1)**.** Note that for $n = 2$ and $R = \mathbb{Z}$ in the Theorem 3.1, we have that

$$\mathrm{H}_2(X, X \setminus \{x\}, \mathbb{Z}) \cong \mathrm{H}_1(U \setminus \{x\}, R) \cong \mathrm{H}_1(S^1, \mathbb{Z}) \cong \mathbb{Z}.$$

In this case, we have two possibilities of generators for $\mathrm{H}_2(X, X \setminus \{x\}, \mathbb{Z}) \cong \mathbb{Z}$: a generator corresponding to the loop α in the clockwise, or the other generator $-\alpha$ in the sense contrary.

Intuitively, the Theorem 3.1 tells us that, in this case, we can think in to define an $\mathbb{Z}$-local orientation in each point x of the 2-variety X as being the choice of a generator of the $\mathrm{H}_2(X, X \setminus \{x\}, \mathbb{Z})$.

Thus, from of the Theorem 3.1, we can consider the following definition.

Definition 3.3 (Local orientation)**.** An R-local orientation of an n-variety X in a point $x \in X$ is a choice of a generator of the R-module

$$\mathrm{H}_n(X, X \setminus \{x\}, R) \cong R,$$

for $n \geq 1$ where the isomorphism it follows of the Theorem 3.1. A such generator, usually, is denoted by α_x.

We end this note with the following remark.

Remark 3.4. An n-variety X has in each point $x \in X$ so many R-local orientations $\alpha_x \in \mathrm{H}_n(X, X \setminus \{x\}, R)$ how many are the units of the ring R.

4 Acknowledgment

The author thanks to ICMC (Instituto de Ciências Matemáticas e de Computação) by the support of conditions adequate of working that were offered during this work.

References

[1] J.M. Lee, *Introduction to topological manifolds*, Springer, Second edition, 2010.

[2] Stichtenoth, H., *Algebraic Function Fields and Codes*, Berlin Heidelberg New York: Springer-Verlag, 1993.