

A Two-Step Levenberg-Marquardt type Method for Solving Weighted Linear Complementarity Problems

Abstract: The weighted linear complementarity problem (WLCP) is important for modeling a variety of economic equilibrium problems. However, most existing numerical methods for WLCP either require the problem to be monotone or rely on nonsingularity conditions to ensure fast convergence. To overcome these limitations, this paper proposes a two-step Levenberg-Marquardt type method (TS-LMM) for solving WLCPs. By incorporating a smooth weighted complementarity function, the WLCP is reformulated as a system of smooth nonlinear equations, which is then solved using the proposed algorithm. The method is proved to be globally convergent without the need for monotonicity. Furthermore, under a local error bound condition, the proposed method achieves a cubic convergence rate, which is faster than the quadratic convergence of existing Levenberg-Marquardt methods. Numerical results demonstrate that the proposed method outperforms existing Levenberg-Marquardt methods in both iteration count and CPU time.

Keywords: Weighted linear complementarity problems; Levenberg-Marquardt method; Cubic convergence

1 Introduction

The weighted linear complementarity problem (WLCP) was introduced by Potra [14] which is to find vectors $x \in \mathbb{R}^n, s \in \mathbb{R}^n, y \in \mathbb{R}^m$ such that

$$\text{(WLCP)} \quad x \geq 0, \quad s \geq 0, \quad Px + Qs + Ry = a, \quad xs = w. \quad (1.1)$$

Here $P \in \mathbb{R}^{(n+m) \times n}, Q \in \mathbb{R}^{(n+m) \times n}, R \in \mathbb{R}^{(n+m) \times m}$ are given matrices, $a \in \mathbb{R}^{n+m}$ is a given vector, $w \geq 0$ is a given weight vector (the data of the problem) and xs is the componentwise product of the vectors x and s . The significance of studying the WLCP lies in the fact that a lot of equilibrium problems in economics can be formulated in a natural way as WLCP [14]. Moreover, those formulations lend themselves to the development of highly efficient algorithms for solving the corresponding equilibrium problems. For example, Fisher market equilibrium problem, which can be written as a nonlinear CP, can also be formulated as a WLCP that can be efficiently solved by interior-point methods [14].

Since Potra introduced the notion of WLCP, various numerical algorithms have been proposed to solve this problem (e.g., [1, 5, 6, 7, 10, 13, 16, 17, 18, 19, 20, 21, 22]). Among them, the smoothing Newton-type method is one class of the most effective algorithms (e.g.,

[16, 17, 20, 21]). The main idea of this class of methods is to use a smoothing function to reformulate the WLCP concerned as a system of smooth nonlinear equations and then solve it by Newton method. Note that in smoothing Newton-type methods, to ensure Newton step be feasible, one usually requires that the WLCP is monotone. Moreover, to obtain the local quadratic convergence, these smoothing Newton-type method need the nonsingularity condition. Lately, by using a smooth weighted complementarity function, Tang and Zhou [19] reformulated the weighted nonlinear complementarity problem as a smooth nonlinear equation and proposed a Levenberg-Marquardt type method to solve it. Their algorithm is globally convergent without any additional condition, and it has local quadratic convergence under the local error bound condition, which is weaker than the nonsingularity condition used in smoothing Newton-type methods.

Recently, Fan [8, 9] studied a modified Levenberg-Marquardt method for nonlinear equations. In each iteration, this method computes not only the standard LM step but also an approximate LM step, which utilizes the function value at the point after the first step. The motivation behind this two-step strategy is to achieve a higher convergence order without significantly increasing computational cost. Under the local error bound condition, Fan proved that the modified LM method possesses cubic convergence, which is faster than the quadratic convergence of the classical LM method. Inspired by this idea, we propose a two-step Levenberg-Marquardt type method (denoted by TS-LMM) for solving the WLCP. By incorporating an additional approximate LM step, our method can accelerate the local convergence while maintaining the global convergence properties. We prove that TS-LMM is globally convergent without requiring monotonicity of the WLCP. Moreover, under the local error bound condition, we establish that TS-LMM achieves cubic convergence, which is superior to the quadratic convergence of the standard LM method studied in [19]. Numerical experiments confirm the efficiency of the proposed method.

This paper is organized as follows. In Sect. 2, we give a weighted complementarity function and use it to reformulate the WLCP as a smooth nonlinear equation. In Sect. 3, we present the two steps Levenberg-Marquardt type method and give its global convergence. In Sect. 4, we analyze the local cubic convergence of the method. Numerical results are reported in Sect. 5. Some conclusions are given in Sect. 6.

2 A weighted complementarity function

To equivalently reformulate the WLCP as a system of the nonlinear equations, we consider the following weighted complementarity function introduced in [19]:

$$\phi_\tau^c(a, b) := (a + b)^3 - \left(\sqrt{a^2 + b^2 + (\tau - 2)ab + (4 - \tau)c}\right)^3, \quad \forall (a, b) \in \mathbb{R}^2, \quad (2.1)$$

where $c \geq 0$ is a constant and $\tau \in [0, 4)$. The following lemma gives some nice properties of ϕ_τ^c whose proof can be found in [19].

Lemma 2.1 *Let ϕ_τ^c be defined by (2.1). Then the following results hold.*

1 (a) ϕ_τ^c satisfies

$$\phi_\tau^c(a, b) = 0 \iff a \geq 0, b \geq 0, ab = c.$$

2 (b) ϕ_τ^c is continuously differentiable at any $(a, b) \in \mathbb{R}^2$ with

$$\nabla \phi_\tau^c(a, b) = \begin{bmatrix} \frac{\partial \phi_\tau^c}{\partial a} \\ \frac{\partial \phi_\tau^c}{\partial b} \end{bmatrix},$$

3 where

$$\begin{aligned} \frac{\partial \phi_\tau^c}{\partial a} &= 3[(a+b)^2 - h_\tau^c(a, b)(a + (\tau/2 - 1)b)], \\ \frac{\partial \phi_\tau^c}{\partial b} &= 3[(a+b)^2 - h_\tau^c(a, b)(b + (\tau/2 - 1)a)], \end{aligned}$$

4 in which

$$h_\tau^c(a, b) := \sqrt{a^2 + b^2 + (\tau - 2)ab + (4 - \tau)c}.$$

5 Let $z := (x, s, y)$. By using ϕ_τ^c , we define the function $F(z) : \mathbb{R}^{2n+m} \rightarrow \mathbb{R}^{2n+m}$ as

$$F(z) := \begin{pmatrix} Px + Qs + Ry - a \\ \phi_\tau^{w_1}(x_1, s_1) \\ \vdots \\ \phi_\tau^{w_n}(x_n, s_n) \end{pmatrix}, \quad (2.2)$$

6 where $w = (w_1, \dots, w_n)^T$ is the weight vector given in the WLCP. Then, due to Lemma 2.1,
7 the function $F(z)$ is continuously differentiable at any $z \in \mathbb{R}^{2n+m}$ and solving the WLCP is
8 equivalent to computing a solution of the nonlinear equations

$$F(z) = 0. \quad (2.3)$$

9 By Lemma 2.1 (b) and (2.2), the Jacobian of $F(z)$ is given as

$$J(z) = \begin{bmatrix} P & Q & R \\ \text{diag}(\frac{\partial \phi_\tau^{w_i}}{\partial x_i}) & \text{diag}(\frac{\partial \phi_\tau^{w_i}}{\partial s_i}) & 0 \end{bmatrix}, \quad (2.4)$$

10 where

$$\begin{aligned} \frac{\partial \phi_\tau^{w_i}}{\partial x_i} &= 3[(x_i + s_i)^2 - h_\tau^{w_i}(x_i, s_i)(x_i + (\tau/2 - 1)s_i)], \\ \frac{\partial \phi_\tau^{w_i}}{\partial s_i} &= 3[(x_i + s_i)^2 - h_\tau^{w_i}(x_i, s_i)(s_i + (\tau/2 - 1)x_i)], \end{aligned}$$

11 in which

$$h_\tau^{w_i}(x_i, s_i) := \sqrt{x_i^2 + s_i^2 + (\tau - 2)x_i s_i + (4 - \tau)w_i}.$$

12 The following theorem shows that the Jacobian $J(z)$ satisfies the Lipschitz continuity.

13 **Theorem 2.1** *Let $\tau \in (0, 4)$. Then the Jacobian $J(z)$ given in (2.4) is Lipschitz continuous on
1 any closed and convex set $N(z) := \{z \in \mathbb{R}^{2n+m} \mid \|z\| \leq \varrho\}$ where $\varrho > 0$ a constant.*

2 *Proof* Obviously, we only need to prove that the gradient $\nabla\phi_\tau^c(a, b)$ given in Lemma 2.1 (b) is
 3 Lipschitz continuous on any closed and convex set $\Omega := \{(a, b) \in \mathbb{R}^2 \mid \|(a, b)\| \leq \theta\}$ for any $\theta > 0$.
 4 We consider the following three cases.

5 **Case 1.** Assume that $c > 0$. Since $h_\tau^c(a, b)$ can be rewritten as

$$\begin{aligned} h_\tau^c(a, b) &= \sqrt{(a + (\tau/2 - 1)b)^2 + \tau(1 - \tau/4)b^2 + (4 - \tau)c} \\ &= \sqrt{(b + (\tau/2 - 1)a)^2 + \tau(1 - \tau/4)a^2 + (4 - \tau)c}, \end{aligned}$$

6 we have from $\tau \in (0, 4)$ that $h_\tau^c(a, b) > 0$ for any $(a, b) \in \Omega$. Thus, ϕ_τ^c is twice continuously
 7 differentiable at any $(a, b) \in \Omega$ with

$$\nabla^2\phi_\tau^c(a, b) = \begin{bmatrix} \frac{\partial^2\phi_\tau^c}{\partial a^2} & \frac{\partial^2\phi_\tau^c}{\partial a\partial b} \\ \frac{\partial^2\phi_\tau^c}{\partial b\partial a} & \frac{\partial^2\phi_\tau^c}{\partial b^2} \end{bmatrix},$$

8 where

$$\begin{aligned} \frac{\partial^2\phi_\tau^c}{\partial a^2} &= 3 \left\{ 2(a + b) - \left(\frac{(a + (\tau/2 - 1)b)^2}{h_\tau^c(a, b)} + h_\tau^c(a, b) \right) \right\}, \\ \frac{\partial^2\phi_\tau^c}{\partial b^2} &= 3 \left\{ 2(a + b) - \left(\frac{(b + (\tau/2 - 1)a)^2}{h_\tau^c(a, b)} + h_\tau^c(a, b) \right) \right\}, \\ \frac{\partial^2\phi_\tau^c}{\partial a\partial b} &= \frac{\partial^2\phi_\tau^c}{\partial b\partial a} \\ &= 3 \left\{ 2(a + b) - \left(\frac{(a + (\tau/2 - 1)b)(b + (\tau/2 - 1)a)}{h_\tau^c(a, b)} + (\tau/2 - 1)h_\tau^c(a, b) \right) \right\}. \end{aligned}$$

9 Since $h_\tau^c(a, b)$ is continuous on Ω , there exists a constant $\xi > 0$ such that $h_\tau^c(a, b) \leq \xi$ for any
 10 $(a, b) \in \Omega$. It follows that

$$\left| \frac{(a + (\tau/2 - 1)b)^2}{h_\tau^c(a, b)} \right| \leq h_\tau^c(a, b) \leq \xi,$$

$$\left| \frac{(b + (\tau/2 - 1)a)^2}{h_\tau^c(a, b)} \right| \leq h_\tau^c(a, b) \leq \xi,$$

$$\left| \frac{(a + (\tau/2 - 1)b)(b + (\tau/2 - 1)a)}{h_\tau^c(a, b)} \right| \leq h_\tau^c(a, b) \leq \xi.$$

13 Thus there exists a constant $C > 0$ independent of $(a, b) \in \Omega$ such that

$$\|\nabla^2\phi_\tau^c(a, b)\| \leq C, \quad \forall (a, b) \in \Omega.$$

14 By Mean Value Theorem, we have that

$$\|\nabla\phi_\tau^c(a_1, b_1) - \nabla\phi_\tau^c(a_2, b_2)\| \leq C\|(a_1, b_1) - (a_2, b_2)\|$$

15 holds for any $(a_1, b_1), (a_2, b_2) \in \Omega$ and prove the desired result.

1 **Case 2.** Assume that $c = 0$ and $(0, 0) \notin \Omega$. Since $\tau \in (0, 4)$ and

$$h_\tau^0(a, b) = \sqrt{(a + (\tau/2 - 1)b)^2 + \tau(1 - \tau/4)b^2}$$

$$= \sqrt{(b + (\tau/2 - 1)a)^2 + \tau(1 - \tau/4)a^2},$$

2 we have $h_\tau^0(a, b) > 0$ for any $(a, b) \in \Omega$. Thus ϕ_τ^0 is twice continuously differentiable at any
3 $(a, b) \in \Omega$. By following exactly the same steps as in Case 1, we can prove the desired result.

4 **Case 3.** Assume that $c = 0$ and $(0, 0) \in \Omega$. Similarly as Case 1, we can prove that there
5 exists a constant $\bar{C} > 0$ independent of (a, b) such that

$$\|\nabla^2 \phi_\tau^0(a, b)\| \leq \bar{C}, \quad \forall (a, b) \neq (0, 0) \in \Omega.$$

6 By [4, Lemma 2.6], we have

$$\|\nabla \phi_\tau^0(a_1, b_1) - \nabla \phi_\tau^0(a_2, b_2)\| \leq \bar{C} \|(a_1, b_1) - (a_2, b_2)\| \quad (2.5)$$

7 holds for all $(a_1, b_1), (a_2, b_2) \in \Omega$ with $(0, 0) \notin [(a_1, b_1), (a_2, b_2)]$. Moreover, since $\nabla \phi_\tau^0(0, 0) =$
8 $(0, 0)$, the inequality (2.5) also holds in case $(a_1, b_1) = (a_2, b_2) = (0, 0)$. Therefore, we can
9 assume $(a_1, b_1) \neq (0, 0) \in \Omega$. Since ϕ_τ^0 is continuously differentiable for all $(a, b) \in \mathbb{R}^2$ with
10 $\nabla \phi_\tau^0(0, 0) = (0, 0)$, by using a continuity argument, we obtain that the inequality (2.5) remains
11 true for all $(a_2, b_2) \in \Omega$. Thus, the inequality (2.5) holds for all $(a_1, b_1), (a_2, b_2) \in \Omega$ which proves
12 the desired result. \square

13 3 A two steps Levenberg-Marquardt type method

14 Our two steps Levenberg-Marquardt type method (TS-LMM) is described as follows.

15

16 **Algorithm TS-LMM:** Choose constants $\mu > 0, \theta, \rho, \sigma \in (0, 1)$ and $z_0 \in \mathbb{R}^n$. Set $k := 0$.

17 **Step 1:** If $\|J(z_k)^T F(z_k)\| = 0$, then stop.

18 **Step 2:** Choose $\delta \in [1, 2]$ and set

$$\lambda_k := \mu \|F(z_k)\|^\delta. \quad (3.1)$$

19 Compute $\bar{d}_k \in \mathbb{R}^{2n+m}$ by solving

$$[J(z_k)^T J(z_k) + \lambda_k I] \bar{d}_k = -J(z_k)^T F(z_k). \quad (3.2)$$

20 Set $t_k := z_k + \bar{d}_k$. Then, compute $\hat{d}_k \in \mathbb{R}^{2n+m}$ by solving

$$[J(z_k)^T J(z_k) + \lambda_k I] \hat{d}_k = -J(z_k)^T F(t_k). \quad (3.3)$$

21 **Step 3:** If

$$\|F(z_k + \bar{d}_k + \hat{d}_k)\| \leq \theta \|F(z_k)\|, \quad (3.4)$$

22 then set $z_{k+1} := z_k + \bar{d}_k + \hat{d}_k$. Otherwise, set $\alpha_k := \rho^{l_k}$, where l_k is the smallest nonnegative
23 integer l satisfying

$$\|F(z_k + \rho^l \bar{d}_k)\|^2 \leq \|F(z_k)\|^2 + \sigma \rho^l F(z_k)^T J(z_k) \bar{d}_k. \quad (3.5)$$

24 Set $z_{k+1} := z_k + \alpha_k \bar{d}_k$.

1 **Step 4:** Set $k := k + 1$ and go to Step 1.

2 **Remark 3.1** We denote the merit function

$$\psi(z) := \frac{1}{2} \|F(z)\|^2. \quad (3.6)$$

3 Then $\psi(z) = 0$ if and only if $z = (x, s, y)$ is a solution of the WLCP. Moreover, $\psi(z)$ is continu-
4 ously differentiable at any $z \in \mathbb{R}^{2n+m}$ with its gradient

$$\nabla\psi(z) = J(z)^T F(z).$$

5 Note that $\bar{d}_k + \hat{d}_k$ is not a descent direction of the merit function ψ at z_k . However, the LM
6 step \bar{d}_k is a descent direction. In fact, if $J(z_k)^T F(z_k) \neq 0$ for some k , then $\bar{d}_k \neq 0$ by (3.2), and
7 $F(z_k) \neq 0$ which yields $\lambda_k = \mu \|F(z_k)\|^\delta > 0$. Hence the matrix $J(z_k)^T J(z_k) + \lambda_k I$ is positive
8 definite which gives

$$\begin{aligned} \nabla\psi(z_k)^T \bar{d}_k &= F(z_k)^T J(z_k) \bar{d}_k \\ &= -(\bar{d}_k)^T [J(z_k)^T J(z_k) + \lambda_k I] \bar{d}_k \\ &< 0. \end{aligned} \quad (3.7)$$

9 Thus in Step 3 we only use the LM step \bar{d}_k so that the Armijo line search procedure can be
10 carried out.

11 In the following, we assume $J(z_k)^T F(z_k) \neq 0$ for all $k \geq 0$ so that Algorithm TS-LMM
12 generates an infinite sequence $\{z_k\}$. We have the following global convergence property.

13 **Theorem 3.1** *Let $\{z_k\}$ be the iteration sequence generated by Algorithm TS-LMM. Then any
14 accumulation point z^* of $\{z_k\}$ satisfies $\nabla\psi(z^*) = 0$.*

15 *Proof* We assume that z^* is the limit of the subsequence $\{z_k\}_{k \in K} \subset \{z_k\}$ where $K \subset \{0, 1, \dots\}$.
16 So, $\lim_{(K \ni) k \rightarrow \infty} z_k = z^*$. By the continuity,

$$\lim_{(K \ni) k \rightarrow \infty} F(z_k) = F(z^*), \quad \lim_{(K \ni) k \rightarrow \infty} J(z_k) = J(z^*),$$

17 and consequently,

$$\lim_{(K \ni) k \rightarrow \infty} \lambda_k = \mu \|F(z^*)\|^\delta := \lambda^*.$$

18 Now we assume $\nabla\psi(z^*) \neq 0$ and will derive a contradiction. For all $k \in K$, since $J(z_k)^T F(z_k) \neq$
19 0 , $F(z_k) \neq 0$ and hence $\lambda_k = \mu \|F(z_k)\|^\delta > 0$. So, the matrix $J(z_k)^T J(z_k) + \lambda_k I$ is positive
20 definite for any $k \in K$. Moreover,

$$\lim_{(K \ni) k \rightarrow \infty} [J(z_k)^T J(z_k) + \lambda_k I] = J(z^*)^T J(z^*) + \lambda^* I. \quad (3.8)$$

21 Since $\nabla\psi(z^*) = J(z^*)^T F(z^*) \neq 0$, $F(z^*) \neq 0$ and so $\lambda^* > 0$. Thus the matrix $J(z^*)^T J(z^*) + \lambda^* I$
1 is also positive definite. By (3.8) we have

$$\lim_{(K \ni) k \rightarrow \infty} [J(z_k)^T J(z_k) + \lambda_k I]^{-1} = [J(z^*)^T J(z^*) + \lambda^* I]^{-1},$$

2 which together with (3.2) yields

$$\lim_{(K\ni)k\rightarrow\infty} \bar{d}_k = -[J(x^*)^T J(x^*) + \lambda^* I]^{-1} J(z^*)^T F(z^*) := d^*. \quad (3.9)$$

3 By (3.4), (3.5) and (3.7), we have that $\|F(z_{k+1})\| < \|F(z_k)\|$ for all $k \geq 0$. This indicates that
 4 there exists a constant $F^* \geq 0$ such that $\lim_{k\rightarrow\infty} \|F(z_k)\| = F^*$. Obviously, $F^* = F(z^*)$. Thus,
 5 if there are infinitely many k for which the condition (3.4) holds, then $\|F(z_{k+1})\| \leq \theta \|F(z_k)\|$
 6 holds for infinitely many k . This gives $F^* \leq \theta F^*$ and so $F^* = 0$ because $\theta \in (0, 1)$. It follows
 7 that $J(z^*)^T F(z^*) = 0$ which contradicts our assumption. Thus there exists an index $\bar{k} > 0$
 8 such that α_k is determined by the Armijo line search (3.5) for all $k \geq \bar{k}$. Now we show that
 9 $F(z^*)^T J(z^*) d^* = 0$. We divide the proof into the following two parts.

10 **Part 1.** $\alpha_k \geq c > 0$ for all $k \in K$ and $k \geq \bar{k}$ where c is a fixed constant. In this case, it
 11 follows from (3.5) and (3.7) that for all $k \in K$ and $k \geq \bar{k}$.

$$0 \leq -\sigma c F(z_k)^T J(z_k) \bar{d}_k \leq -\sigma \alpha_k F(z_k)^T J(z_k) \bar{d}_k \leq \|F(z_k)\|^2 - \|F(z_{k+1})\|^2. \quad (3.10)$$

12 Since $\lim_{k\rightarrow\infty} \|F(z_k)\| = F^*$, by letting $k \rightarrow \infty$ with $k \in K$ in (3.10), we have $F(z^*)^T J(z^*) d^* = 0$.

13 **Part 2.** $\{\alpha_k\}_{k \in K}$ has a subsequence converging to zero. We may pass to the subsequence
 14 and assume that $\lim_{(K\ni)k\rightarrow\infty} \alpha_k = 0$. Let $\hat{\alpha}_k := \rho^{-1} \alpha_k$. Then $\lim_{(\bar{K}\ni)k\rightarrow\infty} \hat{\alpha}_k = 0$. Moreover, by (3.5),
 15 for any sufficiently large $k \in K$,

$$\|F(z_k + \hat{\alpha}_k \bar{d}_k)\|^2 - \|F(z_k)\|^2 > \sigma \hat{\alpha}_k F(z_k)^T J(z_k) \bar{d}_k,$$

16 which gives

$$\frac{\psi(z_k + \hat{\alpha}_k \bar{d}_k) - \psi(z_k)}{\hat{\alpha}_k} > \frac{\sigma}{2} F(z_k)^T J(z_k) \bar{d}_k, \quad (3.11)$$

17 where ψ is defined by (3.6). Since ψ is continuously differentiable at any $z \in \mathbb{R}^{2n+m}$, by letting
 18 $k \rightarrow \infty$ with $k \in K$ in (3.11), we have $\nabla \psi(z^*)^T d^* = F(z^*)^T J(z^*) d^* \geq \frac{\sigma}{2} F(z^*)^T J(z^*) d^*$, i.e.,
 19 $(1 - \frac{\sigma}{2}) F(z^*)^T J(z^*) d^* \geq 0$. This together with $\sigma \in (0, 1)$ gives $F(z^*)^T J(z^*) d^* \geq 0$. On the other
 20 hand, by (3.7) it holds that $F(z^*)^T J(z^*) d^* \leq 0$. Thus $F(z^*)^T J(z^*) d^* = 0$.

21 By $F(z^*)^T J(z^*) d^* = 0$ and (3.9), we have

$$(d^*)^T [J(z^*)^T J(z^*) + \lambda^* I] d^* = 0.$$

22 Since the matrix $J(z^*)^T J(z^*) + \lambda^* I$ is positive definite, we have $d^* = 0$ which together (3.9)
 23 gives $J(z^*)^T F(z^*) = 0$. A contradiction is derived. We complete the proof. \square

24 **Corollary 3.1** *Let z^* be any accumulation point of $\{z_k\}$ generated by Algorithm TS-LMM. If*
 25 *$J(z^*)$ is nonsingular, then $F(z^*) = 0$.*

26 4 Cubic convergence rate

27 In this section, we assume that the solution set Z of the nonlinear equations (2.3) is nonempty.
 1 We further suppose that the sequence $\{z_k\}$ generated by Algorithm TS-LMM converges to a

2 point $z^* \in Z$ and lies in some neighbourhood of z^* . We make the following assumptions.

3

4 **Assumption A** (a) $F(z)$ provides a local error bound in some neighbourhood of z^* , i.e. there
5 exist constants $\kappa > 0$ and $0 < \varepsilon < 1$ such that

$$\kappa \text{dist}(z, Z) \leq \|F(z)\|, \quad \forall z \in N(z^*, \varepsilon), \quad (4.1)$$

6 where $N(z^*, \varepsilon) := \{z \in \mathbb{R}^{2n+m} \mid \|z - z^*\| \leq \varepsilon\}$.

7 (b) The Jacobian $J(z)$ is Lipschitz continuous on $N(z^*, \varepsilon)$, i.e. there exists a constant $M > 0$
8 such that

$$\|J(u) - J(v)\| \leq M\|u - v\|, \quad \forall u, v \in N(z^*, \varepsilon). \quad (4.2)$$

9 Note that by Theorem 2.1, the condition (b) in Assumption A holds for $\tau \in (0, 4)$. By (4.2),
10 we have (see [8])

$$\|F(v) - F(u) - J(u)(v - u)\| \leq M\|v - u\|^2, \quad \forall u, v \in N(z^*, \varepsilon), \quad (4.3)$$

11 and there exists a constant $L > 0$ such that

$$\|F(v) - F(u)\| \leq L\|v - u\|, \quad \forall u, v \in N(z^*, \varepsilon). \quad (4.4)$$

12 In the following, we denote \bar{z}_k as the vector in Z that satisfies

$$\|\bar{z}_k - z_k\| = \text{dist}(z_k, Z).$$

13 Now we suppose the singular value decomposition (SVD) of $J(\bar{z}_k)$ is

$$\begin{aligned} J(\bar{z}_k) &= \bar{U}_k \bar{\Sigma}_k \bar{V}_k^T \\ &= (\bar{U}_{k,1}, \bar{U}_{k,2}) \begin{pmatrix} \bar{\Sigma}_{k,1} & \\ & 0 \end{pmatrix} \begin{pmatrix} \bar{V}_{k,1}^T \\ \bar{V}_{k,2}^T \end{pmatrix} \\ &= \bar{U}_{k,1} \bar{\Sigma}_{k,1} \bar{V}_{k,1}^T, \end{aligned}$$

14 where $\bar{\Sigma}_{k,1} = \text{diag}(\bar{\sigma}_{k,1}, \dots, \bar{\sigma}_{k,r})$ with $\bar{\sigma}_{k,1} \geq \dots \geq \bar{\sigma}_{k,r} > 0$. The corresponding SVD of $J(z_k)$ is

$$\begin{aligned} J(z_k) &= U_k \Sigma_k V_k^T \\ &= (U_{k,1}, U_{k,2}) \begin{pmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{pmatrix} \begin{pmatrix} V_{k,1}^T \\ V_{k,2}^T \end{pmatrix} \\ &= U_{k,1} \Sigma_{k,1} V_{k,1}^T + U_{k,2} \Sigma_{k,2} V_{k,2}^T, \end{aligned}$$

15 where $\Sigma_{k,1} = \text{diag}(\sigma_{k,1}, \dots, \sigma_{k,r})$ with $\sigma_{k,1} \geq \dots \geq \sigma_{k,r} > 0$ and $\Sigma_{k,2} = \text{diag}(\sigma_{k,r+1}, \dots, \sigma_{k,n})$ with
16 $\sigma_{k,r+1} \geq \dots \geq \sigma_{k,n} \geq 0$. In the following, if the context is clear, we neglect the subscript k in
17 $U_{k,i}, \Sigma_{k,i}, V_{k,i} (i = 1, 2)$ and write $J(\bar{z}_k)$ and $J(z_k)$ as

$$J(\bar{z}_k) = \bar{U}_1 \bar{\Sigma}_1 \bar{V}_1^T, \quad J(z_k) = U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T.$$

18 By the matrix perturbation theory [15] and (4.2), we have

$$\|\text{diag}(\Sigma_1 - \bar{\Sigma}_1, \Sigma_2)\| \leq \|J(z_k) - J(\bar{z}_k)\| \leq M\|\bar{z}_k - z_k\|,$$

1 which gives

$$\|\Sigma_1 - \bar{\Sigma}_1\| \leq M\|\bar{z}_k - z_k\|, \quad \|\Sigma_2\| \leq M\|\bar{z}_k - z_k\|. \quad (4.5)$$

2 **Lemma 4.1** Under Assumption A, there exists a constant $c_1 > 0$ such that for all sufficiently
3 large k ,

$$\|\bar{d}_k\| \leq c_1 \|\bar{z}_k - z_k\|. \quad (4.6)$$

4 *Proof* By (3.1) and (4.1), for all sufficiently large k ,

$$\lambda_k = \mu \|F(z_k)\|^\delta \geq \mu \kappa^\delta \|\bar{z}_k - z_k\|^\delta. \quad (4.7)$$

5 Since

$$\|\bar{z}_k - z^*\| \leq \|\bar{z}_k - z_k\| + \|z_k - z^*\| \leq 2\|z_k - z^*\|,$$

6 we have $\bar{z}_k \in N(z^*, \varepsilon)$ for all sufficiently large k . Then, by (4.3) and $F(\bar{z}_k) = 0$, for all sufficiently
7 large k ,

$$\|F(z_k) + J(z_k)(\bar{z}_k - z_k)\| \leq M \|\bar{z}_k - z_k\|^2. \quad (4.8)$$

8 For any $k \geq 0$, we consider the following minimization problem:

$$\min_{d \in \mathbb{R}^{2n+m}} \varphi_k(d) := \|F(z_k) + J(z_k)d\|^2 + \lambda_k \|d\|^2. \quad (4.9)$$

9 Then the LM step \bar{d}_k is a solution of (4.9). By (4.7) and (4.8), for all sufficiently large k ,

$$\begin{aligned} \|\bar{d}_k\|^2 &\leq \frac{\varphi_k(\bar{d}_k)}{\lambda_k} \\ &\leq \frac{\varphi_k(\bar{z}_k - z_k)}{\lambda_k} \\ &= \frac{\|F(z_k) + J(z_k)(\bar{z}_k - z_k)\|^2}{\lambda_k} + \|\bar{z}_k - z_k\|^2 \\ &\leq M^2 \mu^{-1} \kappa^{-\delta} \|\bar{z}_k - z_k\|^{4-\delta} + \|\bar{z}_k - z_k\|^2 \\ &\leq M^2 \mu^{-1} \kappa^{-\delta} \|\bar{z}_k - z_k\|^2. \end{aligned}$$

10 Letting $c_1 := \sqrt{M^2 \mu^{-1} \kappa^{-\delta} + 1}$, we have (4.6). □

11 **Lemma 4.2** Under Assumption A, for all sufficiently large k ,

- 12 (a) $\|U_1 U_1^T F(z_k)\| \leq L \|\bar{z}_k - z_k\|$;
- 13 (b) $\|U_2 U_2^T F(z_k)\| \leq 2M \|\bar{z}_k - z_k\|^2$;
- 14 (c) $\|U_1^T U_1 F(z_k)\| \leq c_2 \|\bar{z}_k - z_k\|^2$;
- 15 (d) $\|U_2 U_2^T F(z_k)\| \leq c_3 \|\bar{z}_k - z_k\|^3$,
- 16 where c_2, c_3 are positive constants.

17 *Proof* The results can be found in [8, Lemma 3.4 and Lemma 3.5]. □

18 **Lemma 4.3** Under Assumption A, there exists a constant $c_4 > 0$ such that for all sufficiently
1 large k ,

$$\|F(z_k) + J(z_k)\bar{d}_k\| \leq c_4 \|\bar{z}_k - z_k\|^2. \quad (4.10)$$

2 *Proof* By (3.2) and the SVD of $J(z_k)$, we have

$$\begin{aligned}\bar{d}_k &= [J(z_k)^T J(z_k) + \lambda_k I]^{-1} J(z_k)^T F(z_k) \\ &= -V_1(\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T F(z_k) - V_2(\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T F(z_k),\end{aligned}\quad (4.11)$$

3 and

$$\begin{aligned}&F(z_k) + J(z_k) \bar{d}_k \\ &= F(z_k) - U_1 \Sigma_1 (\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T F(z_k) - U_2 \Sigma_2 (\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T F(z_k) \\ &= \lambda_k U_1 (\Sigma_1^2 + \lambda_k I)^{-1} U_1^T F(z_k) + \lambda_k U_2 (\Sigma_2^2 + \lambda_k I)^{-1} U_2^T F(z_k).\end{aligned}\quad (4.12)$$

4 Since $\{z_k\}$ converges to z^* , without loss of generality, we assume that $M \|\bar{z}_k - z_k\| \leq \bar{\sigma}/2$ holds
5 for all sufficiently large k . Then it follows from (4.5) that

$$\|(\Sigma_1^2 + \lambda_k I)^{-1}\| \leq \|\Sigma_1^{-2}\| \leq \frac{1}{(\bar{\sigma} - M \|\bar{z}_k - z_k\|)^2} \leq \frac{4}{\bar{\sigma}^2}.\quad (4.13)$$

6 Moreover, by (3.1) and (4.4), for all sufficiently large k ,

$$\lambda_k = \mu \|F(z_k)\|^\delta = \mu \|F(z_k) - F(\bar{z}_k)\|^\delta \leq \mu L^\delta \|\bar{z}_k - z_k\|^\delta.\quad (4.14)$$

Therefore, by Lemma 4.2, (4.12), (4.13), (4.14) and $\|(\Sigma_2^2 + \lambda_k I)^{-1}\| \leq \lambda_k^{-1}$, for all sufficiently large k ,

$$\begin{aligned}\|F(z_k) + J(z_k) \bar{d}_k\| &\leq \lambda_k \|(\Sigma_1^2 + \lambda_k I)^{-1}\| \|U_1^T F(z_k)\| + \|U_2^T F(z_k)\| \\ &\leq \frac{4\mu L^{1+\delta}}{\bar{\sigma}^2} \|\bar{z}_k - z_k\|^{1+\delta} + 2M \|\bar{z}_k - z_k\|^2 \\ &\leq \left(\frac{4\mu L^{1+\delta}}{\bar{\sigma}^2} + 2M \right) \|\bar{z}_k - z_k\|^2.\end{aligned}\quad (4.15)$$

7 Letting $c_4 := 4\mu L^{1+\delta}/\bar{\sigma}^2 + 2M$, we have (4.10). \square

8 **Lemma 4.4** *Under Assumption A, for all sufficiently large k ,*

$$z_{k+1} = z_k + \bar{d}_k + \hat{d}_k.\quad (4.16)$$

9 *Proof* By (3.3) and the SVD of $J(z_k)$, we have

$$\begin{aligned}\hat{d}_k &= [J(z_k)^T J(z_k) + \lambda_k I]^{-1} J(z_k)^T F(t_k) \\ &= -V_1(\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T F(t_k) - V_2(\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T F(t_k),\end{aligned}$$

1 and

$$\begin{aligned}&F(t_k) + J(z_k) \hat{d}_k \\ &= F(t_k) - U_1 \Sigma_1 (\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T F(t_k) - U_2 \Sigma_2 (\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T F(t_k) \\ &= \lambda_k U_1 (\Sigma_1^2 + \lambda_k I)^{-1} U_1^T F(t_k) + \lambda_k U_2 (\Sigma_2^2 + \lambda_k I)^{-1} U_2^T F(t_k),\end{aligned}$$

2 which together with (4.5), (4.7), (4.13), (4.14), Lemma 4.2 and $\|(\Sigma_2^2 + \lambda_k I)^{-1}\| \leq \lambda_k^{-1}$ implies
 3 that for all sufficiently large k ,

$$\begin{aligned} \|\hat{d}_k\| &\leq \|(\Sigma_1^2 + \lambda_k I)^{-1}\| \|U_1^T F(t_k)\| + \|\lambda_k^{-1} \Sigma_2\| \|U_2^T F(t_k)\| \\ &\leq \frac{4c_2}{\bar{\sigma}^2} \|\bar{z}_k - z_k\|^2 + \frac{Mc_3}{\mu\kappa^\delta} \|\bar{z}_k - z_k\|^{4-\delta} \\ &\leq c_5 \|\bar{z}_k - z_k\|^2, \end{aligned} \tag{4.17}$$

4 and

$$\begin{aligned} \|F(t_k) + J(z_k)\hat{d}_k\| &\leq \lambda_k \|(\Sigma_1^2 + \lambda_k I)^{-1}\| \|U_1^T F(t_k)\| + \|U_2^T F(t_k)\| \\ &\leq \frac{4\mu L^\delta c_2}{\bar{\sigma}^2} \|\bar{z}_k - z_k\|^{2+\delta} + c_3 \|\bar{z}_k - z_k\|^3 \\ &\leq c_6 \|\bar{z}_k - z_k\|^3, \end{aligned} \tag{4.18}$$

5 where $c_5 := 4c_2/\bar{\sigma}^2 + Mc_3/\mu\kappa^\delta$ and $c_6 := 4\mu L^\delta c_2/\bar{\sigma}^2 + c_3$. Therefore, by (4.1), (4.2), (4.3), (4.6),
 6 (4.17) and (4.18), for all sufficiently large k ,

$$\begin{aligned} \|F(z_k + \bar{d}_k + \hat{d}_k)\| &= \|F(t_k + \hat{d}_k)\| \\ &\leq \|F(t_k + \hat{d}_k) - F(t_k) - J(t_k)\hat{d}_k\| \\ &\quad + \|F(t_k) + J(z_k)\hat{d}_k\| + \|(J(t_k) - J(z_k))\hat{d}_k\| \\ &\leq M\|\hat{d}_k\|^2 + \|F(t_k) + J(z_k)\hat{d}_k\| + M\|\bar{d}_k\| \|\hat{d}_k\| \\ &\leq Mc_5^2 \|\bar{z}_k - z_k\|^4 + c_6 \|\bar{z}_k - z_k\|^3 + Mc_1 c_5 \|\bar{z}_k - z_k\|^3 \\ &\leq (Mc_5^2 + c_6 + Mc_1 c_5) \text{dist}(z_k, Z^*)^3 \\ &\leq \frac{Mc_5^2 + c_6 + Mc_1 c_5}{\kappa^3} \|F(z_k)\|^3. \end{aligned}$$

7 This implies that

$$\|F(z_k + \bar{d}_k + \hat{d}_k)\| \leq \theta \|F(z_k)\|$$

8 holds for all sufficiently large k . Hence, by Step 3 of Algorithm TS-LMM, (4.16) holds for all
 9 sufficiently large k . We complete the proof.

10 Lemma 4.4 indicates that when k is sufficiently large, Algorithm TS-LMM becomes the
 11 modified Levenberg-Marquardt method without line search. Thus, by following the arguments
 12 in [8], we obtain the cubic convergence of Algorithm TS-LMM as follows.

13 **Theorem 4.1** *Under Assumption A, the sequence $\{z^k\}$ converges to z^* locally cubically.*

14 5 Numerical experiments

15 We would like to perform some numerical experiments in this section. We compare the numerical
 16 performances of the following two algorithms.

- 1 • Algorithm TS-LMM studied in this paper, denoted by **TS-LMM**;

• Levenberg-Marquardt type method studied by Tang and Zhou [19], denoted by **LMM**.

We test the WLCP (1.1) with

$$P = \begin{pmatrix} A \\ M \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ -I \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ -A^T \end{pmatrix}, \quad a = \begin{pmatrix} b \\ -f \end{pmatrix},$$

where $A \in \mathbb{R}^{m \times n}$ is a full row rank matrix with $m < n$, $b \in \mathbb{R}^m$, $f \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ is an $n \times n$ symmetric positive semidefinite matrix. This WLCP is the optimality conditions of the quadratic programming and weighted centering problem [14, Theorem 2.1]. In the experiments, we generate a random matrix $A \in \mathbb{R}^{m \times n}$ with full row rank and set $M = BB^T / \|BB^T\|$ with $B = \mathbf{rand}(n, n)$. Then we choose $\hat{x} = \mathbf{rand}(n, 1)$, $f = \mathbf{rand}(n, 1)$ and set $b := A\hat{x}$, $\hat{s} := M\hat{x} + f$ and $w := \hat{x}\hat{s}$. The parameters used in Algorithm TS-LMM are chosen as $\mu = 10^{-5}$, $\sigma = 10^{-6}$, $\rho = 0.8$, $\delta = 1$, $\theta = 0.5$.

First, to observe the local convergence behavior of **TS-LMM**, we generate one test problem with $n = 500$ and $m = 250$ and solve it by **TS-LMM** and **LMM** respectively. The starting point is chosen as $x_0 = s_0 = (1, \dots, 1)^T$ and $y_0 = (0, \dots, 0)^T$. Table 1 gives the value of $\|F(z_k)\|$ at the k -th iteration in which τ is the parameter value used in the function ϕ_τ^c . From Table 1, we can clearly see that **TS-LMM** has at least super-quadratic convergence rate and it converges faster than **LMM** does.

Table 1 The value of $\|F(z_k)\|$ at the k -th iteration

		TS-LMM	LMM
$\tau = 0$	$k = 1$	13.1456	28.1305
	$k = 2$	0.8915	13.0428
	$k = 3$	0.0333	3.7961
	$k = 4$	4.3500e-05	0.2935
	$k = 5$	3.0706e-13	0.0063
	$k = 6$	0	5.7924e-06
	$k = 7$	0	3.5507e-11
$\tau = 2$	$k = 1$	7.5373	17.6283
	$k = 2$	0.5989	6.4880
	$k = 3$	0.0273	2.3709
	$k = 4$	8.2111e-05	0.2565
	$k = 5$	8.0648e-12	0.0090
	$k = 6$	0	3.3340e-05
	$k = 7$	0	8.6151e-10

Next, for each problem with different sizes $n(= 2m)$, we generate 10 instances and solve them by **TS-LMM** and **LMM** respectively. The starting point is chosen as before. We use $\|F(z_k)\| < 10^{-8}$ as the stopping criterion. Numerical results are listed in Table 2 where **AIT** and **ACPU** denote the average number of iterations and the average CPU time in seconds respectively. From Table 2, we can see that **TS-LMM** always takes fewer iterations and often uses less CPU time than **LMM** to reach the stopping tolerance.

Table 2 Comparison of **TS-LMM** and **LMM**

		TS-LMM		LMM	
τ	n	AIT	ACPU	AIT	ACPU
0	1000	5.0	2.69	8.5	3.81
	1500	5.0	7.18	7.8	9.71
	2000	5.0	14.60	8.4	21.55
	2500	5.0	25.73	8.1	37.97
	3000	5.0	41.57	8.2	62.27
	3500	5.1	63.94	8.1	93.55
	4000	5.0	89.02	8.1	133.47
	2	1000	5.1	2.76	8.1
	1500	5.2	7.53	8.1	10.43
	2000	5.2	15.35	8.3	21.47
	2500	5.1	21.74	8.3	43.24
	3000	5.3	43.85	9.4	76.48
	3500	5.2	62.50	9.0	102.33
	4000	5.6	95.08	9.1	144.82

6 Conclusions

In this paper, by using a smooth weighted complementarity function, we reformulated the WLCP as a smooth nonlinear equation and proposed a two steps Levenberg-Marquardt type method to solve it. We proved that the proposed method is globally convergent without requiring any additional condition, and it has cubic convergence rate under the local error bound condition. Numerical experiments indicated that our method is more efficient than the existing Levenberg-Marquardt type method for solving WLCP. Lately, many researchers studied high-order Levenberg-Marquardt methods for nonlinear equations (e.g., [2, 3]). This class of methods computes the LM step and two approximate LM steps at every iteration. Under the local error bound condition, the convergence order of high-order Levenberg-Marquardt methods is fourth. As a future research issue, it is worth designing high-order Levenberg-Marquardt type methods for solving WLCP. **In addition, extending the three-step modulus-based iterative method and the generalized accelerated overrelaxation method proposed in [11, 12] to solve WLCP is also an interesting topic for our future research.**

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