New Exact Solutions, Similarity Reductions and Wave Phenomena for the extended (2+1)-dimensional Sakovich equation

Abstract

In this paper, we investigate the newly formulated (2+1)-dimensional Sakovich equation, highlighting its utility in describing the dynamics of nonlinear waves. This new equation effectively incorporates increased dispersion and nonlinear effects, thereby enhancing its applicability across various physical scenarios. This model is especially useful when modeling nonlinear phenomena in materials that simpler linear models would not accurately describe. It also serves as a founding model for numerical simulations in computational fluid dynamics and solid mechanics. We employ the Clarkson-Kruskal (CK) direct method to investigate exact solutions of the extended (2+1)-dimensional Sakovich equation. A review of the relevant literature indicates that the CK direct method has not yet been applied to solve the extended (2+1)-dimensional Sakovich equation. In this work, we successfully perform the complex and tedious computations required by the CK direct method. The results are classified into two distinct cases. In the first case, the obtained solutions include rational functions, a Weierstrass elliptic function, and new similarity reductions leading to Painlevé I and II equations. The second case yields new solutions involving trigonometric functions and hyperbolic function solutions. To the best of our knowledge, some of the solutions obtained have not been previously reported. All of these solutions manifest diverse wave phenomena, such as the soliton, bright, and dark solitons. Some of them reveal new wave phenomena governed by the extended (2+1)-dimensional Sakovich equation.

Keywords: The extended (2+1)-dimensional Sakovich equation, Exact solutions, the CK direct method, Similarity reductions

1 Introduction

In this manuscript, we mainly consider the extended (2+1)-dimensional Sakovich equation:

$$u_{xt} + u_{yy} + u_{xx} + u_{xy} + 2uu_{xy} + 6u^2 u_{xx} + 2(u_{xx})^2 = 0. \quad (1)$$

This equation was developed by Wazwaz [1] as an extension of the original foundational equation of Sakovich [2]. The addition of the terms u_{xx} and u_{yy} enhances its applicability by incorporating more dispersion and nonlinear effects, making it suitable for broader scenarios.

The original Sakovich's foundational equation is an NLPDE formulated by Sakovich in 1996. It is characterized by having Korteweg-de Vries (KdV)-type solitons and has been a focus of major research work due to its utility in many applications. This equation is Painlevé integrable, i.e., it can satisfy the Painlevé test, which is a test for whether a nonlinear partial differential equation is integrable. This equation is applied to the study of solitary waves, specifically in nonlinear dispersive systems. Some examples include the examination of rogue waves in oceanography, in which it is utilized to model sudden, large waves. This model finds applications in wide areas of physics, mathematics, and other sciences, especially in the theory of waves, soliton theory, plasma physics, biology and chemistry, and nonlinear phenomena. In various disciplines, the (2+1)-dimensional second-order Sakovich equation is an essential mathematical model that helps to investigate the behavior of water waves within a long, narrow, hollow tube. [3]

Optical solitons are a type of electromagnetic wave that maintains a stable propagation pattern in nonlinear media. This stability arises from a strong balance between the linear effects of diffraction or dispersion and the nonlinear effects of the medium. In the realm of optical fiber communications, solitons are particularly significant as they enhance the efficiency and capacity of communication networks. They achieve this by maintaining their shape and speed over long distances, which is essential for effective data transmission. Furthermore, one can conceptualize the genetic system of living organisms as a tripartite unity that encompasses both structural and functional components. This system includes holographic structures that are capable of transmitting information via solitons, which can operate similarly to magnetic and sound waves. In acupuncture, solitons manifest as high-amplitude, nonlinear solitary pulses that efficiently compress and direct the body's energy. Their mechanism resembles shock waves, particularly due to their hydrophilic jumps, which enable them to influence the environments of nearby smaller waves. As solitons propagate, they draw in these smaller waves, assimilating them into their larger potential waves, thereby allowing them to harness and utilize this energy. This complex interaction highlights the multifaceted role of solitons in both communication technologies and biological systems. The investigation of NLEEs is important because it gives information on a broad array of physical phenomena, ranging from fluid dynamics to optical communications. This paper points out different types of exact solutions, such as solitary and periodic waves, that are important in the understanding of these systems.[4–12]

A variety of powerful methods exist for solving NLPDEs, including the unified method [13], the Hirota bilinear technique [14], the Auto-Bäcklund/Darboux transformations [15, 16], and the inverse scattering transform [17]. These approaches yield diverse solutions, such as solitary waves, solitons, breathers, kinks, and lump solitons [18–22]. Consequently, the extended (2+1)-dimensional Sakovich equation (1) has been studied extensively. For example, Özkan et al. [23] by using the multiwaves method, double exponential form, the homoclinic breather approach, and the Lie symmetry technique to derive multiwave and interaction solutions. Sachin Kumar et al. [24] employed the Lie symmetry technique and the extended Jacobi elliptic functions method to derive solutions more generalized than the previously established results. Arnous et al. [25] implementation of an enhanced extended algebraic framework yielded multiple exact solutions.

It is well known that a powerful method for dealing with similarity reductions and exact solutions to NLPDEs [26–32] is the Clarkson-Kruskal(CK) direct method, which was first proposed by Clarkson and Kruskal [26]. The CK direct method mainly results in finding a solution to a special form of partial differential equation. In general, NLPDEs admit a wide spectrum of solutions, then the CK direct method postulates a solution of the following specific form (taking the $(2+1)$-dimension as an example):

$$u(x,y,t) = \alpha(x,y,t) + \beta(x,y,t)\,\omega(z)\,, \tag{2}$$

where α, β and z are functions of x, y, t to be determined, and ω satisfies a certain reduction equation that is lower than the dimension of the original equation.

The extended (2+1)-dimensional Sakovich equation holds significant theoretical value in describing a class of nonlinear wave propagation phenomena, making the search for its new exact solutions and analytical structures a central topic in nonlinear mathematical physics. However, existing analytical approaches have limitations in uncovering the rich solution structures of the equation. Notably, while the CK direct method is a powerful tool for deriving similarity reductions and exact solutions for nonlinear partial differential equations, the literature reveals that it has not yet been applied to this specific equation, presenting a clear research gap. This study aims to fill this gap, driven by the overarching motivation to validate the applicability and effectiveness of the CK direct method for this equation and to advance the study of its integrability. We systematically implement the method with a focused effort to overcome the attendant computational complexities. The primary objectives and novel contributions of this work are: to successfully derive, for the first time, a series of new similarity reductions for the equation (1); to reduce it systematically to an ordinary differential equation in $\omega(z)$; and consequently to construct several new exact solutions. These results not only enrich the repository of analytical solutions for the equation itself but also provide a new reference and methodology for applying the CK method to investigate other complex nonlinear systems. These exact solutions reveal new wave phenomena governed by (1), such as the soliton and the dark soliton. Among these solutions, some have global finite energy, others have local finite energy, which blow up along some lines. This also indicates some new physical phenomena. The result can be classified into two distinct cases.

- In the first case, where $z_x \neq 0$, we derive rational function solutions (Eqs. (37), (61), (62), (82), (91)), a Weierstrass elliptic function solution (Eq. (62)), and the Painlevé I and Painlevé II similarity reductions (Eq. (83)). Based on our understanding of the established literature, we have found that the solution (62) is almost identical to the solutions (112) and (116) in [24] obtained through the extended Jacobian elliptic function expansion method. Two different methods yield almost the same solutions, which, to some extent, show the validity of our research. Furthermore, solution (62) provides a more general form compared to solutions (42) and (79) in [24], and solution (83) provides a more general form compared to solutions (44), (53) and (58) in the same reference [24]. Because solutions (61) and (82) include more parameters, different parameters can lead to different solutions, which can cover some existing solutions and show the generality of our solutions.

- In the second case with $z_x = 0$, we obtain rational function solutions (Eqs. (115), (116)), a hyperbolic function solution (Eq. (117)), a trigonometric periodic solution (Eq. (118)). These new types of solutions reveal new wave phenomena governed by equation (1), which will be presented in detail in Section 2. It is helpful to explore the significance inherent in the extended (2+1)-dimensional Sakovich equation.

The remainder of this paper is organized as follows. In Section 2, through the CK direct method, we discuss and obtain the similarity reduction and new solutions of the extended (2+1)-dimensional Sakovich equation in the case of $z_x \neq 0, z_y \neq 0$ and $z_x = 0, z_y \neq 0$. For the solutions obtained, we provide both 3D and 2D plots, which collectively enhance the observation of their spatial variations. Finally, the conclusion and discussion will be given in Section 3.

2 Symmetry reductions and exact solutions of the extended (2+1)-dimensional Sakovich equation

In this section, we will perform the calculations.

Substituting the ansatz (2) into (1) and collecting coefficients of like monomials in ω and its derivatives yields:

$$\begin{aligned}&\gamma_0(\omega'')^2 + \gamma_1\omega'\omega'' + \gamma_2\omega^2\omega'' + \gamma_3\omega\omega'' + \gamma_4\omega'' + \gamma_5(\omega')^2+\\&\gamma_6\omega^2\omega' + \gamma_7\omega\omega' + \gamma_8\omega' + \gamma_9\omega^3 + \gamma_{10}\omega^2 + \gamma_{11}\omega + \gamma_{12} = 0,\end{aligned} \tag{3}$$

where

$$
\begin{aligned}
\gamma_0 &= 2\beta^2 z_x^4, \\
\gamma_1 &= 4\beta^2 z_x^2 z_{xx} + 8\beta\beta_x z_x^3, \\
\gamma_2 &= 6\beta^3 z_x^2, \\
\gamma_3 &= 12\alpha\beta^2 z_x^2 + 2\beta^2 z_x z_y + 4\beta\beta_{xx} z_x^2, \\
\gamma_4 &= 6\alpha^2\beta z_x^2 + 4\beta\alpha z_{xx}^2 + 2\alpha\beta z_x z_y + \beta z_x^2 + \beta z_x z_y + \beta z_x z_t + \beta z_y^2, \\
\gamma_5 &= 2\beta(z_{xx})^2 + 8\beta\beta_x z_x z_{xx} + 8\beta^2 z_x^2, \\
\gamma_6 &= 6\beta^3 z_{xx} + 12\beta^2\beta_x z_x, \\
\gamma_7 &= 12\alpha\beta^2 z_{xx} + 24\alpha\beta z_x + 2\beta^2 z_{xy} + 2\beta\beta_x z_y + 2\beta\beta_y z_x + 4\beta\beta_{xx} z_{xx} + 8\beta_x\beta_{xx} z_x, \\
\gamma_8 &= 6\alpha^2\beta z_{xx} + 12\alpha^2\beta_x z_x + 2\alpha\beta z_{xy} + 4\beta\alpha_{xx} z_{xx} + 8\alpha_{xx}\beta_x z_x + 2\alpha\beta_y z_x + \beta z_{xt} \\
&\quad + \beta z_{xy} + \beta z_{yy} + \beta z_{xx} + 2\beta_x z_x + \beta_t z_x + \beta_y z_x + \beta_x z_y + 2\beta_y z_y + \beta_x z_t, \\
\gamma_9 &= 6\beta^2\beta_{xx}, \\
\gamma_{10} &= 6\beta^2\alpha_{xx} + 12\alpha\beta\beta_{xx} + 2\beta\beta_{xy} + 2\beta_{xx}^2, \\
\gamma_{11} &= 12\alpha\beta\alpha_{xx} + 6\alpha^2\beta_{xx} + 2\beta\alpha_{xy} + 2\alpha\beta_{xy} + 4\alpha_{xx}\beta_{xx} + \beta_{xt} + \beta_{xx} + \beta_{yy} + \beta_{xy}, \\
\gamma_{12} &= 6\alpha^2\alpha_{xx} + 2\alpha\alpha_{xy} + 2\alpha_{xx}^2 + \alpha_{xx} + \alpha_{xy} + \alpha_{yy} + \alpha_{xt}.
\end{aligned} \tag{4}
$$

and $' := d/dz$. In order that Eq. (3) be an ordinary differential equation for $\omega(z)$, the ratios of coefficients of different derivatives and powers of $\omega(z)$ must be functions of z only. If $z_x \neq 0$ and $z_y \neq 0$, these conditions are expressed as

$$\gamma_i = \gamma_0 \Gamma_i(z) \ \ (i = 1, 2, \ldots, 12), \tag{5}$$

where $\Gamma_i(z)\,(i = 1, 2, \ldots, 12)$ are some arbitrary functions of z to be determined later.

In the determination of $\alpha(x,y,t), \beta(x,y,t), z(x,y,t)$ and $\Gamma_i(z)\,(i = 1, \ldots, 12)$, the following four remarks may be involved:

Remark (i): if $\alpha(x,y,t)$ has the form $\alpha = \alpha_0(x,y,t) + \beta(x,y,t)\,\Omega(z)$, we can set $\Omega = 0$ without loss of generality by redefining $\omega(z) \to \omega(z) - \Omega(z)$.

Remark (ii): if $\beta(x,y,t)$ takes the form $\beta = \beta_0(x,y,t)\,\Omega(z)$, we can set $\Omega = \Omega_0 = \textit{constant}$ via the redefinition $\omega(z) \to \omega(z)\,\Omega_0/\Omega(z)$.

Remark (iii): if $z(x,y,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,y,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ (by substituting $z \to \Omega^{-1}(z)$).

Remark (iv): we reserve Greek and English letters for undetermined functions of z (or x, y, t) so that after performing operations (differentiation, integration, exponentiation, rescaling, etc.) the result can be denoted by the same letter [for example, the derivative of $\Gamma(z)$ will be called $\Gamma(z)$].

2.1 The case $z_x \neq 0, z_y \neq 0$

In this analysis, we employ the coefficient of $(\omega^{''})^2$, specifically $2\beta^2 z_x^4$, as the normalization factor. This requires that all other coefficients take the form $2\beta^2 z_x^4 \Gamma_i(z)$, as stipulated by (5).

From (5) with $i = 2$, we obtain

$$6\beta^3 z_x^2 = 2\beta^2 z_x^4 \Gamma_2(z). \tag{6}$$

By invoking Remark (ii), we can set:

$$\beta = \frac{1}{3} z_x^2, \qquad \Gamma_2(z) = 1. \tag{7}$$

For $i = 1$, substituting the results from (7) into (6) yields:

$$z_{xx} = z_x^2 \Gamma_1(z). \tag{8}$$

Dividing by z_x and integrating with respect to x, and subsequently applying Remarks (iii) and (iv), we obtain:

$$z = x\theta(y,t) + \sigma(y,t), \tag{9}$$

where $\theta(y,t)$ and $\sigma(y,t)$ are differentiable functions to be determined.

For $i = 3$, combining Eqs. (5), (7), and (9), we obtain:

$$\frac{4}{3}\alpha\theta^6 + \frac{2}{9}\theta^5 \left(x\frac{\mathrm{d}\theta}{\mathrm{d}y} + \frac{\mathrm{d}\sigma}{\mathrm{d}y} \right) = \frac{2}{9}\theta^8 \Gamma_3(z), \tag{10}$$

which simplifies to

$$\alpha = -\frac{1}{6\theta}(x\theta_y + \sigma_y) + 2\beta\Gamma_3(z). \tag{11}$$

Applying Remark (i) allows us to set:

$$\alpha = -\frac{1}{6\theta}(x\theta_y + \sigma_y), \qquad \Gamma_3(z) = 0. \tag{12}$$

From the results in (7), (9), (12), it follows that:

$$\Gamma_1(z) = \Gamma_5(z) = \Gamma_6(z) = \Gamma_9(z) = \Gamma_{10}(z) = 0. \tag{13}$$

Next, taking into account Eq. (5) with $i = 7$ under constraints (7), (9), (12), we get:

$$\frac{2}{3}\theta^4\theta_y = \frac{2}{9}\theta^8 \Gamma_7(z), \tag{14}$$

Since the left-hand side of (14) is at most a function of y and t, $\Gamma_7(z)$ must be a constant, denoted as A.

This leads to two distinct cases, which we analyze separately.

Case 1. A $\neq 0$

In this case, we have the relations:

$$\theta_y = z_{xy} = z_{yx}, \tag{15}$$

and the characteristic derivative:

$$z_x = \theta, \tag{16}$$

substituting (16) into (15) and integrating with respect to x, we obtain:

$$\frac{1}{3}\theta^3(\mathrm{A}z + \mathrm{B}) = x\theta_y + \sigma_y, \tag{17}$$

where B denotes an integration constant. Noting the linear dependence on x on the right-hand side, we equate coefficients to obtain the system:

$$\frac{1}{3}\left(\mathrm{A}x\theta + \mathrm{A}\sigma + \mathrm{B}\right)\theta^3 = x\theta_y + \sigma_y. \tag{18}$$

Balancing the coefficients of x and the constant terms in (18) gives:

$$\theta_y = \frac{1}{3}\mathrm{A}\theta^4, \tag{19}$$

$$\sigma_y = \frac{1}{3}\theta^3(\mathrm{A}\sigma + \mathrm{B}). \tag{20}$$

From Eq. (5) with $i = 11$ under constraints (7), (9), (12), we get

$$\frac{1}{3}\mathrm{A}^2\theta^8 = \frac{2}{9}\theta^8\Gamma_{11}(z). \tag{21}$$

This leads to $\Gamma_{11}(z) = \frac{3}{2}\mathrm{A}^2$.

Similarly, for $i = 4$ and $i = 8$, we obtain:

$$\frac{5}{162}\mathrm{A}^2\theta^{10}x^2 + \frac{5}{27}\mathrm{A}\theta^6\sigma_y x + \frac{5}{18}\theta^2\sigma_y^2 + \frac{1}{3}\theta^3(\sigma_y + \sigma_t + \theta) = \frac{2}{9}\theta^8\Gamma_4(z), \tag{22}$$

$$\left(\theta\theta_y^2 + \frac{1}{3}\theta^2\theta_{yy}\right)x + \theta\theta_y\sigma_y + \frac{1}{3}\theta^2\sigma_{yy} + (\theta_y + \theta_t)\theta^2 = \frac{2}{9}\theta^8\Gamma_8(z). \tag{23}$$

To simplify the subsequent analysis, we impose the constraints:

$$\theta_y + \theta_t = 0, \tag{24}$$

$$\sigma_y + \sigma_t + \theta = 0. \tag{25}$$

Solving (19) together with condition (24) gives:

$$\theta(y, t) = (\mathrm{A}t - \mathrm{A}y + \mathrm{C}_1)^{-1/3}, \tag{26}$$

Substituting (26) into (20) and (25) gives the explicit solution:

$$\sigma(y,t) = \frac{-t + \mathrm{C}_2}{(\mathrm{A}t - \mathrm{A}y + \mathrm{C}_1)^{1/3}} - \frac{\mathrm{B}}{\mathrm{A}}, \tag{27}$$

where C_1 and C_2 are arbitrary integration constants.

A parallel analysis for $i \in \{4, 8, 12\}$ yields:

$$\Gamma_4\left(z\right) = \frac{5}{36}(\mathrm{A}z + \mathrm{B})^2, \tag{28}$$

$$\Gamma_8\left(z\right) = \frac{7\mathrm{A}}{6}(\mathrm{A}z + \mathrm{B}), \tag{29}$$

$$\Gamma_{12}\left(z\right) = -\frac{17\mathrm{A}^2}{36}(\mathrm{A}z + \mathrm{B}). \tag{30}$$

Consequently, the similarity reduction of the extended (2+1)-dimensional Sakovich equation (1) is given by:

$$u(x,y,t) = \frac{\theta^2\omega(z)}{3} - \frac{x\theta_y + \sigma_y}{6\theta}, \tag{31}$$

$$z(x,y,t) = x\theta\left(y,t\right) + \sigma\left(y,t\right), \tag{32}$$

where $\theta(y,t)$ and $\sigma(y,t)$ are defined by (26) and (27), respectively, and $\omega(z)$ satisfies the reduced ordinary differential equation:

$$(\omega'')^2 + \omega^2\omega'' + \frac{5(\mathrm{A}z + \mathrm{B})^2\omega''}{36} + \mathrm{A}\omega\omega' + \frac{7\mathrm{A}(\mathrm{A}z + \mathrm{B})\omega'}{6} + \frac{3\mathrm{A}^2\omega}{2} = \frac{17\mathrm{A}^2(\mathrm{A}z + \mathrm{B})}{36}. \tag{33}$$

Particular solutions of Eq. (33) include:

$$\omega_1(z) = \frac{1}{6}(\mathrm{A}z + \mathrm{B}), \tag{34}$$

$$\omega_2(z) = -\frac{17}{6}(\mathrm{A}z + \mathrm{B}). \tag{35}$$

Substituting these expressions for ω back into the similarity form (31) and (32) yields the following solutions of the extended (2+1)-dimensional Sakovich equation (1) :

$$u_1(x,y,t) \equiv 0, \tag{36}$$

$$u_2(x,y,t) = \frac{x - t + \mathrm{C}_2}{y - t - \frac{\mathrm{C1}}{\mathrm{A}}}. \tag{37}$$

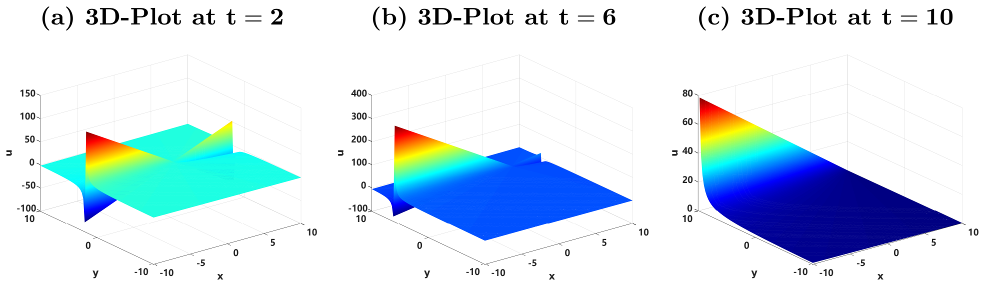

Fig. 1 3D-polts of the solution (37) are depicted at A = 1, $C_1 = 0$ and $C_2 = 1$ within the interval $-10 \leq x, y \leq 10$ for $t = 2$, $t = 6$ and $t = 10$.

Fig. 1 exhibits the evolution of the rational function solution (37), showing the propagation of the rational function wave with the line wave along the positive y -axis.

Case 2. A = 0

The condition A = 0 in (14) implies $\theta_y = 0$, and consequently

$$\theta(y,t) = \theta(t), \tag{38}$$

with the characteristic variable now taking the form:

$$z = x\theta\left(t\right) + \sigma\left(y,t\right). \tag{39}$$

In this case, the non-zero coefficients γ_i simplify to:

$$\begin{aligned} \gamma_0 &= \frac{2}{9}\theta^8, \\ \gamma_2 &= \frac{2}{9}\theta^8, \\ \gamma_4 &= \frac{1}{3}x\theta^3\theta_t + \frac{5}{18}\theta^2\sigma_y^2 + \frac{1}{3}\theta^3(\sigma_y + \sigma_t + \theta), \\ \gamma_8 &= \theta^2\theta_t + \frac{1}{3}\theta^2\sigma_{yy}, \\ \gamma_{12} &= -\frac{\sigma_{yyy}}{6\theta}. \end{aligned} \tag{40}$$

The normalization condition $\Gamma_2(z) = 1$ from (7) still holds.
For $i \in \{4, 8, 12\}$, the determining equations become:

$$\frac{1}{3}x\theta^3\theta_t + \frac{5}{18}\theta^2\sigma_y^2 + \frac{1}{3}\theta^3(\sigma_y + \sigma_t + \theta) = \frac{2}{9}\theta^8\Gamma_4(z), \tag{41}$$

$$\theta^2\theta_t + \frac{1}{3}\theta^2\sigma_{yy} = \frac{2}{9}\theta^8\Gamma_8(z), \tag{42}$$

$$-\frac{\sigma_{yyy}}{6\theta} = \frac{2}{9}\theta^8\Gamma_{12}(z). \tag{43}$$

Since the left-hand sides of Eqs. (42) and (43) depend at most on y and t, whereas the right-hand sides are proportional to $\Gamma_8(z)$ and $\Gamma_{12}(z)$ which depend on $z = x\theta(t) + \sigma(y,t)$, it follows that $\Gamma_8(z)$ and $\Gamma_{12}(z)$ must be constants.

Integrating (43) with respect to y yields:

$$\sigma(y,t) = -\frac{2\theta^9}{9}\Gamma_{12}(z)y^3 + \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t), \tag{44}$$

where Γ_{12} is now a constant, and $c_i(t) \in C^1(\mathbb{R})$, $i \in \{1,2,3\}$.

Substituting (44) into Eq. (41) and analyzing the resulting polynomial in x and y lead to the consistency conditions:

$$\frac{2\theta^8}{9}\Gamma_4(z) = \frac{1}{3}x\theta^3\theta_t + \frac{5\theta^2}{18}\Big(-\frac{2\theta^9}{3}\Gamma_{12}(z)y^2 + c_1(t)y + c_2(t)\Big)^2 + \frac{\theta^3}{3}\Big[-\frac{2\theta^9}{3}\Gamma_{12}(z)y^2 + c_1(t)y + c_2(t)\Big(-2\Gamma_{12}(z)\theta^8\theta_t y^3 + \frac{1}{2}c_1'(t)y^2 + c_2'(t)y + c_3'(t)\Big)\Big]. \tag{45}$$

The requirement for polynomial balance forces $\Gamma_{12}(z) = 0$, and determines that $\Gamma_4(z)$ must be a linear function, $\Gamma_4(z) = \mathrm{A}_1 z + \mathrm{B}_1$, where $\mathrm{A}_1, \mathrm{B}_1$ are arbitrary constants. Consequently, we have

$$\sigma(y,t) = \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t), \tag{46}$$

$$\theta(t) = (\mathrm{C} - \frac{10\mathrm{A}_1 t}{3})^{-1/5}. \tag{47}$$

This results in a reduced system of ordinary differential equations for the functions $c_i(t)$:

$$\frac{5}{18}\theta^2 c_1^2(t) + \frac{1}{6}\theta^3 c_1'(t) = \frac{1}{9}\mathrm{A}_1\theta^8 c_1(t), \tag{48}$$

$$\frac{5}{9}\theta^2 c_1(t)c_2(t) + \frac{1}{3}\theta^3(c_1(t) + c_2'(t)) = \frac{2}{9}\mathrm{A}_1\theta^8 c_2(t), \tag{49}$$

$$\frac{5}{18}\theta^2 c_2^2(t) + \frac{1}{3}\theta^3(c_2(t) + c_3'(t) + \theta) = \frac{2}{9}(\mathrm{A}_1 c_3(t) + \mathrm{B}_1)\theta^8, \tag{50}$$

$$\frac{2}{3}\mathrm{A}_1\theta^8 + \frac{1}{3}\theta^2 c_1(t) = \frac{2}{9}\theta^8\Gamma_8(z). \tag{51}$$

The structure of the solutions depends on the parameters Γ_8 and A_1, necessitating a separate analysis of the following two subcases:

- Non-autonomous case $\mathrm{A}_1 \neq 0$
- Autonomous case $\mathrm{A}_1 = 0$

Case (2a): $\mathbf{A_1 = 0, \Gamma_8(z) = 0}$

In this degenerate case, we obtain $c_1(t) = 0$, $\theta(t) = \mathrm{C}^{-1/5} = \theta_0 \neq 0$. Substituting $c_1(t) = 0$ into Eqs. (48), (49) gives the solutions: $c_2(t) = \mathrm{C}_2$,

$$c_3(t) = \mathrm{C}_3 t + \mathrm{C}_4, \tag{52}$$

$$\sigma(y,t) = \mathrm{C}_2 y + \mathrm{C}_3 t + \mathrm{C}_4, \tag{53}$$

where $\mathrm{C}_3 = \frac{2\mathrm{B}_1\theta_0^5}{3} - \frac{5\mathrm{C}_2^2}{6\theta_0} - \theta_0 - \mathrm{C}_2$, $\mathrm{C}_2, \mathrm{C}_4 \in \mathbb{R}$ are integration constants.

The similarity reduction of the extended (2+1)-dimensional Sakovich equation then becomes:

$$u(x,y,t) = \frac{\theta_0^2 \omega(z)}{3} - \frac{\mathrm{C}_2}{6\theta_0}, \tag{54}$$

$$z(x,y,t) = \theta_0 x \ + \mathrm{C}_2 y + \mathrm{C}_3 t + \mathrm{C}_4, \tag{55}$$

where $\omega(z)$ satisfies the reduced ordinary differential equation:

$$(\omega'')^2 + \omega^2\omega'' + \mathrm{B}_1\omega'' = 0. \tag{56}$$

Equation (56) admits the following set of solutions:

$$\omega_1(z) = \mathrm{K}_1 z + \mathrm{K}_2, \tag{57}$$

$$\omega_2(z) = -\frac{6}{(z+\mathrm{C}_5)^2}, \tag{58}$$

$$\omega_3(z) = -6\,\wp\left(z - z_0;\, g_2 = -\frac{\mathrm{B}_1}{3}, g_3 = \mathrm{C}_6\right), \tag{59}$$

where $\wp$ denotes the Weierstrass elliptic function with invariants g_2, g_3.

Substituting these expressions for $\omega(z)$ back into the similarity form (54) and (55) yields the corresponding solutions of Eq. (1):

$$u_1(x,y,t) = \frac{\mathrm{K}_1\theta_0^3 x + \mathrm{K}_1\theta_0^2\mathrm{C}_2 y + \mathrm{K}_1\theta_0^2\mathrm{C}_3 t + \theta_0^2\mathrm{K}_2}{3} - \frac{\mathrm{C}_2}{6\theta_0}, \tag{60}$$

$$u_2(x,y,t) = -\frac{2\theta_0^2}{(\theta_0 x + \mathrm{C}_2 y + \mathrm{C}_3 t + \mathrm{C}_5)^2} - \frac{\mathrm{C}_2}{6\theta_0}, \tag{61}$$

$$u_3(x,y,t) = -2\theta_0^2\wp\left(\theta_0 x + \mathrm{C}_2 y + \mathrm{C}_3 t + \mathrm{C}_4 - z_0; -\frac{\mathrm{B}_1}{3}, \mathrm{C}_6\right) - \frac{\mathrm{C}_2}{6\theta_0}. \tag{62}$$

We relate the Weierstrass elliptic function to a Jacobi elliptic function by the expression

$$\wp\,(z - z_0; g_2, g_3) = \mathrm{A}_0\,\mathrm{sn}^2(\mathrm{k}_1\,(z - z_0), k) + \mathrm{B}_0, \tag{63}$$

where A_0 and B_0 are constants to be determined, k_1 is a scale factor, and k is the elliptic modulus. Using the differential identity $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ and equating the coefficients of like powers of sn in the equation, we determine the constants:

$$\mathrm{A}_0 = \mathrm{k}_1^2 k^2, \qquad \mathrm{B}_0 = \frac{2\mathrm{k}_1^2\theta_0^2(k^2+1)}{3}. \tag{64}$$

Therefore, a Jacobi elliptic function solution is obtained:

$$\begin{aligned} u_4(x,y,t) &= -2\theta_0^2 \mathrm{k}_1^2 k^2 \operatorname{sn}^2\left(\mathrm{k}_1\theta_0 x + \mathrm{k}_1\mathrm{C}_2 y + \mathrm{k}_1\mathrm{C}_3 t + \mathrm{k}_1(\mathrm{C}_4 - z_0), k\right) \\ &\quad + \frac{2\mathrm{k}_1^2\theta_0^2(k^2+1)}{3} - \frac{\mathrm{C}_2}{6\theta_0}, \end{aligned} \tag{65}$$

where the invariants g_2, g_3 of the corresponding Weierstrass form are given by

$$g_2 = \frac{4\mathrm{k}_1^4(k^4 - k^2 + 1)}{3}, \qquad g_3 = \frac{4\mathrm{k}_1^6(k^2+1)(k^2-2)(2k^2-1)}{27}. \tag{66}$$

In the limit $k \to 1$, $\operatorname{sn}(\cdot, k) \to \tanh(\cdot)$, yielding the hyperbolic solution:

$$\begin{aligned} u_5(x,y,t) &= -2\theta_0^2 \mathrm{k}_1^2 \tanh^2\left(\mathrm{k}_1\theta_0 x + \mathrm{k}_1\mathrm{C}_2 y + k_1\mathrm{C}_3 t + \mathrm{k}_1(\mathrm{C}_4 - z_0)\right) \\ &\quad + \frac{4\mathrm{k}_1^2\theta_0^2}{3} - \frac{\mathrm{C}_2}{6\theta_0}. \end{aligned} \tag{67}$$

Similarly,

$$\begin{aligned} u_6(x,y,t) &= -2\theta_0^2 \mathrm{k}_2^2 \operatorname{ns}^2\left(\mathrm{k}_2\theta_0 x + \mathrm{k}_2\mathrm{C}_2 y + \mathrm{k}_2\mathrm{C}_3 t + k_2(\mathrm{C}_4 - z_0), k\right) \\ &\quad + \frac{2\mathrm{k}_2^2\theta_0^2(k^2+1)}{3} - \frac{\mathrm{C}_2}{6\theta_0}, \end{aligned} \tag{68}$$

where g_2, g_3 satisfy:

$$g_2 = \frac{4\mathrm{k}_2^4(k^4 - k^2 + 1)}{3}, \qquad g_3 = \frac{4\mathrm{k}_2^6(k^2+1)(k^2-2)(2k^2-1)}{27}. \tag{69}$$

Taking the limit $k \to 1$ (where $\operatorname{ns}(\cdot, k) \to \coth(\cdot)$) gives the hyperbolic solution

$$\begin{aligned} u_7(x,y,t) &= -2\theta_0^2 \mathrm{k}_2^2 \coth^2\left(\mathrm{k}_2\theta_0 x + \mathrm{k}_2\mathrm{C}_2 y + \mathrm{k}_2\mathrm{C}_3 t + \mathrm{k}_2(\mathrm{C}_4 - z_0)\right) \\ &\quad + \frac{4\mathrm{k}_2^2\theta_0^2}{3} - \frac{\mathrm{C}_2}{6\theta_0}. \end{aligned} \tag{70}$$

From the results of the solution (65) and the solution (68), we have found that the solution (62) is almost identical to the solutions (112) and (116) in [24] obtained by the extended Jacobian elliptic function expansion method. Two different methods yield almost the same solutions, which, to some extent, shows the validity of our research. Furthermore, (61) provides a more general form compared to Eqs. (42) and (79) in [24] since the solution (62) includes parameters, different parameters can lead to different solutions. In Fig. 2, Fig. 3, Fig. 4, and Fig. 5, we show the corresponding solutions (61), (65), (68), (70) with different parameters.

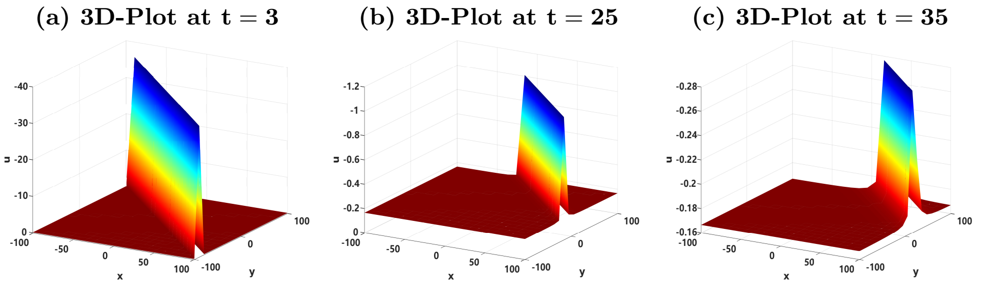

Fig. 2 3D-plots of the solution (61) are depicted at $\theta_0 = 1$, $\mathrm{C}_2 = 1$, $\mathrm{B}_1 = 0$ and $\mathrm{C}_5 = 0$ within the interval $-100 \leq x, y \leq 100$ for $t = 3$, $t = 25$ and $t = 35$.

The solution (61) is expressed in the inverse square solution. Fig. 2 exhibits this solution, showing the propagation of the inverse square wave with the line wave along the direction of the vector $[\mathbf{1}, \mathbf{1}, \mathbf{0}]$.

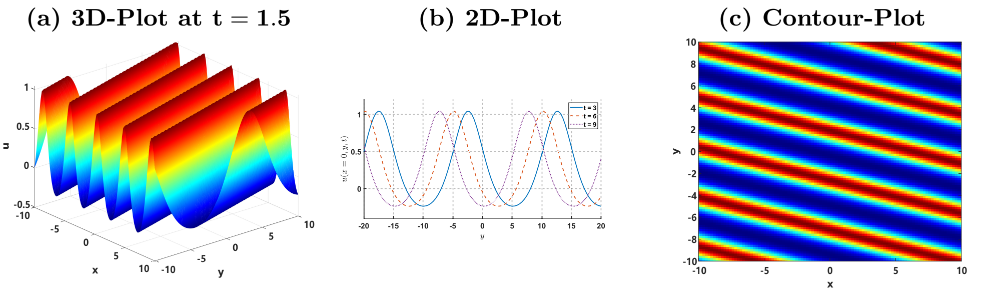

Fig. 3 Evolution plot of singular periodic wave for solution (65) at $t = 1.5$. The figure is delineated for $\theta_0 = \mathrm{k}_1 = 1, k = 0.8, \mathrm{C}_2 = 0.3, \mathrm{B}_1 = 0.5$ and $C_4 = z_0 = 0$ with in the interval $-10 \leq x, y \leq 10$. Corresponding 2D and contour plot are represented in part (b) and part (c), respectively.

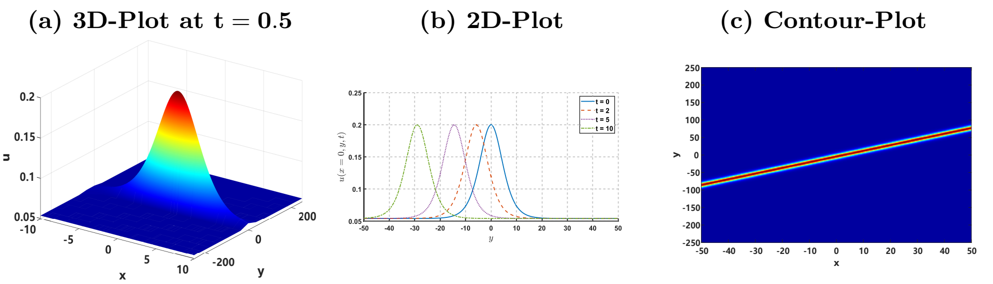

Fig. 4 Soliton for solution (67) at $t = 0.5$. Also the 2D and contour plot of this soliton are drawn in part (b) and part (c) , respectively. The figure is delineated for $\theta_0 = 0.27, \mathrm{k}_1 = 1, \mathrm{C}_2 = -0.166, \mathrm{B}_1 = -4.9$ and $\mathrm{C}_4 = z_0 = 0$ with in the interval $-10 \leq x \leq 10$ and $-300 \leq y \leq 300$.

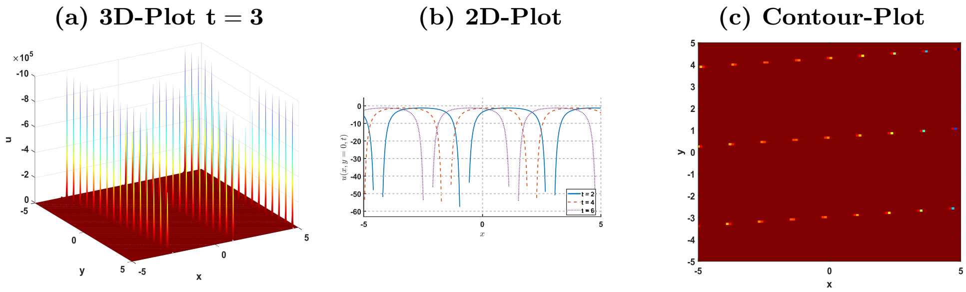

Fig. 5 Design of the multiple singular periodic soliton for the solution (68) at $t = 3$. The values of free parameters are taken as $\theta_0 = \mathrm{k}_2 = 1, k = 0.45, \mathrm{C}_2 = -0.083, \mathrm{B}_1 = -2.286$ and $\mathrm{C}_4 = z_0 = 0$ within the range $-5 \leq x, y \leq 5$. Corresponding 2D plot and contour plot are depicted in part (b) and part (c), respectively.

Case (2b): $\mathbf{A_1 \neq 0, \Gamma_8(z) = 0}$

In this non-autonomous case, solving the system (48) - (51) yields the coupled solutions:

$$c_1(t) = -2\mathrm{A}_1\theta^6(t), \tag{71}$$

$$c_2(t) = (2\mathrm{A}_1 t + \mathrm{C}_7)\theta^6(t), \tag{72}$$

$$c_3(t) = -\frac{(3\mathrm{C} + 5\mathrm{C}_7)^2}{100\mathrm{A}_1}\theta^6(t) + \mathrm{C}_8\theta^3(t) + \frac{21}{100\mathrm{A}_1}\theta^{-4}(t) - \frac{\mathrm{B}_1}{\mathrm{A}_1}, \tag{73}$$

and consequently,

$$\sigma(y, t) = \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t), \tag{74}$$

where $\mathrm{C}_7, \mathrm{C}_8 \in \mathbb{R}$ are the integration constants.

The corresponding similarity reduction is given by

$$u(x, y, t) = \frac{\theta^2(t)\omega(z)}{3} - \frac{c_1(t)y + c_2(t)}{6\theta(t)}, \tag{75}$$

$$z(x, y, t) = \theta(t)x + \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t). \tag{76}$$

Here, $\omega(z)$ satisfies the reduced ordinary differential equation:

$$(\omega'')^2 + \omega^2\omega'' + (\mathrm{A}_1 z + \mathrm{B}_1)\omega'' = 0. \tag{77}$$

Equation (77) admits, among others, the following solutions:

$$\omega_1(z) = \mathrm{K}_3 z + \mathrm{K}_4, \tag{78}$$

$$\omega_2(z) = \frac{216}{\mathrm{A}_1^2}P_k\left((-1)^{k+1}\frac{\mathrm{A}_1}{6}\left(z + \frac{\mathrm{B}_1}{\mathrm{A}_1}\right)\right), \tag{79}$$

where the function P_k satisfies either the first (P_I) or the second (P_{II}) Painlevé equation:

$$P_I : P_1'' = 6P_1^2 + \tau, \tag{80}$$

$$P_{II}: P_2'' = 2P_2^3 + \tau P_2 + \alpha, \tag{81}$$

Here, P_1 and P_2 are the first and second Painlevé transcendents, respectively, τ is the independent variable, and α is a constant parameter.

Substituting the expressions for $\omega(z)$, $\theta(t)$, $c_i(t)$ and $z(x,y,t)$ back into the similarity form (75) and (76) yields the corresponding solutions of (1) .

$$\begin{aligned} u_1(x,y,t) = & \frac{\mathrm{K}_3}{3}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-3/5} x - \frac{\mathrm{A}_1\mathrm{K}_3}{3}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-8/5} y^2 \\ & + \left(\frac{2\mathrm{A}_1\mathrm{K}_3 t + \mathrm{C}_7\mathrm{K}_4}{3}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-8/5} + \frac{\mathrm{A}_1}{-10\mathrm{A}_1 t + 3\mathrm{C}}\right) y \\ & - \frac{\mathrm{K}_3(3\mathrm{C} + 5\mathrm{C}_7)}{300\mathrm{A}_1}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-8/5} + \frac{-2\mathrm{A}_1 t + 2\mathrm{K}_3\mathrm{C}\mathrm{C}_8 - \mathrm{C}_7}{-20\mathrm{A}_1 t + 6C} \\ & - \frac{27\mathrm{B}_1\mathrm{K}_3}{\mathrm{A}_1(-10\mathrm{A}_1 t + 3\mathrm{C})^2} + \frac{7\mathrm{K}_3(-10\mathrm{A}_1 t + 3\mathrm{C})^2}{900\mathrm{A}_1}, \end{aligned} \tag{82}$$

$$\begin{aligned} u_2(x,y,t) = & \frac{72}{\mathrm{A}_1^2}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-2/5} P_k\left((-1)^{k+1}\frac{\mathrm{A}_1}{6}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-2/5} x\right. \\ & - \mathrm{A}_1\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-6/5} y^2 + (2\mathrm{A}_1 t + \mathrm{C}_7)\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-6/5} y \\ & - \frac{3\mathrm{C} + 5\mathrm{C}_7}{100\mathrm{A}_1}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-6/5} + \mathrm{C}_8\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-3/5} \\ & \left. + \frac{21}{100\mathrm{A}_1}\left(\frac{-10\mathrm{A}_1 t + 3\mathrm{C}}{3}\right)^{-4/5}\right). \end{aligned} \tag{83}$$

The solution (82) provides a more general form compared to solutions (44), (53) and (58) in [24], since the solution (82) includes more parameters, different parameters can lead to different solutions, which can cover some existing solutions and show the generality of our solutions.

Case (2c): $\mathbf{A_1 \neq 0, \Gamma_8(z) = 3A_1}$

Under this constraint, solving the system gives:

$$c_1(t) = 0, \tag{84}$$

$$c_2(t) = C_9\theta(t), \tag{85}$$

$$c_3(t) = \mathrm{C}_{10}\theta(t) + \frac{5\mathrm{C}_9^2 + 6\mathrm{C}_9 + 6}{20\mathrm{A}_1}\theta^{-4}(t) - \frac{\mathrm{B}_1}{\mathrm{A}_1}. \tag{86}$$

.

The similarity reduction in this case simplifies to

$$u(x,y,t) = \frac{\theta^2(t)\omega(z)}{3} - \frac{C_9}{6}, \tag{87}$$

$$z(x,y,t) = \theta(t)x + c_2(t)y + c_3(t), \tag{88}$$

The characteristic ordinary differential equation for $\omega(z)$ is now

$$(\omega'')^2 + \omega^2\omega'' + (A_1 z + B_1)\omega'' + 3A_1\omega' = 0. \tag{89}$$

A particular rational function solution to this ODE is

$$\omega(z) = -\frac{6A_1^2}{(A_1 z + B_1)^2}, \tag{90}$$

Substituting this, along with (47), (85), (86), (88) and (90), back into (87) yields the following explicit solution of Eq. (1):

$$u(x,y,t) = -\frac{2A_1^2}{\left(A_1 x + A_1 C_9 y + \frac{5C_9^2+6C_9+6}{60}(-10A_1 t + 3C) + A_1 C_{10}\right)^2} - \frac{C_9}{6}. \tag{91}$$

2.2 The case $z_x = 0, z_y \neq 0$

Under the condition $z_x = 0$, substituting the ansatz (2) into (1) reduces the governing equation to the following nonlinear ordinary differential equation:

$$\gamma_0\omega'' + \gamma_1\omega\omega' + \gamma_2\omega' + \gamma_3\omega^3 + \gamma_4\omega^2 + \gamma_5\omega + \gamma_6 = 0, \tag{92}$$

where the coefficients γ_i are given by

$$\begin{aligned}
\gamma_0 &= \beta z_y^2, \\
\gamma_1 &= 2\beta\beta_x z_y, \\
\gamma_2 &= 2\alpha\beta_x z_y + \beta z_{yy} + \beta_x z_y + 2\beta_y z_y + \beta_x z_t, \\
\gamma_3 &= 6\beta^2\beta_{xx}, \\
\gamma_4 &= 6\beta^2\alpha_{xx} + 12\alpha\beta\beta_{xx} + 2\beta\beta_{xy} + 2\beta_{xx}^2 \\
\gamma_5 &= 12\alpha\beta\alpha_{xx} + 6\alpha^2\beta_{xx} + 2\beta\alpha_{xy} + 2\alpha\beta_{xy} + 4\alpha_{xx}\beta_{xx} + \beta_{tx} + \beta_{xx} + \beta_{xy} + \beta_{yy}, \\
\gamma_6 &= 6\alpha^2\alpha_{xx} + 2\alpha\alpha_{xy} + 2\alpha_{xx}^2 + \alpha_{xx} + \alpha_{xy} + \alpha_{yy} + \alpha_{xt}.
\end{aligned} \tag{93}$$

The normalization constraint in this case is

$$\gamma_i = \gamma_0\Gamma_i(z), \ (i = 1, 2, \ldots, 6). \tag{94}$$

The subsequent analysis of the compatibility conditions yields the following:

$$2\beta_x = z_y\Gamma_1(z). \tag{95}$$

Integrating (95) with respect to x gives

$$\beta = \frac{1}{2}\Gamma_1(z)z_y x + \Sigma(y,t). \tag{96}$$

Without loss of generality, we set $\Sigma(y,t) = 0$ by invoking Remark (ii), which leads to

$$\beta = \frac{1}{2}xz_y, \qquad \Gamma_1(z) = 1. \tag{97}$$

For $i = 2$, condition (94) gives

$$2\alpha\beta_x z_y + \beta z_{yy} + \beta_x z_y + 2\beta_y z_y + \beta_x z_t = \beta z_y^2\Gamma_2\left(z\right). \tag{98}$$

Substituting the results from (98) into (99) and solving for α yields

$$\alpha = -\frac{z_{yy}}{z_y}x - \frac{z_t}{2z_y} - \frac{1}{2} + \frac{1}{2}z_y^3\Gamma_2(z) \tag{99}$$

Applying Remark (i) allows us to set $\Gamma_2(z) = 0$, which simplifies α to

$$\alpha = -\frac{z_{yy}}{z_y}x - \frac{z_t}{2z_y} - \frac{1}{2}. \tag{100}$$

For $i = 3$ in (94), direct substitution gives: $\Gamma_3(z) = 0$.
The condition for $i = 4$ is

$$6\beta^2\alpha_{xx} + \beta z_{yy} = \beta z_y^2\Gamma_4\left(z\right). \tag{101}$$

Substituting the expressions for β and α from (99) and (100) into (101) leads to

$$z_{yy} = z_y^2\Gamma_4(z) \tag{102}$$

Applying Remark (iv) to Eq. (102) yields the form

$$z = y\mu(t) + \nu(t). \tag{103}$$

Consequently, from (97), (100) and (103), we obtain the simplified expressions

$$\alpha = -\frac{y\mu'(t) + \nu'(t)}{2\mu(t)}, \qquad \beta = \frac{1}{2}x\mu(t) \tag{104}$$

Similarly, substituting the results from (2.2) into the condition (93) for $i = 5$ and $i = 6$ gives:

$$\frac{1}{2}\mu^3(t)x\Gamma_5(z) = \frac{1}{2}\mu'(t), \tag{105}$$

$$\Gamma_6(z) = 0. \tag{106}$$

Since $\Gamma_5(z)$ is a function of z only, while the right-hand side of (105) is a function of t, and the left-hand side has an explicit linear dependence on x through $z = y\mu(t) + \nu(t)$, consistency requires that $\mu'(t) = 0$. Thus, $\mu(t)$ is a constant, which we denote as L ($\mathrm{L} \neq 0$). The characteristic variable then simplifies to $z = \mathrm{L}y + \nu(t)$, where L is a nonzero arbitrary constant.

With $\mu(t) = \mathrm{L}$, the functions α and β in (103) simplify to

$$\alpha(x, y, t) = -\frac{1}{2\mathrm{L}}\nu'(t) - \frac{1}{2}, \tag{107}$$

$$\beta(x, y, t) = \frac{\mathrm{L}}{2}x, \tag{108}$$

. Therefore, the similarity reduction takes the final form

$$u(x, y, t) = \frac{\mathrm{L}}{2}x\omega(z) - \frac{1}{2} - \frac{1}{2\mathrm{L}}\nu'(t), \tag{109}$$

where $\omega(z)$ satisfies the reduced ordinary differential equation

$$\omega'' + \omega\omega' = 0. \tag{110}$$

Equation (110) admits the following set of solutions:

$$\omega_1(z) = \mathrm{C}_1, \tag{111}$$

$$\omega_2(z) = \frac{2}{z + \mathrm{C}_2}, \tag{112}$$

$$\omega_3(z) = \mathrm{l}_1 \tanh\left(\frac{\mathrm{l}_1}{2}z + \mathrm{C}_3\right), \tag{113}$$

$$\omega_4(z) = -\mathrm{l}_2 \tan\left(\frac{\mathrm{l}_2}{2}z + \mathrm{C}_4\right). \tag{114}$$

where $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4 \in \mathbb{R}$ are the integration constant, and both l_1 and l_2 are positive constants.

Substituting these solutions for $\omega(z)$, along with $z = \mathrm{L}y + \nu(t)$, into the similarity form (109) yields the following exact solutions of (1):

$$u_1(x, y, t) = -\frac{1}{2} - \frac{1}{2\mathrm{L}}\nu'(t) + \frac{\mathrm{C}_1\mathrm{L}}{2}x, \tag{115}$$

$$u_2(x, y, t) = -\frac{1}{2} - \frac{1}{2\mathrm{L}}\nu'(t) + \frac{\mathrm{L}x}{\mathrm{L}y + \nu(t) + \mathrm{C}_2}, \tag{116}$$

$$u_3(x,y,t) = -\frac{1}{2} - \frac{1}{2\mathrm{L}}\nu'(t) + \frac{\mathrm{l}_1\mathrm{L}}{2}x\tanh\left(\frac{\mathrm{l}_1}{2}\left(\mathrm{L}y + \nu(t)\right) + \mathrm{C}_3\right), \tag{117}$$

$$u_4(x,y,t) = -\frac{1}{2} - \frac{1}{2\mathrm{L}}\nu'(t) - \frac{\mathrm{l}_2\mathrm{L}}{2}x\tan\left(\frac{\mathrm{l}_2}{2}(\mathrm{L}y + \nu(t)) + \mathrm{C}_4\right). \tag{118}$$

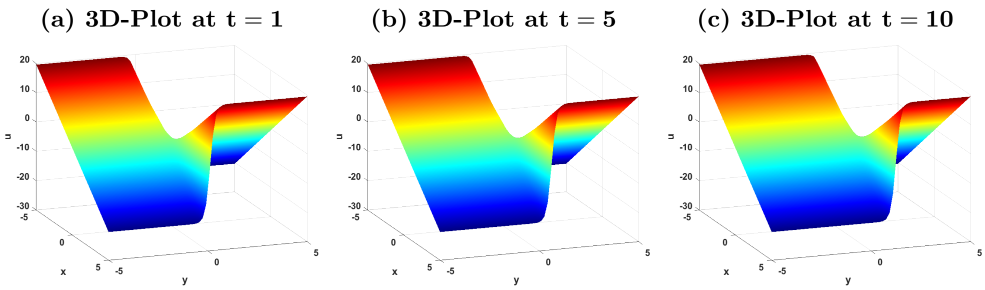

Fig. 6 3D-plots of the solution (117) are depicted at $\nu(t) = -t$, $\mathrm{L} = 0.8$, $\mathrm{l}_1 = 10$, and $\mathrm{C}_3 = 0$ within the interval $-5 \leq x, y \leq 5$ for $t = 1$, $t = 5$ and $t = 10$.

The solution (117) includes a dark soliton structure, as illustrated in Fig. 6. This dark soliton propagates steadily in the positive y direction while maintaining its form without distortion, thereby demonstrating a stable state.

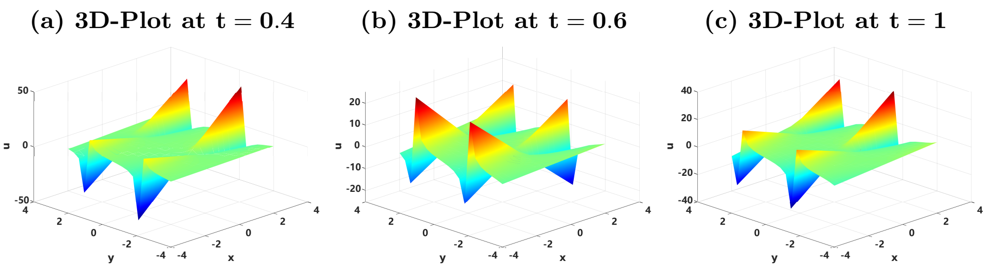

Fig. 7 3D-plots of the solution (118) are depicted at $\nu(t) = -t$, $\mathrm{L} = 1$, $\mathrm{l}_2 = 2$ and $\mathrm{C}_4 = 0$ within the interval $-3 \leq x, y \leq 3$ for $t = 0.4$, $t = 0.6$, $t = 1$.

The solution (118) is expressed in the trigonometric solution. Fig. 7 exhibits this solution, which consists of discrete surfaces, showing the propagation of the trigonometric wave with wave crest along the positive y direction.

3 Conclusion and discussion

In this work, we have, for the first time, successfully applied the Clarkson–Kruskal (CK) direct method to the extended (2+1)-dimensional Sakovich equation. Our systematic investigation yields a wide variety of exact solutions and similarity reductions, which are comprehensively categorized and assessed in Table 1. These results demonstrate the efficacy of the CK method in uncovering the intricate solution structure of this nonlinear model and underscore the computational challenges inherent in the process.

The obtained solutions naturally separate into two families based on the condition of the similarity variable z_x, reflecting underlying symmetries of the equation. A significant outcome is the independent recovery of the Weierstrass elliptic function solution (62) via a different methodological route, which corroborates previous results and validates the reliability of our approach. More importantly, the CK method enabled us to derive more general forms of known rational solutions, such as (61) and (82). The additional free parameters in these solutions extend the solution families, allowing for the description of waves under broader initial conditions and providing a continuous bridge between distinct wave types.

Furthermore, this study reports new types of solutions previously undocumented for this equation. The hyperbolic function solution (117) represents a stable dark soliton—a localized wave characterized by an intensity dip. The trigonometric solution (118) indicates that the equation supports periodic wave trains, expanding the known repertoire of its dynamical behaviors. The rational function solution (37) describes a singular line wave, whose structure may offer insights into wave-breaking or focusing phenomena in related physical contexts. The new similarity reduction to Painlevé equations (83) also suggests deeper integrability properties worthy of further exploration.

Future work may focus on several promising directions. First, a numerical study of the evolution of the dark soliton (117) and periodic wave (118) could confirm their stability and interaction properties. Second, the methodology employed here could be tested on other members of the extended Sakovich equation family. Finally, exploring potential connections between the free parameters in our generalized solutions and specific physical boundary conditions would enhance the applicability of these results.

In conclusion, this study not only enriches the set of known analytical solutions for the extended (2+1)-dimensional Sakovich equation but also provides a clearer map of its solution space through systematic classification. The new and generalized solutions we present may serve as a basis for interpreting complex wave phenomena in applicable physical systems.

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Data Availability Statement

NO Data associated in the manuscript.

Table 1 Summary and new assessment of exact solutions obtained for for the extended (2+1)-dimensional Sakovich equation via the CK direct method

Solution	**Math Form / Key Feature**	**Relationship to Prior Work (Ref. [15])**	**New Declaration & Remarks**
Case 1: Solutions obtained under the condition $z_x \neq 0$			
(61)	Rational function (inverse square)	More general than Eqs. (42) and (79) in [24].	**Generalized form.** Extra parameters $(\theta_0, C_2, C_3, C_5)$ yield a broader family, encompassing known solutions as special cases (see Fig. 2).
(62)	Weierstrass elliptic function	Essentially identical to Eqs. (112) and (116) in [15].	**Independent verification.** Cross-validates the result via the CK method, confirming its correctness and method effectiveness (see Fig. 3,Fig. 4,Fig. 5.
(82)	Rational function (complex form with time-dependent coefficients)	More general than Eqs. (44), (53), and (58) in [24].	**Generalized form.** Additional parameters C_1, C, C_7, C_8 reveal richer solution structures.
(83)	Similarity reduction to Painlevé I/II equations ((85))	Not mentioned in [15].	**New similarity reduction.** Transforms PDE into ODE with Painlevé transcendents, uncovering new integrability aspects.
Case 2: Solutions obtained under the condition $z_x = 0$			
(117)	Hyperbolic function (tanh form)	Not mentioned in [24].	**New solution.** Represents a stable **dark soliton** propagating along y-direction.
(118)	Trigonometric function (tan form)	Not mentioned in [24].	**New solution.** Describes a periodic wave train.

Conflict of interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

References

[1] G. Wang, A.M. Wazwaz, Symmetry and Painlevé analysis for the extended Sakovich equation, *International Journal of Numerical Methods for Heat & Fluid Flow*, **31**(1), 541-547 (2021). https://doi.org/10.1108/HFF-04-2020-0235

[2] S. Sakovich, A new Painlev e-integrable equation possessing KdV-type solitons, Nonlinear Phenomena Complex Syst, **22**, 299-304 (2019). https://doi.org/10.48550/arXiv.1907.01324

[3] Ahmad, J., Hameed, M., Mustafa, Z. et al. A study of traveling wave solutions and modulation instability in the (3+1)-dimensional Sakovich equation employing advanced analytical techniques. *Sci Rep*, **15**, 23332 (2025). https://doi.org/10.1038/s41598-025-00503-7

[4] Q. Cao, S. Yu, L. Xu, Y.-Y. Zhai, Spin-exchange relaxation-free magnetometer enhanced by biased weak measurement, *Results in Physics*, **60**, 107627 (2024). https://doi.org/10.1016/j.rinp.2024.107627

[5] R. Hossain, M. Sagib, M. A. Hossain, S. M. R. Islam, Soliton solutions, bifurcation, sensitivity and chaotic analysis of the nonlinear combined Kairat-II-X wave model, *Physics Letters A*, **562**, 131010 (2025). https://doi.org/10.1016/j.physleta.2025.131010

[6] M. H. Bashar, S. M. R. Islam, Exact solutions to the (2 + 1)-Dimensional Heisenberg ferromagnetic spin chain equation by using modified simple equation and improve F-expansion methods, *Physics Open*, **5**, 100027 (2020). https://doi.org/10.1016/j.physo.2020.100027

[7] M. M. Rahman, S. M. R. Islam, A. Hoque, Investigations of soliton structures and dynamical behaviors of the Westervelt equation with two analytical techniques, *AIP Advances*, **15**(5), 055234 (2025). https://doi.org/10.1063/5.0271307

[8] S. M. R. Islam, S. M. Y. Arafat, M. Inc, Exploring novel optical soliton solutions for the stochastic chiral nonlinear Schrödinger equation: Stability analysis and impact of parameters, *Journal of Nonlinear Optical Physics & Materials*, **34**(05), 2450009 (2025). https://doi.org/10.1142/S0218863524500097

[9] S. M. R. Islam, M. E. Islam, M. A. Akbar, D. Kumar, The stretch coordinate effect, bifurcation, and stability analysis of the nonlinear Hamiltonian amplitude equation, *Partial Differential Equations in Applied Mathematics*, **13**, 101126 (2025). https://doi.org/10.1016/j.padiff.2025.101126

[10] S. M. Y. Arafat, S. M. R. Islam, Bifurcation analysis and soliton structures of the truncated M-fractional Kuralay-II equation with two analytical techniques, *Alexandria Engineering Journal*, **105**, 70-87 (2024). https://doi.org/10.1016/j.aej.2024.06.079

[11] S. M. R. Islam, Bifurcation analysis and soliton solutions to the doubly dispersive equation in elastic inhomogeneous Murnaghan's rod, *Scientific Reports*, **14**, 11428 (2024). https://doi.org/10.1038/s41598-024-62113-z

[12] S. M. Y. Arafat, M. Asif, S. M. R. Islam, M. A. Saklayen, M. M. Rahman, Investigating travelling wave solutions of the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation through the two analytical techniques, *Physica Scripta*, **100**(1), 015285 (2025). https://doi.org/10.1088/1402-4896/ad9ee9

[13] Zhang B, Gao B. Investigation of exact solutions to the (2+ 1)-dimensional Sakovich equation with time dependent coefficients using analytical methods[J]. *Physica Scripta*, **99(11)**: 115203, (2024). https://doi.org/10.1088/1402-4896/ad7d4c

[14] N. Raza, F. Salman, A. R. Butt, et al. Lie symmetry analysis, soliton solutions and qualitative analysis concerning to the generalized q-deformed Sinh-Gordon equation, *Communications in Nonlinear Science and Numerical Simulation*, **116**, 106824 (2023). https://doi.org/10.1016/j.cnsns.2022.106824

[15] S. Zhang, X. Zheng, N-soliton solutions and nonlinear dynamics for two generalized Broer–Kaup systems, *Nonlinear Dynamics*, **107**(1), 1179-1193 (2022). https://doi.org/10.1007/s11071-021-07030-w

[16] M. Wang, J. Zhang, X. Li, N-dimensional Auto-Bäcklund transformation and exact solutions to n-dimensional Burgers system, *Applied Mathematics Letters*, **63**, 46-52 (2017). https://doi.org/10.1016/j.aml.2016.07.019

[17] Y. Zhang, Y. Liu, Darboux transformation and explicit solutions for 2+ 1-dimensional nonlocal Schrödinger equation, *Applied Mathematics Letters*, **92**, 29-34 (2019). https://doi.org/10.1016/j.aml.2018.12.014

[18] Y. Li, B. Hu, L. Zhang, et al., The exact solutions for the nonlocal Kundu-NLS equation by the inverse scattering transform, *Chaos, Solitons & Fractals*, **180**, 114603 (2024). https://doi.org/10.1016/j.chaos.2024.114603

[19] H. Wang, Numerical simulation for the solitary wave of Zakharov–Kuznetsov equation based on lattice Boltzmann method, *Applied Mathematical Modelling*, **45**, 1-13 (2017). https://doi.org/10.1016/j.apm.2016.12.002

[20] Z. G. Liu, J. Zhang, Y. S. Wang, et al., Analytical solutions of solitary waves and their collision stability in a pre-compressed one-dimensional granular crystal, *Nonlinear Dynamics*, **104**(4), 4293-4309 (2021). https://doi.org/10.1007/s11071-021-06534-9

[21] Y. Wang, L. Tian, Y. Zhang, Negative order mKdV equation and N-solitons, *Applied Mathematics Letters*, **146**, 108802 (2023). https://doi.org/10.1016/j.aml.2023.108802

[22] Y. Sun, X. Y. Wu, Studies on the breather solutions for the (2+ 1)-dimensional variable-coefficient Kadomtsev–Petviashvili equation in fluids and plasmas, *Nonlinear Dynamics*, **106**(3), 2485-2495 (2021). https://doi.org/10.1007/s11071-021-06917-y

[23] M. A. Ilyas, A. Javid, A. M. Wazwaz, Painlevé analysis, multi-kinks and hybrid nonlinear waves for an extended (3+ 1)-dimensional B-type Kadomtsev-Petviashvili equation,*Physica Scripta*, **98**(9), 095229 (2023). https://doi.org/10.1088/1402-4896/acecb2

[24] Y.S. Özkan, E. Yaşar, Multiwave and interaction solutions and Lie symmetry analysis to a new (2+ 1)-dimensional Sakovich equation, *Alexandria Engineering Journal*, **59**(6), 5285-5293 (2020). https://doi.org/10.1016/j.aej.2020.10.014

[25] S. Kumar, S. Rani, N. Mann, Diverse analytical wave solutions and dynamical behaviors of the new (2+1)-dimensional Sakovich equation emerging in fluid dynamics, *The European Physical Journal Plus*, **137** (2022) 1226. https://doi.org/10.1140/epjp/s13360-022-03397-w

[26] A. H. Arnous, M. S. Hashemi, K. S. Nisar, et al., Investigating solitary wave solutions with enhanced algebraic method for new extended Sakovich equations in fluid dynamics, *Results in Physics*, **57**, 107369 (2024). https://doi.org/10.1016/j.rinp.2024.107369

[27] P. A. Clarkson, M. D. Kruskal, New similarity reductions of the Boussinesq equation, *Journal of Mathematical Physics*, **30**(10), 2201-2213 (1989). https://doi.org/10.1063/1.528613

[28] S. Y. Lou, X. Y. Tang, L. Ji, Similarity and conditional similarity reductions of a (2+ 1)-dimensional KdV equation via a direct method, *Journal of Mathematical Physics*, **41**(12), 8286–8303 (2000). https://doi.org/10.1063/1.1320859

[29] H. Wang, H. Y. Tian, H. L. Chen, Non-Lie symmetry group and new exact solutions for the two-dimensional KdV-Burgers equation, *Chinese Physics Letters*, **28**(2), 020225 (2011). https://doi.org/10.1088/0256-307x/28/2/020205

[30] H. Liu, B. Sang, X. Xin et al., CK transformations, symmetries, exact solutions and conservation laws of the generalized variable-coefficient KdV types of equations, *Journal of Computational and Applied Mathematics*, **(**345), 127–134 (2019). https://doi.org/10.1016/j.cam.2018.06.019

[31] B. Q. Zhao, S. W. Liu, Extended direct method and new similarity solutions of Kadomtsev Petviashvili equation, *Computational Mathematics and Mathematical Physics*, **64**(9), 2045-2065 (2024). https://doi.org/10.1134/S0965542524701148

[32] K. Zhang, S. Tang, Z. Wang, Bifurcation of travelling wave solutions for the generalized Camassa–Holm–KPequations, *Communications*

in Nonlinear Science and Numerical Simulation, **15**(3), 564-572 (2010). https://doi.org/10.1016/j.cnsns.2009.04.027

[33] M. M. A. Khater, S. H. Alfalqi, J. F. Alzaidi et al, Computational and numerical simulations; the generalized (2+1)-dimensional Camassa–Holm–Kadomtsev–Petviashvili equation, *Results in Physics*, **(**52), 106876 (2023). https://doi.org/10.1016/j.rinp.2023.106876