

New Exact Solutions, Similarity Reductions and Wave Phenomena for the extended (2+1)-dimensional Sakovich equation

Abstract

This study employs the Clarkson-Kruskal (CK) direct method to investigate exact solutions of the extended (2+1)-dimensional Sakovich equation. By reviewing the relevant literature, no one has used the CK direct method to solve the extended (2+1)-dimensional Sakovich equation. In this work, we have successfully overcome this complex and tedious computations of CK direct method. We implement a decoupling strategy that effectively reduces complexity and obtains new similarity reductions and new exact solutions. The results are classified into two distinct cases. In the first case, the obtained solutions encompass rational function solutions, the Weierstrass elliptic function solution, and the new similarity reductions for Painlevé I and Painlevé II. The second case yields new solutions that include trigonometric functions, logarithmic expressions, and components of the Bessel function. To the best of our knowledge, some of the solutions obtained have not been previously reported. Among these new solutions, in particular, the logarithmic function type and the Bessel function type solutions exhibit two new types, which do not appear in similar integrable equations. All of these solutions manifest diverse wave behaviors, such as soliton, dark soliton, and Bessel function type. Some of them reveal new wave phenomena governed by the the extended (2+1)-dimensional Sakovich equation.

Keywords: The extended (2+1)-dimensional Sakovich equation, Exact solutions, the CK direct method, Similarity reductions

1 Introduction

In this manuscript, we mainly consider the extended (2+1)-dimensional Sakovich equation:

$$u_{xt} + u_{yy} + u_{xx} + u_{xy} + 2uu_{xy} + 6u^2u_{xx} + 2(u_{xx})^2 = 0. \quad (1)$$

Eq. (1) was developed by Wazwaz [1] as an extension of the original Sakovich's foundational equation [2]. By adding two new terms u_{xx} and u_{yy} , (1) has a wider applicability since it can describe more dispersion and nonlinear effects to adapt to more general application scenarios. The original Sakovich's foundational equation finds applications in wide areas of plasma physics, mathematics, especially in the theory of waves, soliton theory, and nonlinear dispersive systems, which include the examination of rogue waves in oceanography. It also helps to investigate the model of the sudden large waves and the behavior of water waves within a long, narrow, hollow tube. [3]

Effective methods for solving NLPDEs include the unified method [4], the Hirota bilinear technique [5], the Auto-Bäcklund/Darboux transformations [6, 7], and the inverse scattering transform [8]. These approaches yield diverse solutions, such as solitary waves, solitons, breathers, kinks, and lump solitons [9–13]. Today, many researchers have studied the extended (2+1)-dimensional Sakovich equation (1) using a variety of powerful methods and the corresponding exact solutions have been obtained, such as Özkan et al. [14] by using the multiwaves method, double exponential form, the homoclinic breather approach and the Lie symmetry technique to obtain multiwave and interaction solutions. Sachin Kumar et al. [15] employed the Lie symmetry technique and the extended Jacobi elliptic functions method to derive solutions of more generalized than the previous established results. Arnous et al. [16] implementation of an enhanced extended algebraic framework yielded multiple exact solutions.

It is well known that a powerful method for dealing with similarity reductions and exact solutions to NLPDEs [17–23] is the Clarkson-Kruskal(CK) direct method, which was first proposed by Clarkson and Kruskal [17]. Compared to other methods, such as the Lie symmetry method, the unified method, the Hirota bilinear technique, etc, the CK direct method is capable of yielding more types of solutions. Despite the complex and tedious computations of the CK direct method, the solutions obtained by the CK method are more diverse, especially including rational-form solutions, lump solutions, soliton solutions, etc. In this work, by employing the CK direct method, we have obtained solutions of logarithmic and Bessel function types, which, to the best of our knowledge, have not been reported via other approaches thus far. The CK direct method mainly results in finding a solution to a special form of partial differential equation. In general, NLPDEs admit a wide spectrum of solutions, then the CK direct method postulates a solution of the following specific form (taking the (2 + 1)-dimension as an example):

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)\omega(z), \quad (2)$$

where α, β and z are functions of x, y, t to be determined and ω satisfies a certain reduction equation that is lower than the dimension of the original equation.

In this paper, we mainly use the CK direct method to find new similarity reductions and exact solutions of the equation (1). Based on existing literature,

the CK direct method has not been applied to the study of the extended (2+1)-dimensional Sakovich equation. In this work, we have successfully overcome this complex and tedious computation of the CK direct method. For general partial differential equations, even low-order ones, it is often challenging to directly determine β (or z) in the form of (2) using the CK direct method. A key challenge may arise when the coefficient of the highest-order term is used as the normalizing coefficient to derive the coefficients of the remaining terms. When the remaining terms consist entirely of polynomial combinations, their coupled nature may complicate direct resolution. To address this, we adopt the method proposed in [24], which decomposes the polynomial combinations into individual components. This method significantly reduces computational cost and complexity. By sequentially solving the forms of β (or z) from each decomposed term and ultimately requiring the original equation to hold, we simply take the intersection of all possible solutions of β (or z).

Next, we reduced the extended (2+1)-dimensional Sakovich equation (1) to an ordinary differential equation of ω with respect to z , and obtained new similarity reductions and some exact solutions, through a large number of complex and tedious calculations. To our knowledge, some of these solutions have not been reported in the previous literature. These exact solutions show the richness of solutions to (1) and reveal new wave phenomena governed by (1), such as soliton, dark soliton, and the Bessel function type. Among these solutions, some have global finite energy, others have local finite energy which blow up along some line. This also indicates some new physical phenomena. The result can be classified into two distinct cases.

- In the first case where $z_x \neq 0$, we derive rational function solutions (Eqs. (38), (62), (63), (84), (93)), a Weierstrass elliptic function solution (Eq. (64)), and the Painlevé I and Painlevé II similarity reductions (Eq. (85)). Based on our understanding of the established literature, we have found that the solution (64) is almost identical to the solutions (112) and (116) in [15] obtained through the extended Jacobian elliptic function expansion method. Two different methods yield almost the same solutions, which, to some extent, shows the validity of our research. Furthermore, solution (63) provides a more general form compared to solutions (42) and (79) in [15], and solution (84) provides a more general form compared to solutions (44), (53) and (58) in the same reference [15]. Because solutions (63) and (84) include more parameters, different parameters can lead to different solutions, which can cover some existing solutions and show the generality of our solutions.
- In the second case with $z_x = 0$, we obtain rational function solutions (Eqs. (129), (130)), a hyperbolic function solution (Eq. (131)), a trigonometric periodic solution (Eq. (132)), a logarithmic function solution (Eq. (151)) and Bessel function solutions (Eqs. (143), (148)). Specifically, As far as we are aware, two new types, solutions of the logarithmic type and solutions of the Bessel function type, are obtained, which have not been previously reported in similar integrable equations. These new types of solutions reveal new wave phenomena governed by equation (1), which will be presented in detail in Section 2. It is helpful to explore the significance inherent in the extended (2+1)-dimensional Sakovich equation.

The remainder of this paper is organized as follows. In Section 2, through the CK direct method, we discuss and obtain the similarity reduction and new solutions of the extended (2+1)-dimensional Sakovich equation in the case of $z_x \neq 0, z_y \neq 0$ and $z_x = 0, z_y \neq 0$. For the solutions obtained, we provide both 3D and 2D plots, which collectively enhance the observation of their spatial variations. Finally, the conclusion and discussion will be given in Section 3.

2 Symmetry reductions and exact solutions of the extended (2+1)-dimensional Sakovich equation

In this section, we will perform the calculations.

Substituting (2) into (1) and collecting coefficients of monomials of ω and its derivatives yields the following:

$$\begin{aligned} & \gamma_0(\omega'')^2 + \gamma_1\omega'\omega'' + \gamma_2\omega^2\omega'' + \gamma_3\omega\omega'' + \gamma_4\omega'' + \gamma_5(\omega')^2 + \\ & \gamma_6\omega^2\omega' + \gamma_7\omega\omega' + \gamma_8\omega' + \gamma_9\omega^3 + \gamma_{10}\omega^2 + \gamma_{11}\omega + \gamma_{12} = 0, \end{aligned} \quad (3)$$

where

$$\begin{aligned} \gamma_0 &= 2\beta^2 z_x^4, \\ \gamma_1 &= 4\beta^2 z_x^2 z_{xx} + 8\beta\beta_x z_x^3, \\ \gamma_2 &= 6\beta^3 z_x^2, \\ \gamma_3 &= 12\alpha\beta^2 z_x^2 + 2\beta^2 z_x z_y + 4\beta\beta_{xx} z_x^2, \\ \gamma_4 &= 6\alpha^2 \beta z_x^2 + 4\beta\alpha z_{xx}^2 + 2\alpha\beta z_x z_y + \beta z_x^2 + \beta z_x z_y + \beta z_x z_t + \beta z_y^2, \\ \gamma_5 &= 2\beta(z_{xx})^2 + 8\beta\beta_x z_x z_{xx} + 8\beta^2 z_x^2, \\ \gamma_6 &= 6\beta^3 z_{xx} + 12\beta^2 \beta_x z_x, \\ \gamma_7 &= 12\alpha\beta^2 z_{xx} + 24\alpha\beta z_x + 2\beta^2 z_{xy} + 2\beta\beta_x z_y + 2\beta\beta_y z_x + 4\beta\beta_{xx} z_{xx} + 8\beta_x \beta_{xx} z_x, \\ \gamma_8 &= 6\alpha^2 \beta z_{xx} + 12\alpha^2 \beta_x z_x + 2\alpha\beta z_{xy} + 4\beta\alpha_{xx} z_{xx} + 8\alpha_{xx} \beta_x z_x + 2\alpha\beta_y z_x + \beta z_{xt} \\ & \quad + \beta z_{xy} + \beta z_{yy} + \beta z_{xx} + 2\beta_x z_x + \beta_t z_x + \beta_y z_x + \beta_x z_y + 2\beta_y z_y + \beta_x z_t, \\ \gamma_9 &= 6\beta^2 \beta_{xx}, \\ \gamma_{10} &= 6\beta^2 \alpha_{xx} + 12\alpha\beta\beta_{xx} + 2\beta\beta_{xy} + 2\beta_{xx}^2, \\ \gamma_{11} &= 12\alpha\beta\alpha_{xx} + 6\alpha^2 \beta_{xx} + 2\beta\alpha_{xy} + 2\alpha\beta_{xy} + 4\alpha_{xx}\beta_{xx} + \beta_{xt} + \beta_{xx} + \beta_{yy} + \beta_{xy}, \\ \gamma_{12} &= 6\alpha^2 \alpha_{xx} + 2\alpha\alpha_{xy} + 2\alpha_{xx}^2 + \alpha_{xx} + \alpha_{xy} + \alpha_{yy} + \alpha_{xt}. \end{aligned} \quad (4)$$

and $' := d/dz$. In order that Eq. (3) be an ordinary differential equation for $\omega(z)$, the ratios of coefficients of different derivatives and powers of $\omega(z)$ have to be functions of z only. If $z_x \neq 0, z_y \neq 0$, these conditions read

$$\gamma_i = \gamma_0 \Gamma_i(z) \quad (i = 1, 2, \dots, 12), \quad (5)$$

where $\Gamma_i(z)$ ($i = 1, 2, \dots, 12$) are some arbitrary functions of z to be determined later.

In the determination of $\alpha(x, y, t)$, $\beta(x, y, t)$, $z(x, y, t)$ and $\Gamma_i(z)$ ($i = 1, \dots, 12$), the following four remarks may be involved:

Remark (i): if $\alpha(x, y, t)$ has the form $\alpha = \alpha_0(x, y, t) + \beta(x, y, t)\Omega(z)$, then we can take $\Omega = 0$ (by substituting $\omega(z) \rightarrow \omega(z) - \Omega(z)$).

Remark (ii): if $\beta(x, y, t)$ has the form $\beta = \beta_0(x, y, t)\Omega(z)$, then we can take $\Omega = \Omega_0 = \text{constant}$ (by substituting $\omega(z) \rightarrow \omega(z)\Omega_0/\Omega(z)$).

Remark (iii): if $z(x, y, t)$ is determined by an equation of the form $\Omega(z) = z_0(x, y, t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ (by substituting $z \rightarrow \Omega^{-1}(z)$).

Remark (iv): we reserve Greek and English letters for undetermined functions of z (or x, y, t) so that after performing operations (differentiation, integration, exponentiation, rescaling, etc.) the result can be denoted by the same letter [e.g., the derivative of $\Gamma(z)$ will be called $\Gamma(z)$].

2.1 The case $z_x \neq 0, z_y \neq 0$

In this analysis, we employ the coefficient of $(\omega'')^2$ (specifically $2\beta^2 z_x^4$) as the normalization factor. This requires that all other coefficients adopt the form $2\beta^2 z_x^4 \Gamma_i(z)$, as established in Eq. (5).

Combining (5) with $i = 2$, we have the following

$$6\beta^3 z_x^2 = 2\beta^2 z_x^4 \Gamma_2(z). \quad (6)$$

Applying Remark (ii), we directly obtain:

$$\beta = \frac{1}{3} z_x^2, \quad \Gamma_2(z) = 1. \quad (7)$$

Substituting into (5) for $i = 1$, one can immediately get the following

$$4\beta^2 z_x^2 z_{xx} + 8\beta\beta_x z_x^3 = 2\beta^2 z_x^4 \Gamma_1(z). \quad (8)$$

Substituting (7) into (6), we have

$$z_{xx} = z_x^2 \Gamma_1(z). \quad (9)$$

Dividing by z_x and integrating (9) with respect to x , also applying Remark (iii) and Remark (iv) yields:

$$z = x\theta(y, t) + \sigma(y, t), \quad (10)$$

where θ and σ denote differentiable functions of y and t .

For $i = 3$, combining (5), (7) and (10), we derive:

$$\frac{4}{3}\alpha\theta^6 + \frac{2}{9}\theta^5 \left(x \frac{d\theta}{dy} + \frac{d\sigma}{dy} \right) = \frac{2}{9}\theta^8 \Gamma_3(z), \quad (11)$$

namely,

$$\alpha = -\frac{1}{6\theta}(x\theta_y + \sigma_y) + 2\beta\Gamma_3(z). \quad (12)$$

Applying Remark (iv) yields the following result:

$$\alpha = -\frac{1}{6\theta}(x\theta_y + \sigma_y), \quad \Gamma_3(z) = 0. \quad (13)$$

From the solutions in (7), (8), (9), we establish:

$$\Gamma_1(z) = \Gamma_5(z) = \Gamma_6(z) = \Gamma_9(z) = \Gamma_{10}(z) = 0. \quad (14)$$

Considering (5) with $i = 7$ under constraints (7), (10), (13), we obtain the following.

$$\frac{2}{3}\theta^4\theta_y = \frac{2}{9}\theta^8\Gamma_7(z), \quad (15)$$

the left-hand side of (15) is at most a function of y and t , which implies $\Gamma_7(z) = A$, where A is a constant. Two distinct cases emerge from this result.

Case 1. $A \neq 0$

In this case, we observe the fundamental relations:

$$\theta_y = z_{xy} = z_{yx}, \quad (16)$$

with the characteristic derivative:

$$z_x = \theta, \quad (17)$$

substituting (16) into (15) and integrating with respect to x , we obtain:

$$\frac{1}{3}\theta^3(Az + B) = z_y = x\theta_y + \sigma_y, \quad (18)$$

where B denotes an integration constant. Given the linear dependence of x on the right-hand side, we establish the component equations:

$$\frac{1}{3}(Ax\theta + A\sigma + B)\theta^3 = x\theta_y + \sigma_y. \quad (19)$$

The balance of coefficients of x yields the following:

$$\theta_y = \frac{1}{3}A\theta^4, \quad (20)$$

$$\sigma_y = \frac{1}{3}\theta^3(A\sigma + B). \quad (21)$$

Considering (5) with $i = 11$ under constraints (7), (10), (13), we have

$$\frac{1}{3}A^2\theta^8 = \frac{2}{9}\theta^8\Gamma_{11}(z), \quad (22)$$

we derive $\Gamma_{11}(z) = \frac{3}{2}A^2$.

Similarly, for $i = 4$ and $i = 8$, we have the following result:

$$\frac{5}{162}A^2\theta^{10}x^2 + \frac{5}{27}A\theta^6\sigma_yx + \frac{5}{18}\theta^2\sigma_y^2 + \frac{1}{3}\theta^3(\sigma_y + \sigma_t + \theta) = \frac{2}{9}\theta^8\Gamma_4(z), \quad (23)$$

$$\left(\theta\theta_y^2 + \frac{1}{3}\theta^2\theta_{yy}\right)x + \theta\theta_y\sigma_y + \frac{1}{3}\theta^2\sigma_{yy} + (\theta_y + \theta_t)\theta^2 = \frac{2}{9}\theta^8\Gamma_8(z). \quad (24)$$

To simplify the subsequent analysis, we impose the following.

$$\theta_y + \theta_t = 0, \quad (25)$$

$$\sigma_y + \sigma_t + \theta = 0. \quad (26)$$

Combining (20) and (25), we can derive the following.

$$\theta(y, t) = (At - Ay + C_1)^{-1/3}, \quad (27)$$

Substituting (27) into (21) and (26) admits an explicit solution:

$$\sigma(y, t) = \frac{-t + C_2}{(At - Ay + C_1)^{1/3}} - \frac{B}{A}, \quad (28)$$

where C_1 and C_2 are arbitrary integration constants.

Parallel analysis of $i = 7$ with $i \in \{4, 8, 12\}$ reveals the following.

$$\Gamma_4(z) = \frac{5}{36}(Az + B)^2, \quad (29)$$

$$\Gamma_8(z) = \frac{7A}{6}(Az + B), \quad (30)$$

$$\Gamma_{12}(z) = -\frac{17A^2}{36}(Az + B). \quad (31)$$

The similarity reduction of the extended (2+1)-dimensional Sakovich equation (1) consequently takes the form:

$$u(x, y, t) = \frac{\theta^2\omega(z)}{3} - \frac{x\theta_y + \sigma_y}{6\theta}, \quad (32)$$

$$z(x, y, t) = x\theta(y, t) + \sigma(y, t), \quad (33)$$

where $\theta(y, t)$ and $\sigma(y, t)$ satisfy (27), (28) and $\omega(z)$ governed by:

$$(\omega'')^2 + \omega^2\omega'' + \frac{5(Az + B)^2\omega''}{36} + A\omega\omega' + \frac{7A(Az + B)\omega'}{6} + \frac{3A^2\omega}{2} = \frac{17A^2(Az + B)}{36}. \quad (34)$$

Particular solutions of (34) include:

$$\omega_1(z) = \frac{1}{6}(Az + B), \quad (35)$$

$$\omega_2(z) = -\frac{17}{6}(Az + B). \quad (36)$$

Corresponding to these solutions, we obtain the extended (2+1)-dimensional Sakovich equation (1) solutions:

$$u_1(x, y, t) \equiv 0, \quad (37)$$

$$u_2(x, y, t) = \frac{x - t + C_2}{y - t - \frac{C_1}{A}}. \quad (38)$$

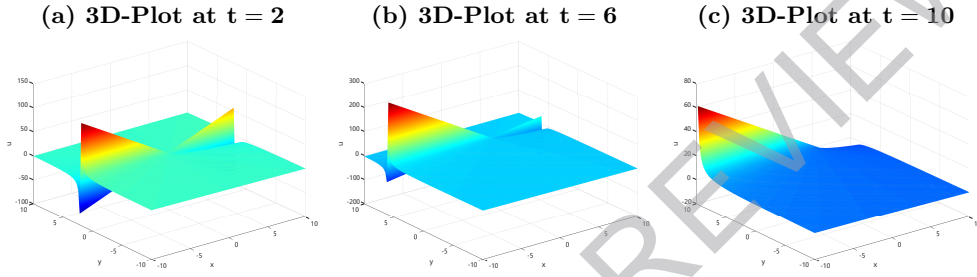


Fig. 1 3D-plots of the solution (38) are depicted at $A = 1$, $C_1 = 0$ and $C_2 = 1$ within the interval $-10 \leq x, y \leq 10$ for $t = 2$, $t = 6$ and $t = 10$.

Fig. 1 exhibits the evolution of the rational function solution (38), showing the propagation of the rational function wave with the line wave along the positive y -axis.

Case 2. $A = 0$

The vanishing coefficient condition in (15) implies: $\theta_y = 0$, and thus

$$\theta(y, t) = \theta(t), \quad (39)$$

with characteristic variable:

$$z = x\theta(t) + \sigma(y, t). \quad (40)$$

The nonlinear coefficients reduce to:

$$\begin{aligned}
\gamma_0 &= \frac{2}{9}\theta^8, \\
\gamma_2 &= \frac{2}{9}\theta^8, \\
\gamma_4 &= \frac{1}{3}x\theta^3\theta_t + \frac{5}{18}\theta^2\sigma_y^2 + \frac{1}{3}\theta^3(\sigma_y + \sigma_t + \theta), \\
\gamma_8 &= \theta^2\theta_t + \frac{1}{3}\theta^2\sigma_{yy}, \\
\gamma_{12} &= -\frac{\sigma_{yyy}}{6\theta}.
\end{aligned} \tag{41}$$

The normalization condition remains:

$$\Gamma_2(z) = 1. \tag{42}$$

For indices $i \in \{4, 8, 12\}$, the governing equations become:

$$\frac{1}{3}x\theta^3\theta_t + \frac{5}{18}\theta^2\sigma_y^2 + \frac{1}{3}\theta^3(\sigma_y + \sigma_t + \theta) = \frac{2}{9}\theta^8\Gamma_4(z), \tag{43}$$

$$\theta^2\theta_t + \frac{1}{3}\theta^2\sigma_{yy} = \frac{2}{9}\theta^8\Gamma_8(z), \tag{44}$$

$$-\frac{\sigma_{yyy}}{6\theta} = \frac{2}{9}\theta^8\Gamma_{12}(z). \tag{45}$$

Since the left-hand sides of both (44) and (45) are at most a function of y and t , while the right-hand side may involve x , it follows that $\Gamma_8(z)$ and $\Gamma_{12}(z)$ must be constant.

Integrating (45) over y yields the following.

$$\sigma(y, t) = -\frac{2\theta^9}{9}\Gamma_{12}(z)y^3 + \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t), \tag{46}$$

where $c_i(t) \in C^1(\mathbb{R})$, $i \in \{1, 2, 3\}$.

Substituting (46) into (43) reveals the polynomial consistency conditions:

$$\begin{aligned}
\frac{2\theta^8}{9}\Gamma_4(z) &= \frac{1}{3}x\theta^3\theta_t + \frac{5\theta^2}{18}\left(-\frac{2\theta^9}{3}\Gamma_{12}(z)y^2 + c_1(t)y + c_2(t)\right)^2 + \frac{\theta^3}{3}\left[-\frac{2\theta^9}{3}\Gamma_{12}(z)y^2\right. \\
&\quad \left.+ c_1(t)y + c_2(t)\left(-2\Gamma_{12}(z)\theta^8\theta_t y^3 + \frac{1}{2}c_1'(t)y^2 + c_2'(t)y + c_3'(t)\right)\right].
\end{aligned} \tag{47}$$

Polynomial balance requires $\Gamma_{12}(z) = 0$, $\Gamma_4(z) = A_1 z + B_1$, where A_1, B_1 are arbitrary constants, leading to:

$$\sigma(y, t) = \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t), \quad (48)$$

$$\theta(t) = (C - \frac{10A_1 t}{3})^{-1/5}. \quad (49)$$

The reduced system for $c_i(t)$ becomes:

$$\frac{5}{18}\theta^2 c_1^2(t) + \frac{1}{6}\theta^3 c_1'(t) = \frac{1}{9}A_1 \theta^8 c_1(t), \quad (50)$$

$$\frac{5}{9}\theta^2 c_1(t)c_2(t) + \frac{1}{3}\theta^3(c_1(t) + c_2'(t)) = \frac{2}{9}A_1 \theta^8 c_2(t), \quad (51)$$

$$\frac{5}{18}\theta^2 c_2^2(t) + \frac{1}{3}\theta^3(c_2(t) + c_3'(t) + \theta) = \frac{2}{9}(A_1 c_3(t) + B_1)\theta^8, \quad (52)$$

$$\frac{2}{3}A_1 \theta^8 + \frac{1}{3}\theta^2 c_1(t) = \frac{2}{9}\theta^8 \Gamma_8(z). \quad (53)$$

The structure of the bifurcations is based on the selection of the parameters Γ_8 and A_1 , requiring separate analysis for:

- Non-autonomous case $A_1 \neq 0$
- Autonomous case $A_1 = 0$

Case (2a): $A_1 = 0, \Gamma_8(z) = 0$

The degenerate condition yields: $c_1(t) = 0$, $\theta(t) = C^{-1/5} = \theta_0 \neq 0$. Substituting $c_1(t) = 0$ into (50), (51) produces solutions: $c_2(t) = C_2$,

$$c_3(t) = C_3 t + C_4, \quad (54)$$

$$\sigma(y, t) = C_2 y + C_3 t + C_4, \quad (55)$$

where $C_3 = \frac{2B_1 \theta_0^5}{3} - \frac{5C_2^2}{6\theta_0} - \theta_0 - C_2$, $C_2, C_4 \in \mathbb{R}$ are integration constants.

The similarity reduction of the extended (2+1)-dimensional Sakovich equation becomes:

$$u(x, y, t) = \frac{\theta_0^2 \omega(z)}{3} - \frac{C_2}{6\theta_0}, \quad (56)$$

$$z(x, y, t) = \theta_0 x + C_2 y + C_3 t + C_4, \quad (57)$$

where $\omega(z)$ satisfies the reduced ODE:

$$(\omega'')^2 + \omega^2 \omega'' + B_1 \omega'' = 0. \quad (58)$$

The set of solutions in (58) comprises:

$$\omega_1(z) = K_1 z + K_2, \quad (59)$$

$$\omega_2(z) = -\frac{6}{(z + C_5)^2}, \quad (60)$$

$$\omega_3(z) = -6\wp\left(z - z_0; g_2 = -\frac{B_1}{3}, g_3 = C_6\right), \quad (61)$$

where \wp denotes the Weierstrass elliptic function with invariants g_2, g_3 .

Corresponding to these solutions, we obtain the following:

$$u_1(x, y, t) = \frac{K_1\theta_0^3x + K_1\theta_0^2C_2y + K_1\theta_0^2C_3t + \theta_0^2K_2}{3} - \frac{C_2}{6\theta_0}, \quad (62)$$

$$u_2(x, y, t) = -\frac{2\theta_0^2}{(\theta_0x + C_2y + C_3t + C_5)^2} - \frac{C_2}{6\theta_0}, \quad (63)$$

$$u_3(x, y, t) = -2\theta_0^2\wp\left(\theta_0x + C_2y + C_3t + C_4 - z_0; -\frac{B_1}{3}, C_6\right) - \frac{C_2}{6\theta_0}. \quad (64)$$

We let

$$\wp(z - z_0; g_2, g_3) = A_0 \operatorname{sn}^2(k_1(z - z_0), k) + B_0, \quad (65)$$

where A_0 and B_0 are undetermined constants, k_1 is the scale factor, and k is the elliptic modulus. Using $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ and equating the coefficients of like powers of sn in the equation, we obtain the following:

$$A_0 = k_1^2k^2, B_0 = \frac{2k_1^2\theta_0^2(k^2 + 1)}{3}. \quad (66)$$

Thus,

$$u_4(x, y, t) = -2\theta_0^2k_1^2k^2 \operatorname{sn}^2(k_1\theta_0x + k_1C_2y + k_1C_3t + k_1(C_4 - z_0), k) + \frac{2k_1^2\theta_0^2(k^2 + 1)}{3} - \frac{C_2}{6\theta_0}, \quad (67)$$

where g_2, g_3 satisfy:

$$g_2 = \frac{4k_1^4(k^4 - k^2 + 1)}{3}, g_3 = \frac{4k_1^6(k^2 + 1)(k^2 - 2)(2k^2 - 1)}{27}. \quad (68)$$

As $k \rightarrow 1$, the hyperbolic form solution obtained is:

$$u_5(x, y, t) = -2\theta_0^2k_1^2 \tanh^2(k_1\theta_0x + k_1C_2y + k_1C_3t + k_1(C_4 - z_0)) + \frac{4k_1^2\theta_0^2}{3} - \frac{C_2}{6\theta_0}. \quad (69)$$

Similarly,

$$u_6(x, y, t) = -2\theta_0^2k_2^2 \operatorname{ns}^2(k_2\theta_0x + k_2C_2y + k_2C_3t + k_2(C_4 - z_0), k) + \frac{2k_2^2\theta_0^2(k^2 + 1)}{3} - \frac{C_2}{6\theta_0}, \quad (70)$$

where g_2, g_3 satisfy:

$$g_2 = \frac{4k_2^4(k^4 - k^2 + 1)}{3}, g_3 = \frac{4k_2^6(k^2 + 1)(k^2 - 2)(2k^2 - 1)}{27}. \quad (71)$$

As $k \rightarrow 1$, the hyperbolic form solution obtained is:

$$u_7(x, y, t) = -2\theta_0^2 k_2^2 \coth^2(k_2 \theta_0 x + k_2 C_2 y + k_2 C_3 t + k_2(C_4 - z_0)) + \frac{4k_2^2 \theta_0^2}{3} - \frac{C_2}{6\theta_0}. \quad (72)$$

From the results of the solution (67) and the solution (71), we have found that the solution (64) is almost identical to the solutions (112) and (116) in [15] obtained by the extended Jacobian elliptic function expansion method. Two different methods yield almost the same solutions, which, to some extent, shows the validity of our research. Furthermore, (63) provides a more general form compared to Eqs. (42) and (79) in [15] since the solution (63) includes parameters, different parameters can lead to different solutions. In Fig. 2, Fig. 3, Fig. 4, and Fig. 5, we show the corresponding solutions (63), (67), (69), (70) with different parameters.

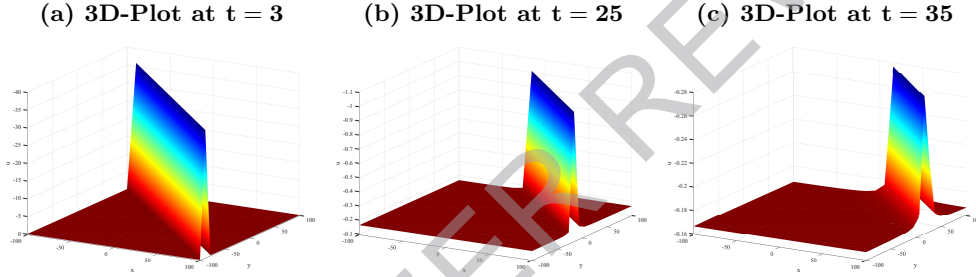


Fig. 2 3D-plots of the solution (63) are depicted at $\theta_0 = 1$, $C_2 = 1$, $B_1 = -2.7$ and $C_5 = 0$ within the interval $-100 \leq x, y \leq 100$ for $t = 3$, $t = 25$ and $t = 35$.

The solution (63) is expressed in the inverse square solution. Fig. 2 exhibits this solution, showing the propagation of the inverse square wave with the line wave along the direction of the vector $[1, 1, 0]$.

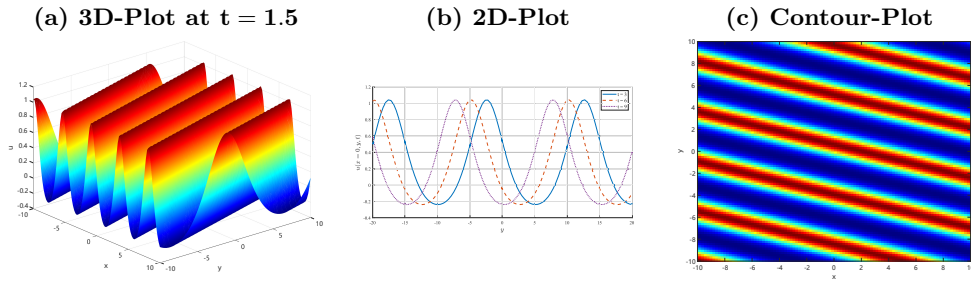


Fig. 3 Evolution plot of singular periodic wave for solution (67) at $t = 1.5$. The figure is delineated for $\theta_0 = k_1 = 1, k = 0.8, C_2 = 0.3, B_1 = 0.5$ and $C_4 = z_0 = 0$ with in the interval $-10 \leq x, y \leq 10$. Corresponding 2D and contour plot are represented in part (b) and part (c), respectively.

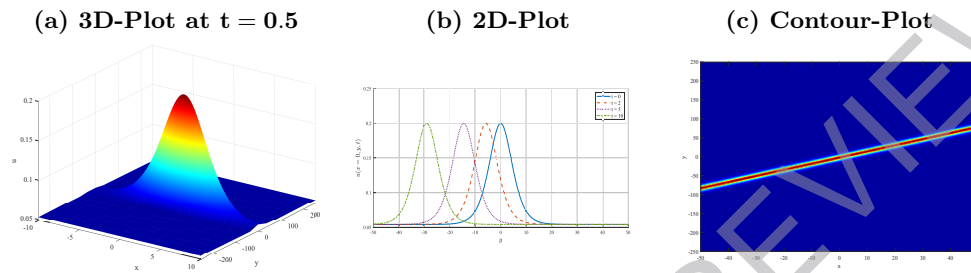


Fig. 4 Soliton for solution (69) at $t = 0.5$. Also the 2D and contour plot of this soliton are drawn in part (b) and part (c), respectively. The figure is delineated for $\theta_0 = 0.27, k_1 = 1, C_2 = -0.166, B_1 = -4.9$ and $C_4 = z_0 = 0$ with in the interval $-10 \leq x \leq 10$ and $-300 \leq y \leq 300$.

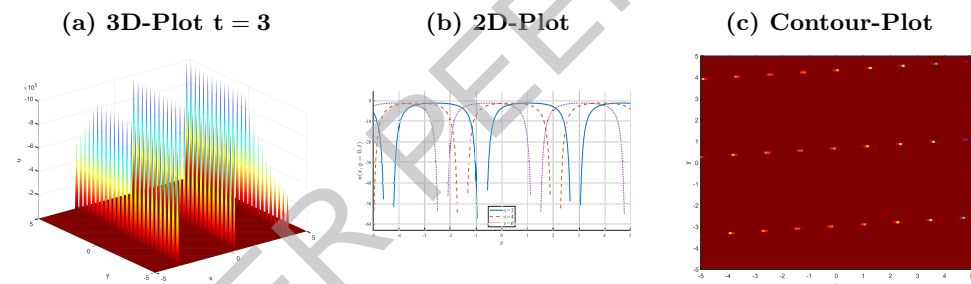


Fig. 5 Design of the multiple singular periodic soliton for the solution (70) at $t = 3$. The values of free parameters are taken as $\theta_0 = k_2 = 1, k = 0.45, C_2 = -0.083, B_1 = -2.286$ and $C_4 = z_0 = 0$ within the range $-5 \leq x, y \leq 5$. Corresponding 2D plot and contour plot are depicted in part (b) and part (c), respectively.

Case (2b): $A_1 \neq 0, \Gamma_8(z) = 0$

The non-autonomous condition generates coupled solutions:

$$c_1(t) = -2A_1\theta^6(t), \quad (73)$$

$$c_2(t) = (2A_1t + C_7)\theta^6(t), \quad (74)$$

$$c_3(t) = -\frac{(3C + 5C_7)^2}{100A_1}\theta^6(t) + C_8\theta^3(t) + \frac{21}{100A_1}\theta^{-4}(t) - \frac{B_1}{A_1}, \quad (75)$$

thus

$$\sigma(y, t) = \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t), \quad (76)$$

where $C_7, C_8 \in \mathbb{R}$ are the integration constants.

The similarity reduction takes the form:

$$u(x, y, t) = \frac{\theta^2(t)\omega(z)}{3} - \frac{c_1(t)y + c_2(t)}{6\theta(t)}, \quad (77)$$

$$z(x, y, t) = \theta(t)x + \frac{1}{2}c_1(t)y^2 + c_2(t)y + c_3(t). \quad (78)$$

The governing ODE for $\omega(z)$ is

$$(\omega'')^2 + \omega^2\omega'' + (A_1z + B_1)\omega'' = 0. \quad (79)$$

The solution contains:

$$\omega_1(z) = K_3z + K_4, \quad (80)$$

$$\omega_2(z) = \frac{216}{A_1^2}P_k \left((-1)^{k+1} \frac{A_1}{6} \left(z + \frac{B_1}{A_1} \right) \right), \quad (81)$$

where P_k satisfies:

$$P_I : P_1'' = 6P_1^2 + \tau, \quad (82)$$

$$P_{II} : P_2'' = 2P_2^3 + \tau P_2 + \alpha, \quad (83)$$

P_1 and P_2 denote the first and second Painlevé transcendents, respectively, where τ is the independent variable, and α is a constant parameter.

The corresponding solutions of the extended (2+1)-dimensional Sakovich equation (1) are as follows:

$$\begin{aligned} u_1(x, y, t) = & \frac{K_3}{3} \left(\frac{-10A_1t + 3C}{3} \right)^{-3/5} x - \frac{A_1K_3}{3} \left(\frac{-10A_1t + 3C}{3} \right)^{-8/5} y^2 \\ & + \left(\frac{2A_1K_3t + C_7K_4}{3} \left(\frac{-10A_1t + 3C}{3} \right)^{-8/5} + \frac{A_1}{-10A_1t + 3C} \right) y \\ & - \frac{K_3(3C + 5C_7)}{300A_1} \left(\frac{-10A_1t + 3C}{3} \right)^{-8/5} + \frac{-2A_1t + 2K_3C_8 - C_7}{-20A_1t + 6C} \\ & - \frac{27B_1K_3}{A_1(-10A_1t + 3C)^2} + \frac{7K_3(-10A_1t + 3C)^2}{900A_1}, \end{aligned} \quad (84)$$

$$\begin{aligned}
u_2(x, y, t) = & \frac{72}{A_1^2} \left(\frac{-10A_1t + 3C}{3} \right)^{-2/5} P_k \left((-1)^{k+1} \frac{A_1}{6} \left(\frac{-10A_1t + 3C}{3} \right)^{-2/5} x \right. \\
& - A_1 \left(\frac{-10A_1t + 3C}{3} \right)^{-6/5} y^2 + (2A_1t + C_7) \left(\frac{-10A_1t + 3C}{3} \right)^{-6/5} y \\
& - \frac{3C + 5C_7}{100A_1} \left(\frac{-10A_1t + 3C}{3} \right)^{-6/5} + C_8 \left(\frac{-10A_1t + 3C}{3} \right)^{-3/5} \\
& \left. + \frac{21}{100A_1} \left(\frac{-10A_1t + 3C}{3} \right)^{-4/5} \right). \tag{85}
\end{aligned}$$

The solution (84) provides a more general form compared to solutions (44), (53) and (58) in [15], since the solution (84) includes more parameters, different parameters can lead to different solutions, which can cover some existing solutions and show the generality of our solutions.

Case (2c): $A_1 \neq 0, \Gamma_8(z) = 3A_1$

The constrained condition yields:

$$c_1(t) = 0, \tag{86}$$

$$c_2(t) = C_9\theta(t), \tag{87}$$

$$c_3(t) = C_{10}\theta(t) + \frac{5C_9^2 + 6C_9 + 6}{20A_1} \theta^{-4}(t) - \frac{B_1}{A_1}. \tag{88}$$

The reduced system becomes:

$$u(x, y, t) = \frac{\theta^2(t)\omega(z)}{3} - \frac{C_9}{6}, \tag{89}$$

$$z(x, y, t) = \theta(t)x + c_2(t)y + c_3(t), \tag{90}$$

The characteristic ODE transforms to:

$$(\omega'')^2 + \omega^2\omega'' + (A_1z + B_1)\omega'' + 3A_1\omega' = 0. \tag{91}$$

This nonlinear ODE admits a special function solution:

$$\omega(z) = -\frac{6A_1^2}{(A_1z + B_1)^2}, \tag{92}$$

we obtain:

$$u(x, y, t) = -\frac{2A_1^2}{\left(A_1x + A_1C_9y + \frac{5C_9^2 + 6C_9 + 6}{60}(-10A_1t + 3C) + A_1C_{10} \right)^2} - \frac{C_9}{6}. \tag{93}$$

2.2 The case $z_x = 0, z_y \neq 0$

The governing equation reduces to the nonlinear ordinary differential equation:

$$\gamma_0 \omega'' + \gamma_1 \omega \omega' + \gamma_2 \omega' + \gamma_3 \omega^3 + \gamma_4 \omega^2 + \gamma_5 \omega + \gamma_6 = 0, \quad (94)$$

where

$$\begin{aligned} \gamma_0 &= \beta z_y^2, \\ \gamma_1 &= 2\beta \beta_x z_y, \\ \gamma_2 &= 2\alpha \beta_x z_y + \beta z_{yy} + \beta_x z_y + 2\beta_y z_y + \beta_x z_t, \\ \gamma_3 &= 6\beta^2 \beta_{xx}, \\ \gamma_4 &= 6\beta \alpha_{xx} + 12\alpha \beta \beta_{xx} + 2\beta \beta_{xy} + 2\beta_{xx}^2, \\ \gamma_5 &= 12\alpha \beta \alpha_{xx} + 6\alpha^2 \beta_{xx} + 2\beta \alpha_{xy} + 2\alpha \beta_{xy} + 4\alpha_{xx} \beta_{xx} + \beta_{tx} + \beta_{xx} + \beta_{xy} + \beta_{yy}, \\ \gamma_6 &= 6\alpha^2 \alpha_{xx} + 2\alpha \alpha_{xy} + 2\alpha_{xx}^2 + \alpha_{xx} + \alpha_{xy} + \alpha_{yy} + \alpha_{xt}. \end{aligned} \quad (95)$$

Under the normalization constraint:

$$\gamma_i = \gamma_0 \Gamma_i(z), \quad (i = 1, 2, \dots, 6). \quad (96)$$

The subsequent analysis of the compatibility conditions yields the following.

$$2\beta_x = z_y \Gamma_1(z). \quad (97)$$

Integration with respect to x produces the following:

$$\beta = \frac{1}{2} \Gamma_1(z) z_y x + \Sigma(y, t). \quad (98)$$

Applying Remark (ii) by setting $\Sigma(y, t) = 0$, we obtain the canonical form:

$$\beta = \frac{1}{2} x z_y, \quad \Gamma_1(z) = 1. \quad (99)$$

For $i = 3$ in (96), direct substitution gives: $\Gamma_3(z) = 0$. The critical case for $i = 4$ requires solving:

$$6\beta \alpha_{xx} + \beta z_{yy} = \beta z_y^2 \Gamma_4(z). \quad (100)$$

We adopt a decoupling approach by separating:

$$6\beta \alpha_{xx} = \beta z_y^2 \Gamma_{4i}(z) \quad (101)$$

and

$$\beta z_{yy} = \beta z_y^2 \Gamma_{4ii}(z), \quad (102)$$

where $\Gamma_4(z) = \Gamma_{4i}(z) + \Gamma_{4ii}(z)$.

Implementing Remarks (iii)-(iv), the general solutions emerge as:

$$z = y\mu(t) + \nu(t), \quad (103)$$

$$\beta = \frac{1}{2}x\mu(t), \quad (104)$$

$$\alpha = \frac{1}{12}\Gamma_{4i}(z)\mu^2(t)x^2 + a_1(y, t)x + a_0(y, t). \quad (105)$$

Substituting (103) and (104) into (96) for $i = 2, 5, 6$ generates:

$$\begin{aligned} x\mu^3(t)\Gamma_2(z) &= \left(\frac{1}{12}\Gamma_{4i}(z)\mu^2(t)x^2 + a_1(y, t)x + a_0(y, t) + \frac{1}{2} \right) \mu^2(t) \\ &+ \frac{1}{2}\mu(t)(y\mu'(t) + \nu'(t)), \end{aligned} \quad (106)$$

$$x\mu^3(t)\Gamma_5(z) = \Gamma_{4i}(z)\alpha\mu^3(t)x + \left(\frac{1}{6}\Gamma'_{4i}(z)\mu^3(t)x + \frac{\partial a_1(y, t)}{\partial y} \right) \mu(t)x + \frac{1}{2}\mu'(t), \quad (107)$$

$$\begin{aligned} x\mu^3(t)\Gamma_6(z) &= \frac{1}{144}\Gamma_{4i}^3(z)\mu^6(t)x^4 + \left(\frac{1}{6}\Gamma_{4i}^2(z)\mu^4(t)a_1(y, t) + \frac{1}{36}\Gamma_{4i}(z)\Gamma'_{4i}(z)\mu^5(t) \right) x^3 \\ &+ \left(\frac{1}{6}\Gamma_{4i}^2(z)\mu^4(t)a_0(y, t) + \Gamma_{4i}(z)a_1^2(y, t)\mu^2(t) + \frac{1}{6}\Gamma_{4i}(z)\mu^2(t)\frac{\partial a_1(y, t)}{\partial y} \right. \\ &+ \left. \frac{1}{3}\Gamma'_{4i}(z)\mu^3(t)a_1(y, t) + \frac{1}{12}\Gamma''_{4i}(z)\mu^4(t) \right) x^2 \\ &+ \left(\frac{1}{6}\Gamma'_{4i}(z)\mu^2(t)(y\mu'(t) + (2a_0(y, t) + 1)\mu(t) + \nu'(t)) + \frac{1}{3}\Gamma_{4i}(z)\mu(t)\mu'(t) \right. \\ &+ \left. 2\Gamma_{4i}(z)\mu(t)a_1(y, t)a_0(y, t) + \frac{\partial^2 a_1(y, t)}{\partial y^2} + 2a_1(y, t)\frac{\partial a_1(y, t)}{\partial y} \right) x \\ &+ \left(\Gamma_{4i}(z)\mu^2(t)a_0^2(y, t) + \frac{1}{18}\Gamma_{4i}^2(z)\mu^4(t) + \frac{1}{6}\Gamma'_{4i}(z)\mu^2(t) \right. \\ &+ \left. \frac{\partial^2 a_0(y, t)}{\partial y^2} + \frac{\partial a_1(y, t)}{\partial t} + (2a_0(y, t) + 1)\frac{\partial a_1(y, t)}{\partial y} \right). \end{aligned} \quad (108)$$

Polynomial balance in (106) necessitates:

$$\Gamma_{4i}(z) = 0. \quad (109)$$

Substituting (105) and (109) into (100) demonstrates that (100) holds for arbitrary z . Combining this with (95), the intersection of their solution sets yields $z = y\mu(t) + \nu(t)$. Meanwhile, the residual equation system then simplifies to:

$$\alpha(x, y, t) = a_1(y, t)x + a_0(y, t), \quad (110)$$

$$x\mu^3(t)\Gamma_2(z) = (a_1(y, t)x + a_0(y, t) + \frac{1}{2})\mu^2(t) + \frac{1}{2}\mu(t)(y\mu'(t) + \nu'(t)), \quad (111)$$

$$x\mu^3(t)\Gamma_5(z) = \frac{\partial a_1(y,t)}{\partial y}\mu(t)x + \frac{1}{2}\mu'(t), \quad (112)$$

$$\begin{aligned} x\mu^3(t)\Gamma_6(z) &= (2a_1(y,t)\frac{\partial a_1(y,t)}{\partial y} + \frac{\partial^2 a_1(y,t)}{\partial y^2})x \\ &+ \frac{\partial^2 a_0(y,t)}{\partial y^2} + \frac{\partial a_1(y,t)}{\partial t} + (2a_0(y,t) + 1)\frac{\partial a_1(y,t)}{\partial y}. \end{aligned} \quad (113)$$

Combining (96) and (98) with $i = 4$, we have

$$\Gamma_4(z) = 0. \quad (114)$$

Since $\Gamma_5(z)$ is the function only for y and t , comparing the coefficients of (112) with the powers of x on both sides of the equations, we obtain $\mu'(t) = 0$. Let $L = \mu(t)$, then $z = Ly + \nu(t)$, where L is a nonzero arbitrary constant of integration.

Substituting it into (111), (112), (113) comparing the coefficients of like powers of x on both sides of the equations, we obtain:

$$a_1(y,t) = L\Gamma_2(z), \quad (115)$$

$$\frac{\partial a_1(y,t)}{\partial y} = L^2\Gamma_5(z), \quad (116)$$

$$2a_1(y,t)\frac{\partial a_1(y,t)}{\partial y} + \frac{\partial^2 a_1(y,t)}{\partial y^2} = L^3\Gamma_6(z), \quad (117)$$

$$(a_0(y,t) + \frac{1}{2})L^2 + \frac{L}{2}\nu'(t) = 0, \quad (118)$$

$$\frac{\partial^2 a_0(y,t)}{\partial y^2} + \frac{\partial a_1(y,t)}{\partial y} + (2a_0(y,t) + 1)\frac{\partial a_1(y,t)}{\partial t} = 0. \quad (119)$$

The (118) reveals that $a_0(y,t)$ depends exclusively on t . Then, inserting (118) into (119) yields the following:

$$\frac{\partial a_1(y,t)}{\partial y} - \frac{\nu'(t)}{L} \frac{a_1(y,t)}{\partial t} = 0. \quad (120)$$

We let $a_1(y,t) = f(z)$, and inserting it into (120) yields

$$(L + \nu'(t))(L - \nu'(t))\frac{\partial f}{\partial z} = 0. \quad (121)$$

Two distinct cases emerge from this result.

Case 1. $\frac{\partial f}{\partial z} = 0$

In this case, because $a_1(y,t) = f(z) = 0$, which leads to $a_1(y,t) = 0$. Then we obtain

$$\Gamma_2(z) = \Gamma_5(z) = \Gamma_6(z) = 0, \quad (122)$$

$$\alpha(x, y, t) = -\frac{1}{2L}\nu'(t) - \frac{1}{2}, \quad (123)$$

$$u(x, y, t) = \frac{L}{2}x\omega(z) - \frac{1}{2} - \frac{1}{2L}\nu'(t), \quad (124)$$

where $\omega(z)$ satisfies

$$\omega'' + \omega\omega' = 0. \quad (125)$$

The set of solutions in (125) comprises:

$$\omega_1(z) = C_1, \quad (126)$$

$$\omega_2(z) = \frac{2}{z + C_2}, \quad (127)$$

$$\omega_3(z) = l_1 \tanh\left(\frac{l_1}{2}z + C_3\right), \quad (128)$$

$$\omega_4(z) = l_2 \tan\left(\frac{l_2}{2}z + C_4\right). \quad (129)$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are the integration constant, and both l_1 and l_2 are positive.

The corresponding extended (2+1)-dimensional Sakovich equation solutions are:

$$u_1(x, y, t) = -\frac{1}{2} - \frac{1}{2L}\nu'(t) + \frac{C_1L}{2}x, \quad (130)$$

$$u_2(x, y, t) = -\frac{1}{2} - \frac{1}{2L}\nu'(t) + \frac{Lx}{Ly + \nu(t) + C_2}, \quad (131)$$

$$u_3(x, y, t) = -\frac{1}{2} - \frac{1}{2L}\nu'(t) + \frac{l_1L}{2}x \tanh\left(\frac{l_1}{2}(Ly + \nu(t)) + C_3\right), \quad (132)$$

$$u_4(x, y, t) = -\frac{1}{2} - \frac{1}{2L}\nu'(t) - \frac{l_2L}{2}x \tan\left(\frac{l_2}{2}(Ly + \nu(t)) + C_4\right). \quad (133)$$

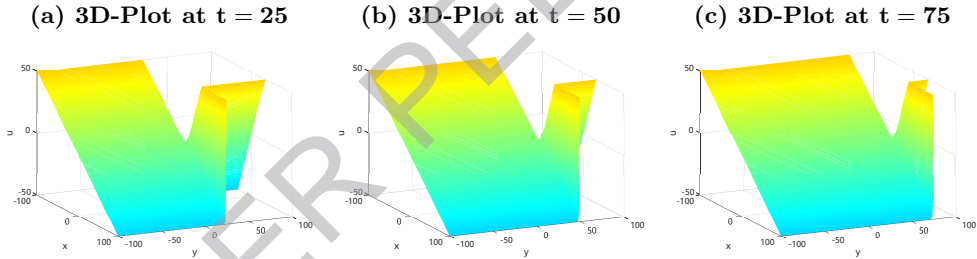


Fig. 6 3D-plots of the solution (132) are depicted at $\nu(t) = -t$, $L = 1$, $l_1 = 2$, $f_0 = 1$ and $C_3 = 0$ within the interval $-100 \leq x, y \leq 100$ for $t = 25$, $t = 50$ and $t = 75$.

The solution (132) includes a dark soliton structure, as illustrated in Fig. 6. This dark soliton propagates steadily in the positive y direction while maintaining its form without distortion, thereby demonstrating a stable state.

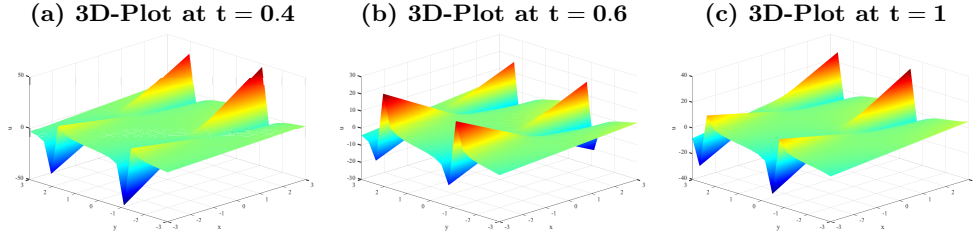


Fig. 7 3D-plots of the solution (133) are depicted at $\nu(t) = -t$, $L = 1$, $l_2 = 2$, $f_0 = 1$ and $C_4 = 0$ within the interval $-3 \leq x, y \leq 3$ for $t = 0.4$, $t = 0.6$, $t = 1$.

The solution (133) is expressed in the trigonometric solution (133). Fig. 7 exhibits this solution, which consists of discrete surfaces, showing the propagation of the trigonometric wave with wave crest along the positive y direction.

Case 2. $\frac{\partial f}{\partial z} \neq 0$

In this case, setting $\nu(t) = -Lt + C_5$ and inserting $a_1(y, t) = f(z)$ into (94) yields

$$\omega'' + \omega\omega' + \frac{f(z)}{L}\omega' + \frac{f'(z)}{L}\omega + \frac{2f(z)f'(z)}{L^2} + \frac{f''(z)}{L} = 0. \quad (134)$$

The solution of (134) is as follows:

$$\omega(z) = \frac{2Q'(z)}{Q(z)} - \frac{f(z)}{z}, \quad (135)$$

where $Q(z)$ satisfies:

$$Q'' + \left(\frac{f^2(z)}{4L^2} - \frac{C}{2} \right) Q = 0, \quad (136)$$

where C is the integration constant.

Here, since $f(z)$ is an arbitrary function to be determined, the solution encompasses a wide range of possibilities. Here, we try to constrain the problem and find a meaningful and intuitive solution. After repeated attempts, we eventually selected $f(z) = \frac{1}{z}$. Then (134) becomes:

$$\omega'' + \omega\omega' + \frac{1}{z}\omega' - \frac{1}{z^2}\omega = 0. \quad (137)$$

Also,

$$\alpha(x, y, t) = \frac{Lx}{Ly - Lt + C_5}, \quad (138)$$

$$u(x, y, t) = \frac{Lx}{Ly - Lt + C_5} + \frac{L}{2}x\omega(z). \quad (139)$$

Integrating (137) once with respect to z yields the Riccati equation:

$$\omega' + \frac{1}{2}\omega^2 + \frac{\omega}{z} = C, \quad (140)$$

where C is the integration constant.

Two distinct cases emerge from this result.

Case 2.1 $C = 0$

Substitution of $v = \frac{1}{\omega}$ into (140), followed by multiplication by the integration factor $\frac{1}{z}$ and integration with respect to z , yields the following result:

$$v = \frac{z \ln z}{2} + C_0, \quad (141)$$

namely,

$$\omega_1(z) = \frac{2}{z \ln(z + C_6)}, \quad (142)$$

where C_0 and C_6 are the integration constants.

As a result, we obtain a set of solutions characterized by logarithmic terms:

$$u_1(x, y, t) = \frac{Lx}{Ly - Lt + C_5} + \frac{Lx}{(Ly - Lt + C_5) \ln(Ly - Lt + C_6)}. \quad (143)$$

Case 2.2 $C \neq 0$

In this case, (140) is a Riccati equation, we apply the standard Riccati transform: $\omega(z) = \frac{2P(z)}{P'(z)}$, then we obtain that:

$$P'' + \frac{1}{z}P' - \frac{C}{2}P = 0. \quad (144)$$

Case 2.2.1 $C > 0$

In this case, (144) can be rewritten as:

$$P'' + \frac{1}{z}P' - k^2P = 0, \quad (145)$$

where $k = \sqrt{\frac{C}{2}}$.

The general solution to (145) can be expressed in terms of the modified Bessel functions:

$$P(z) = C_7 I_0\left(\sqrt{\frac{C}{2}}z\right) + C_8 K_0\left(\sqrt{\frac{C}{2}}z\right), \quad (146)$$

thus,

$$\omega_2(z) = \frac{2P'(z)}{P(z)} = \frac{2\sqrt{\frac{C}{2}} \left[C_7 I_1\left(\sqrt{\frac{C}{2}}z\right) - C_8 K_1\left(\sqrt{\frac{C}{2}}z\right) \right]}{C_7 I_0\left(\sqrt{\frac{C}{2}}z\right) + C_8 K_0\left(\sqrt{\frac{C}{2}}z\right)}. \quad (147)$$

The corresponding solution of the extended (2+1)-dimensional Sakovich equation (1) is as follows:

$$u_2(x, y, t) = \frac{Lx}{Ly - Lt + C_5} + \frac{2\sqrt{\frac{C}{2}} \left[C_7 I_1 \left(\sqrt{\frac{C}{2}} (Ly - Lt) \right) - C_8 K_1 \left(\sqrt{\frac{C}{2}} (Ly - Lt) \right) \right]}{C_7 I_0 \left(\sqrt{\frac{C}{2}} (Ly - Lt) \right) + C_8 K_0 \left(\sqrt{\frac{C}{2}} (Ly - Lt) \right)} Lx, \quad (148)$$

where C_5 , C_7 and C_8 are arbitrary constants, $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and second kind.

Case 2.2.2 $C < 0$

In this case, (144) can be rewritten as:

$$P'' + \frac{1}{z}P' + k^2P = 0, \quad (149)$$

where $k = \sqrt{\frac{-C}{2}}$.

The general solution to (149) can be expressed in terms of the Bessel functions:

$$P(z) = C_9 J_0 \left(\sqrt{\frac{-C}{2}} z \right) + C_{10} Y_0 \left(\sqrt{\frac{-C}{2}} z \right), \quad (150)$$

thus,

$$\omega_3(z) = \frac{2P'(z)}{P(z)} = \frac{2\sqrt{\frac{-C}{2}} \left[C_9 J_1 \left(\sqrt{\frac{-C}{2}} z \right) - C_{10} Y_1 \left(\sqrt{\frac{-C}{2}} z \right) \right]}{C_9 J_0 \left(\sqrt{\frac{-C}{2}} z \right) + C_{10} Y_0 \left(\sqrt{\frac{-C}{2}} z \right)}. \quad (151)$$

The corresponding solution of the extended (2+1)-dimensional Sakovich equation (1) is as follows:

$$u_3(x, y, t) = \frac{Lx}{Ly - Lt + C_5} + \frac{2\sqrt{\frac{-C}{2}} \left[C_9 J_1 \left(\sqrt{\frac{-C}{2}} (Ly - Lt) \right) - C_{10} Y_1 \left(\sqrt{\frac{-C}{2}} (Ly - Lt) \right) \right]}{C_9 J_0 \left(\sqrt{\frac{-C}{2}} (Ly - Lt) \right) + C_{10} Y_0 \left(\sqrt{\frac{-C}{2}} (Ly - Lt) \right)} Lx, \quad (152)$$

where C_5 , C_9 and C_{10} are arbitrary constants, $J_n(z)$ and $Y_n(z)$ are the Bessel functions of the first and second kind, respectively.

As a result, we obtain a set of solutions characterized by Bessel function terms.

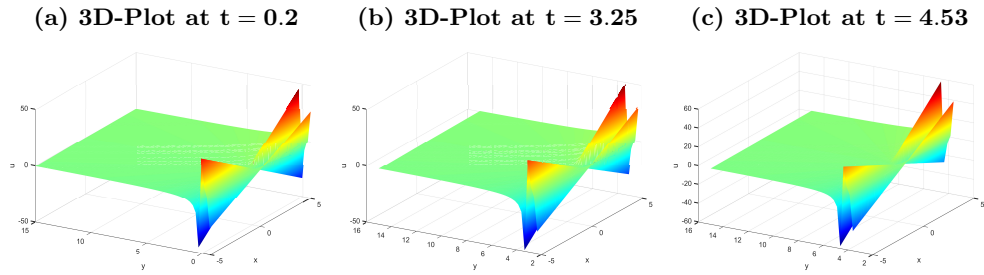


Fig. 8 3D-plots of the solution (143) are depicted at $L = 1$, $C_5 = 1$ and $C_6 = 1$ within the interval $-5 \leq x \leq 5$, $-5 \leq y \leq 15$ for $t = 0.2$, $t = 3.25$, $t = 4.53$.

The solution (143) is expressed in the logarithmic functional form, which includes two movable line waves. Fig. 8 exhibits this solution, showing the propagation of the logarithmic wave with a wave crest along the positive y direction. To the best of the authors' knowledge, this specific solution has not been previously reported in the literature.

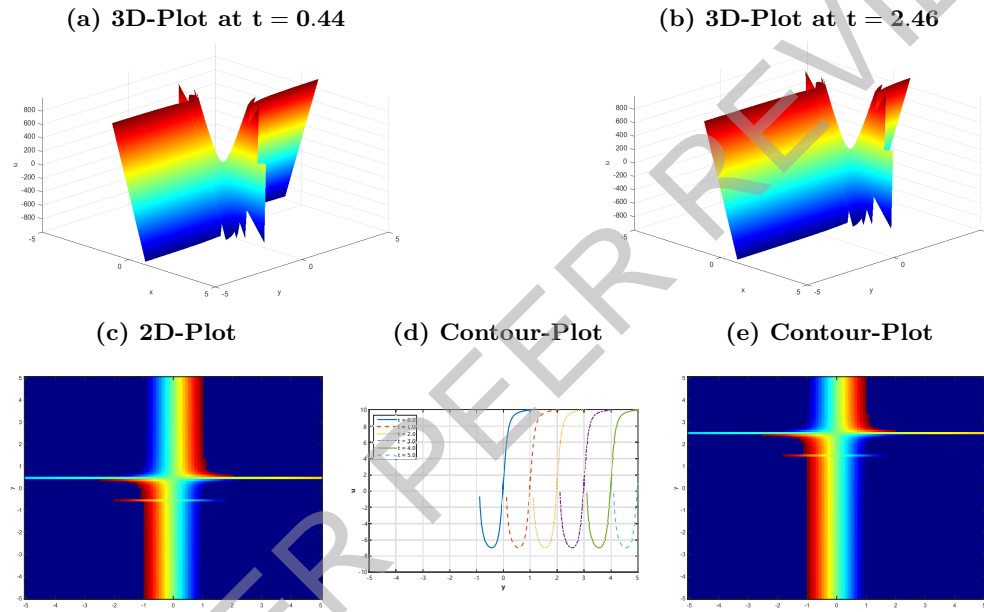


Fig. 9 3D-plots of the solution (148) are depicted in part (a) and (b). The values of free parameters are taken as $L = 10$, $C_5 = 10$, $C = 2$, $C_7 = 1$ and $C_8 = 0$ within the range $-5 \leq x, y \leq 5$. Corresponding 2D-plots are depicted in part (d), and the contour plot is depicted in part (c) and part (e), respectively.

The solution (148), expressed in the Bessel functional form, shows a distinctive structure and a new wave phenomenon in Fig. 9. In Fig. 9(a) and Fig. 9(b), the solution manifests the dark soliton structure characterized by a central depression.

Remarkably, the form of this dark soliton persists unchanged throughout its temporal evolution. In addition, a movable and continuous blow-up on one side occurs due to the presence in the rational function term, as depicted in Fig. 9(c) and Fig. 9(e). To elucidate the temporal behavior within the dark soliton, Fig. 9(d) shows the evolution of the solution at a fixed position $x = 1$. This cross-section in Fig. 9(d) reveals a progressive deepening of the dark soliton over time. We are not aware of any prior work that, such a solution with these characteristic features – namely the persistent progressive deepening dark soliton depression with movable and continuous blow-up – has not been reported in the literature. Its evolutionary profile exhibits the potential significance of equation (1).

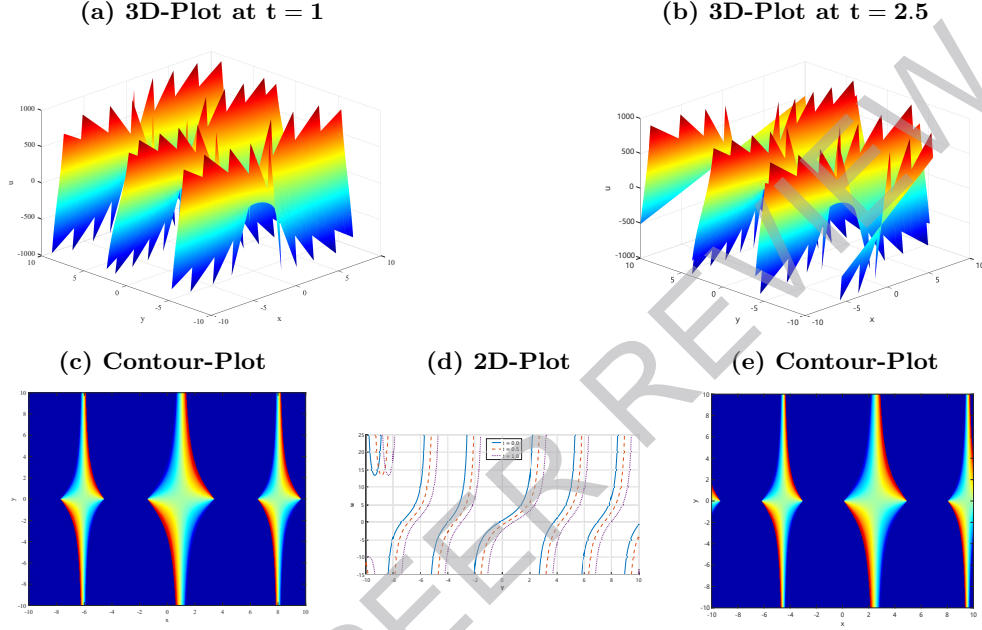


Fig. 10 3D-plots of the solution (152) are depicted in part (a) and part (b). The values of free parameters are taken as $L = 1, C_5 = 10, C = -2, C_9 = 1$ and $C_{10} = 0$ within the range $-10 \leq x, y \leq 10$. Corresponding 2D-plots are depicted in part (d), and the contour plot is depicted in part (c) and part (e), respectively.

The solution (152), expressed in the Bessel functional form, exhibits a new wave phenomenon in Fig. 10. This solution consists of a series of discrete surfaces. Specifically, every individual surface is accompanied by central symmetry, as further corroborated by Fig. 10(c) and Fig. 10(e). Furthermore, this solution includes discrete surfaces propagating along the positive y direction with a finite constant velocity, as depicted in the Fig. 10(a), Fig. 10(b), and Fig. 10(d). During propagation, every individual surface continuously deforms over time while maintaining its structural integrity. To the best of our knowledge, this solution and its distinctive structural and

dynamical features have not been reported in the previous literature. Its evolutionary profile also exhibits the potential significance of equation (1).

3 Conclusion and discussion

This investigation systematically explores the extended (2+1)-dimensional Sakovich equation (1) through the Clarkson-Kruskal direct method, establishing a complete symmetry reduction framework in two cases: $z_x \neq 0$ and $z_x = 0$. The methodology successfully reduces the original PDE to an integrable ODE system, yielding 14 classes of exact solutions as enumerated in Eqs. (38), (62), (63), (64), (84), (85), (93), (130), (131), (132), (133), (143), (148), (152).

Specifically, in the first case, where $z_x \neq 0$, we derive rational function solutions (Eqs. (38), (62), (63), (84), (93)), a Weierstrass elliptic function solution (Eq. (64)), and the Painlevé I and Painlevé II similarity reductions (Eq. (85)). In the second case with $z_x = 0$, we obtain rational function solutions (Eqs. (130), (131)), a hyperbolic function solution (Eq. (132)), a trigonometric periodic solution (Eq. (133)), a logarithmic function solution (Eq. (143)), and Bessel function solutions (Eqs. (148), (152)). Furthermore, these derived solutions exhibit diverse new wave phenomena, to some extent, providing a basis for a deeper understanding of the equation. Compared to other methods, such as the Lie symmetry method, the unified method, the Hirota bilinear technique, etc., the CK direct method is capable of yielding more types of solutions. Despite the complex and tedious computations of the CK direct method, the solutions obtained by the CK method are more diverse, especially including rational-form solutions, lump solutions, soliton solutions, etc.

There are still some questions worth considering. Firstly, without increasing computational complexity, is there a better way to replace the separation method that would allow one to obtain more solutions? Secondly, are there more functional forms of $f(z)$ that yield meaningful solutions in the second case?

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