The Normalized Ground State Solutions for the Kirchhoff Equation with Sobolev-Hardy Critical Exponent

Original Research Article

Abstract

This paper studies the existence of ground state normalized solutions for a modified Kirchhoff equation with the quasilinear term and the Sobolev-Hardy critical exponent. By developing variational methods on the Pohozaev manifold, we prove the existence of solutions for large masses, extending previous results to the challenging case involving the Hardy critical nonlinearity. From a computational perspective, establishing the existence of ground states is crucial for the stability analysis of numerical algorithms used in simulating singular physical systems. Furthermore, the theoretical bounds derived for the mass threshold provide essential constraints for the convergence of iterative schemes in numerical validations.

Keywords: Sobolev-Hardy critical exponent; Kirchhoff equation; Ground state Normalized solutions

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1 Introduction

Consider the following modified Kirchhoff equation with a Hardy term:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u - u\Delta(u^2) - \lambda u = |x|^{-s}|u|^{2^*(s)-2}u, & x\in\mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2dx = c, & u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a, b > 0$. $\lambda \in \mathbb{R}$ is a Lagrange multiplier, $s \in [0, \frac{1}{3})$ is the Hardy potential parameter, and $2^*(s) = 6-2s$ is the Sobolev-Hardy critical exponent, and $c > 0$ is the prescribed L^2-norm constraint.

The study of equations involving the singular Hardy potential $|x|^{-s}$ has attracted considerable attention. Fundamentally based on the core Hardy inequality Hardy et al. (1952) and the weighted Sobolev space theory Maz'ya (2011), the analysis of such singular problems has been extended to various nonlinear structures. For instance, the existence and nonexistence of solutions for the fractional Laplacian with Hardy terms were widely studied in Wang (2017). Rui (2020) investigated the existence of ground state solutions for the Kirchhoff-type Choquard problem involving the Hardy-Littlewood-Sobolev critical exponent. This study rigorously addresses the compactness issues arising from the interaction between the nonlocal Kirchhoff term and the critical nonlinearity, providing a parallel theoretical framework to the singular problems examined in this work.

Recent developments have further broadened the scope of Kirchhoff-type equations involving singular potentials and critical growth. The interaction between the Hardy potential and critical Choquard type nonlinearity was extensively analyzed by Saini and Goyal (2025). In the context of nonlocal operators, Goel et al. (2024) established the existence of high energy solutions for p-Kirchhoff problems with Hardy-Littlewood-Sobolev nonlinearity, while Zeng et al. (2025) focused on the existence of infinitely many small energy solutions for p-Laplacian problems of Kirchhoff type with Hardy potential. Furthermore, Chung (2020) offers valuable insights into handling non-local Kirchhoff operators through variational arguments, paralleling the structural analysis performed in this work for critical singular problems. More closely related to the constraints considered in this work, recent studies have also addressed normalized solutions for Kirchhoff equations featuring upper critical exponents Shang et al. (2025).

In the specific context of Kirchhoff equations involving Hardy terms, Lü (2019) utilized variational arguments to determine the existence of positive solutions. However, the problem encounters a central difficulty when the critical exponent is involved due to the loss of compactness, a challenge famously addressed in classical Sobolev settings by Brézis and Nirenberg (1983) and Lions (1984). Building on these foundations, the study of normalized solutions for singular equations has become a burgeoning field. Soave (2020) analyzed normalized ground state solutions for NLS equations with combined nonlinearities, providing a framework often adapted to singular problems. For Kirchhoff-type problems, Chen et al. (2021) established the existence of normalized solutions for nonautonomous Kirchhoff equations, covering both sub- and super-critical cases. By introducing novel analytical techniques to exclude the vanishing and dichotomy of minimizing sequences, this study provides a robust framework for handling the lack of homogeneity and the presence of potentials in Kirchhoff-type problems.

Another significant feature of the equation under study is the quasilinear term $u\Delta(u^2)$, which appears naturally in mathematical physics. Kurihara Kurihara (1981) first derived the quasilinear Schrödinger equation to model the dynamics of the condensate wave function in superfluid films. Inspired by this physical model, Poppenberg et al. Poppenberg et al. (2002) and Liu et al. Liu et al. (2003) established the fundamental variational framework for such problems. To rigorously handle the non-smoothness introduced by the term $u\Delta(u^2)$, Colin, Jeanjean, and Squassina Colin et al. (2010) developed a celebrated change of variables approach. In recent years, normalized solutions for quasilinear equations have become a focal point. Ye and Yu (2020) investigated the existence of critical points for the L^2-critical quasilinear Schrödinger equation, proving that the constrained minimization problem admits no solution for any prescribed mass. This study establishes a threshold value separating existence from nonexistence results, offering significant insight into the variational structure of critical quasilinear problems, while Li and Zou (2023) extended the results to the ground state and infinitely many solutions.

Recently, significant progress has been made regarding normalized solutions for modified Kirchhoff equations. Wang and Chang (2025) and He et al. (2023) investigated normalized solutions for Kirchhoff equations involving critical exponents and potential terms. While Han (2005) have discussed general solutions for quasilinear equations with Hardy-Sobolev critical exponents, the simultaneous combination of the nonlocal Kirchhoff term, the quasilinear term $u\Delta(u^2)$, and the singular Hardy potential under the normalized constraint remains unexplored.

To the best of our knowledge, there are no results in the literature concerning normalized solutions for problems that simultaneously contain the Kirchhoff nonlocal term, the quasilinear term $u\Delta(u^2)$, and the singular Hardy potential. The introduction of the Hardy term not only leads to a singularity at the origin but also further destroys the compactness of the embedding via the Hardy-Sobolev critical exponent. This makes the construction of ground state solutions satisfying the mass constraint a highly challenging and novel task.

This paper constitutes an extension of the normalized solutions theory for the modified Kirchhoff equation, successfully surmounting the complex analytical hurdles posed by the Sobolev-Hardy critical exponent. The presence of the singular Hardy potential $|x|^{-s}$ drastically complicates the critical exponent $2^*(s)$. In order to overcome the loss of compactness of the critical Sobolev-Hardy term, we analyze the asymptotic behavior of the ground state energy m_c. This strict inequality, $m_c < m_\infty$, ensures the strong convergence of the minimizing sequence. The main result of this paper is as follows. s

1.1 Main Result

In order to introduce our main result, we first define the constraint set

$$S(c) := \{u \in H^1(\mathbb{R}^3) \mid \|u\|_2 = c\}.$$

Let U be the extremal function for the Hardy-Sobolev inequality. We fix a standard reference function $\phi_0 \in H^1(\mathbb{R}^3)$ as a smooth truncation of U, satisfying $\phi_0(x) = U(x)$ for $|x| \leq 1$ and $\phi_0(x) = 0$ for $|x| \geq 2$. We denote three positive structural constants as follows,

$$\mathcal{A} := \|\nabla \phi_0\|_2^2, \quad \mathcal{B} := \int_{\mathbb{R}^3} |x|^{-s} |\phi_0|^p dx, \quad \mathcal{M} := \|\phi_0\|_2^2.$$

Using these constants, we define the critical mass threshold

$$c^* := \left(\frac{b(6-2s)\mathcal{A}^2}{4\mathcal{B}} \right)^{\frac{1}{1-s}} \mathcal{M}.$$

Proposition 1.1. Let $a, b > 0$, and assume that $s \in [0, \frac{1}{3})$. Then equation (1.1) has a ground state solution $(u_c, \lambda_c) \in S_c \times \mathbb{R}$ for all $c > c^*$.

2 Preliminaries and Proof of Theorem 1.1

To establish our main results, we first introduce the following fundamental definitions.

Definition 2.1. Let $p = 2^*(s)$, the energy of equation (1.1) is defined as

$$I(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |x|^{-s} |u|^p dx.$$

Definition 2.2. The Pohozaev set are defined as $\mathbb{P}_c = \{u \in S_c \mid P(u) = 0\}$, where

$$P(u) = a \int_{\mathbb{R}^3} |\nabla u|^2 dx + b \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + 5 \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx - \int_{\mathbb{R}^3} |x|^{-s} |u|^p dx.$$

Definition 2.3. The limiting energy functional $I_\infty : D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$I_\infty(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |x|^{-s} |u|^p dx.$$

And the threshold m_∞ is the infimum of $I_\infty(u)$ over $\mathbb{P}_\infty$ which is defined as

$$m_\infty = \inf_{u \in \mathbb{P}_\infty} I_\infty(u),$$

where the manifold $\mathbb{P}_\infty$ is given by

$$\mathbb{P}_\infty = \left\{ u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\} : \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |x|^{-s} |u|^p dx = 0 \right\}.$$

Definition 2.4. The best Sobolev-Hardy constant $S_{3,s}$ is defined as

$$S_{3,s} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |x|^{-s}|u|^p dx\right)^{2/p}},$$

and the optimal function $U \in D^{1,2}(\mathbb{R}^3)$ satisfies

$$S_{3,s} = \frac{\int_{\mathbb{R}^3} |\nabla U|^2 dx}{\left(\int_{\mathbb{R}^3} |x|^{-s}|U|^p dx\right)^{2/p}},$$

and U also satisfies the Euler-Lagrange Equation of our problem

$$-\Delta U = |x|^{-s} U^{p-1} \quad \text{in } \mathbb{R}^3.$$

Lemma 2.1. Assume that $s \in [0, \frac{1}{3})$, then $I(u)$ is coercive and bounded below on $\mathbb{P}_c$. Specifically, there exists a constant $C_1 = C_1(c, s, a, b) > 0$ such that $I(u) \geq C_1$ for all $u \in \mathbb{P}_c$.
Proof. From the definition of the Pohozaev set $\mathbb{P}_c$, we solve for the integral of the Hardy term and we have

$$\int_{\mathbb{R}^3} |x|^{-s}|u|^p dx = a\int_{\mathbb{R}^3} |\nabla u|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + 5\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx. \tag{2.1}$$

Substitute (2.1) into $I(u)$ to eliminate the Hardy term. Let $\Theta(u) = a\int_{\mathbb{R}^3} |\nabla u|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + 5\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx$, then

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx - \frac{1}{p}\Theta(u).$$

Rewrite $I(u)$ into three terms which is

$$I(u) = K_1 a\int_{\mathbb{R}^3} |\nabla u|^2 dx + K_2 b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + K_3\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx,$$

where

$$K_1 = \frac{1}{2} - \frac{1}{6-2s}, K_2 = \frac{1}{4} - \frac{1}{6-2s}, K_3 = 1 - \frac{5}{6-2s}.$$

Since $s \in [0, \frac{1}{3})$ and $p = 6 - 2s$, by simple calculation we get all K_1, K_2, K_3 are positive.
By the Sobolev-Hardy inequality, for $u \in S_c$,

$$\int_{\mathbb{R}^3} |x|^{-s}|u|^p dx \leq C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{p}{2}},$$

where $C > 0$ is a constant independent of u. Combining with equation (2), we obtain $\Theta(u) \leq Cc^\theta \left(\int |\nabla u|^2 dx\right)^{\frac{p}{2}}$.

If $\int_{\mathbb{R}^3} |\nabla u|^2 dx \to +\infty$, then $\Theta(u) \to +\infty$. Since

$$I(u) = K_1 a\int_{\mathbb{R}^3} |\nabla u|^2 dx + K_2 b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + K_3\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx$$

and all coefficients are positive, we have $I(u) \to +\infty$.

If $\int_{\mathbb{R}^3} |\nabla u|^2 dx$ is bounded, by Hölder inequality and $u \in H^1(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx$ is also bounded. Thus, $I(u)$ is bounded below by a positive constant.

In conclusion, there exists a constant $C_1(c, s, a, b) > 0$ such that $I(u) \geq C_1$ for all $u \in \mathbb{P}_c$. $\square$

Lemma 2.2. Let $p = 6 - 2s$ with $s \in [0, \frac{1}{3})$, and let $u^t := e^{\frac{3}{2}t}u(e^t x)$ for $t \in \mathbb{R}$, then for each fixed $u \in S_c$, there exists a unique $t_0 \in \mathbb{R}$ such that $I(u^{t_0}) = \max_{t\in\mathbb{R}} I(u^t)$ and $u^{t_0} \in \mathbb{P}_c$.

Proof. First, let $u^t = e^{\frac{3}{2}t}u(e^t x)$ and note that

$$-\frac{1}{p}\int_{\mathbb{R}^3} |x|^{-s}|u^t|^p dx = -\frac{1}{p}e^{pt}\int_{\mathbb{R}^3} |x|^{-s}|u|^p dx.$$

Let $A = \int_{\mathbb{R}^3} |\nabla u|^2 dx$, $B = \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx$, $D = \int_{\mathbb{R}^3} |x|^{-s}|u|^p dx$, and we have

$$I(u^t) = \frac{a}{2}e^{2t}A + \frac{b}{4}e^{4t}A^2 + e^{5t}B - \frac{1}{p}e^{pt}D.$$

Taking the derivative of $I(u^t)$ with respect to t,

$$\frac{d}{dt}I(u^t) = ae^{2t}A + be^{4t}A^2 + 5e^{5t}B - e^{pt}D.$$

By definition of $P(u)$ from Lemma 2.1, this derivative equals $P(u^t)$.

Analyze the behavior of $I(u^t)$ at infinity. As $t \to +\infty$, since $s \in [0, \frac{1}{3})$, we have $p > \frac{16}{3}$, then the negative Hardy term dominates, leading to $I(u^t) \to -\infty$. As $t \to -\infty$, all exponential terms decay to 0, and we have $I(u^t) \to 0$.

The second derivative with respect to t is

$$\frac{d^2}{dt^2}I(u^t) = 2ae^{2t}A + 4be^{4t}A^2 + 25e^{5t}B - pe^{pt}D.$$

At the critical point t_0 with $G(u^{t_0}) = 0$, substitute $e^{pt_0}D$ and we get

$$\frac{d^2}{dt^2}I(u^{t_0}) = ae^{2t_0}A(2-p) + be^{4t_0}A^2(4-p) + 5e^{5t_0}B(5-p).$$

By calculation, we directly have $\frac{d^2}{dt^2}I(u^{t_0}) < 0$.

Thus, $I(u^t)$ is continuous on $\mathbb{R}$, tends to 0 as $t \to -\infty$ and $-\infty$ as $t \to +\infty$, and has a unique strict global maximum at t_0 satisfying $P(u^{t_0}) = 0$, and $u^{t_0} \in \mathbb{P}_c$. $\square$

These two lemmas lay the foundation for the subsequent analysis of the existence of ground state normalized solutions, ensuring the coerciveness and boundedness from below of the energy functional on the Pohozaev manifold, and the possibility of mapping arbitrary functions to the Pohozaev manifold through scaling transformations.

Lemma 2.3. Let $p = 6 - 2s$ with $s \in [0, \frac{1}{2})$, and let $u \in \mathbb{P}_c$ be fixed; denote $w := t^\alpha u(t^\beta x)$ for $t > 1$ where $2\alpha - 3\beta = 1$. Then there exists a unique pair $(\bar{\alpha}, \bar{\beta})$ satisfying $2\bar{\alpha} - 3\bar{\beta} = 1$ such that $P(w) = 0$ for all $c > 0$, and it holds that $(2-p)\bar{\alpha} + (2-s)\bar{\beta} > 0 > (4-p)\bar{\alpha} + (2-s)\bar{\beta}$.

Proof. Using the scaling transformation $w = t^\alpha u(t^\beta x)$ with $2\alpha - 3\beta = 1$, we recall that $A = \int_{\mathbb{R}^3} |\nabla u|^2 dx$, $B = \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx$, and $D = \int_{\mathbb{R}^3} |x|^{-s}|u|^p dx$. Since $u \in \mathbb{P}_c$, we have $P(u) = 0$, which implies

$$D = aA + bA^2 + 5B.$$

Substituting into $P(w)$ and we obtain

$$\begin{aligned} P(w) =& at^{2\alpha-\beta}A + bt^{4\alpha-2\beta}A^2 + 5t^{4\alpha-\beta}B \\ &- t^{\alpha p-\beta(3-s)}(aA + bA^2 + 5B). \end{aligned}$$

Case 1. Equate exponents of quasilinear and Hardy terms:

$$4\alpha - \beta = \alpha p - \beta(3-s). \tag{2.2}$$

Combining $2\alpha - 3\beta = 1$ and we get

$$\alpha^{(1)} = \frac{2-s}{4s-2}, \quad \beta^{(1)} = \frac{1-s}{2s-1}.$$

For $p = 2^*(s)$ and $s \in [0, \frac{1}{3})$, we have $\alpha^{(1)} < 0$ and $\beta^{(1)} < 0$. By calculation we get

$$P(w) = t^{\alpha^{(1)}p - \beta^{(1)}(3-s)} \left[at^{E_1} A + bt^{E_2} A^2 + 5B - (aA + bA^2 + 5B) \right].$$

where $E_1 = (2-p)\alpha^{(1)} + (2-s)\beta^{(1)}$ and $E_2 = (4-p)\alpha^{(1)} + (1-s)\beta^{(1)}$. From equation (2.2), we get $E_1, E_2 > 0$. Thus for $t > 1$ we have $t^{E_1} = 1$, $t^{E_2} > 1$, and we can get

$$aA + bt^{E_2} A^2 + 5B - (aA + bA^2 + 5B) = bA^2(t^{E_2} - 1) > 0.$$

Therefore, $P(w) > 0$ in Case 1.

Case 2. Equate exponents of linear Kirchhoff and Hardy terms:

$$2\alpha - \beta = \alpha p - \beta(3-s). \tag{2.3}$$

Combining $2\alpha - 3\beta = 1$ and we get

$$\alpha^{(2)} = -\frac{1}{4}, \quad \beta^{(2)} = -\frac{1}{2}.$$

By calculation we get

$$P(w) = at^{E_1'} A + bt^{E_2'} A^2 + 5t^{E_3'} B - (aA + bA^2 + 5B).$$

where $E_1' = (2-p)\alpha^{(2)} + (2-s)\beta^{(2)}$, $E_2' = (4-p)\alpha^{(2)} + (1-s)\beta^{(2)}$, $E_3' = (4-p)\alpha^{(2)} + (2-s)\beta^{(2)}$. From equation (2.3), we get $E_1' = 0$, $E_2' = 0$ and $E_3' = -\frac{1}{2} < 0$ since $p > \frac{16}{3}$. Thus for $t > 1$, we have $t^{E_1'} = 1$, $t^{E_2'} = 1$, $t^{E_3'} < 1$ and

$$aA + bt^{E_2'} A^2 + 5t^{E_3'} B - (aA + bA^2 + 5B) = 5B(t^{E_3'} - 1) < 0.$$

Therefore, $P(w) < 0$ in Case 2. Since $P(w)$ is continuous in α (with $\beta = \frac{2\alpha - 1}{3}$), and at $\alpha^{(1)}$ we have $P(w) > 0$; at $\alpha^{(2)}$ we have $P(w) < 0$, there exists $\bar{\alpha} \in (\alpha^{(1)}, \alpha^{(2)})$ such that $P(w) = 0$. Uniqueness follows from the strict monotonicity of $P(w)$ with respect to α, which can be verified by computing the derivative. We now prove $(2-p)\bar{\alpha} + (2-s)\bar{\beta} > 0 > (4-p)\bar{\alpha} + (2-s)\bar{\beta}$. From the constraint $2\bar{\alpha} - 3\bar{\beta} = 1$, we have $\bar{\beta} = \frac{2\bar{\alpha} - 1}{3}$. Then

$$(2-p)\bar{\alpha} + (2-s)\bar{\beta} = \frac{s-2}{3}(4\bar{\alpha} + 1);$$

$$(4-p)\bar{\alpha} + (2-s)\bar{\beta} = \frac{1}{3}\left[(4s-2)\bar{\alpha} + (s-2)\right].$$

At $\alpha^{(1)}$ we have $\frac{s-2}{3}(4\alpha^{(1)} + 1) = 0$ and $\frac{1}{3}\left[(4s-2)\alpha^{(1)} + (s-2)\right] > 0$; at $\alpha^{(2)}$ we have $\frac{s-2}{3}(4\alpha^{(2)} + 1) = 0$ and $\frac{1}{3}\left[(4s-2)\alpha^{(2)} + (s-2)\right] < 0$. By linearity and since $\bar{\alpha} \in (\alpha^{(1)}, \alpha^{(2)})$, the inequalities hold. □

Now we have established the existence and uniqueness of the scaling parameters and verifying the required inequalities.

Lemma 2.4. Let $p = 6 - 2s$ with $s \in [0, \frac{1}{3})$. Define $m_c = \inf_{\mathbb{P}_c} I(u)$ and $S_c = \{u \in H^1(\mathbb{R}^3) \mid \|u\|_2^2 = c\}$. Then m_c is strictly decreasing with respect to $c > 0$.

Proof. Let $\{u_n\} \subset \mathbb{P}_c$ be a minimizing sequence for m_c, i.e., $\lim_{n\to\infty} I(u_n) = m_c$. By Lemma 2.1, $I(u)$ is coercive on $\mathbb{P}_c$, which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

For any $c' > c$, let $t = c'/c > 1$. By Lemma 2.3, for each u_n, there exists a unique pair of scaling parameters $(\bar{\alpha}, \bar{\beta})$ satisfying $2\bar{\alpha} - 3\bar{\beta} = 1$ such that the scaled function

$$w_n(x) := t^{\bar{\alpha}} u_n(t^{\bar{\beta}} x)$$

belongs to the Pohozaev manifold $\mathbb{P}_{c'}$. Note that the mass scales as

$$\|w_n\|_2^2 = t^{2\bar{\alpha}-3\bar{\beta}} \|u_n\|_2^2 = t\|u_n\|_2^2 = \frac{c'}{c} c = c'.$$

Since $w_n \in \mathbb{P}_{c'}$, we have the Pohozaev identity $P(w_n) = 0$. This allows us to eliminate the Hardy term in the energy functional. Recall from the proof of Lemma 2.1 that $I(w_n)$ can be rewritten as a sum of positive terms,

$$I(w_n) = K_1 \cdot a \int_{\mathbb{R}^3} |\nabla w_n|^2 dx + K_2 \cdot b \left(\int_{\mathbb{R}^3} |\nabla w_n|^2 dx \right)^2 + K_3 \cdot \int_{\mathbb{R}^3} |\nabla w_n|^2 w_n^2 dx,$$

where $K_1, K_2, K_3 > 0$.

Now we analyze the scaling behavior of each term for $t > 1$. From the proof of Lemma 2.3, we know that $\bar{\beta} \in (\beta^{(1)}, \beta^{(2)})$. By calculation we verified that $\beta^{(2)} = -1/2$. Therefore, we have

$$\bar{\beta} < -\frac{1}{2}.$$

Using $2\bar{\alpha} = 1 + 3\bar{\beta}$, we calculate that for $t > 1$,

$$\int_{\mathbb{R}^3} |\nabla w_n|^2 dx = t^{1+2\bar{\beta}} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx < \int_{\mathbb{R}^3} |\nabla u_n|^2 dx,$$

$$\left(\int_{\mathbb{R}^3} |\nabla w_n|^2 dx \right)^2 = t^{2+4\bar{\beta}} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 < \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2,$$

$$\int_{\mathbb{R}^3} |\nabla w_n|^2 w_n^2 dx = t^{2+5\bar{\beta}} \int_{\mathbb{R}^3} |\nabla u_n|^2 u_n^2 dx < \int_{\mathbb{R}^3} |\nabla u_n|^2 u_n^2 dx.$$

Since a, b, K_i are positive we can conclude that

$$I(w_n) < I(u_n).$$

Taking the limit as $n \to \infty$, we obtain

$$m_{c'} \le \lim_{n\to\infty} I(w_n) < \lim_{n\to\infty} I(u_n) = m_c.$$

Therefore, m_c is strictly decreasing with respect to $c > 0$. □

In the study of critical nonlinear problems, the lack of compact embedding is the central difficulty in proving the existence of minimizers. For our problem, the minimizing sequence $\{u_n\}$ may fail to converge strongly in $H^1(\mathbb{R}^3)$ due to energy concentration at infinity. To rule out this non-compactness, we introduce the critical energy threshold m_∞.

The value of m_∞ is directly expressible in terms of the best Sobolev-Hardy constant $S_{3,s}$, which is defined in Definition 2.4. By minimizing $I_\infty(u)$ on $\mathbb{P}_\infty$, the critical energy threshold m_∞ is explicitly given by

$$m_\infty = \left(\frac{1}{2} - \frac{1}{p} \right) S_{3,s}^{\frac{p}{p-2}}.$$

In our problem where $p = 2^*(s)$, the strong compactness of the minimizing sequence $\{u_n\}$ is guaranteed if and only if the ground state energy m_c satisfies $m_c < m_\infty$.

To establish the existence of solutions for large masses, we first construct a specific test function based on the extremal function of the Hardy-Sobolev inequality. Let $U(x)$ be the optimizer satisfying the Euler-Lagrange equation

$$-\Delta U = |x|^{-s}U^{p-1} \quad \text{in } \mathbb{R}^3.$$

For $N = 3$, it is well known that $U(x) \sim |x|^{-1}$ as $|x| \to \infty$. Consequently, $U \notin L^2(\mathbb{R}^3)$. To overcome this, we introduce a cut-off function $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x| \leq R$ and $\eta(x) = 0$ for $|x| \geq 2R$, where $R > 0$ is a fixed constant. We define the truncated test function as

$$u_R(x) := \eta(x)U(x) \in H^1(\mathbb{R}^3).$$

With this test function u_R, we have the following result concerning the energy level for large masses.

Lemma 2.5. Let U be the extremal function given in Definition 2.4. We fix $\phi_0 \in H^1(\mathbb{R}^3)$ satisfies $\phi_0(x) = U(x)$ for $|x| \leq 1$ and $\phi_0(x) = 0$ for $|x| \geq 2$.

We introduce three positive structural constants determined solely by this reference profile:

$$\mathcal{A} := \|\nabla\phi_0\|_2^2, \quad \mathcal{B} := \int_{\mathbb{R}^3} |x|^{-s}|\phi_0|^p dx, \quad \mathcal{M} := \|\phi_0\|_2^2.$$

Define the critical mass threshold c^* as the unique constant:

$$c^* := \left(\frac{b(6-2s)\mathcal{A}^2}{4\mathcal{B}}\right)^{\frac{1}{1-s}} \mathcal{M}.$$

Then for any $c > c^*$, we have $m_c < 0 < m_\infty$.

Proof. Let $U(x)$ be the optimizer for the Hardy-Sobolev inequality. Since $N = 3$, $U(x) \sim |x|^{-1}$ at infinity, so $U \notin L^2(\mathbb{R}^3)$. We introduce a cut-off function $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x| \leq R$ and $\eta(x) = 0$ for $|x| \geq 2R$, and define the test function $u_R(x) = \eta(x)U(x) \in H^1(\mathbb{R}^3)$. We fix $R = 1$, so $\mathcal{A}, \mathcal{B}, \mathcal{M}$ are fixed constants.

For any mass $c > 0$, we define the scaled function $v_c(x) = \sqrt{\frac{c}{\mathcal{M}}}u_R(x)$, which satisfies $\|v_c\|_2^2 = c$. We project v_c onto the Pohozaev manifold $\mathbb{P}_c$. By Lemma 2.2, there exists a unique $t_c \in \mathbb{R}$ such that $w_c := (v_c)^{t_c} \in \mathbb{P}_c$, which implies

$$\frac{1}{p}\int_{\mathbb{R}^3} |x|^{-s}|w_c|^p dx = \frac{1}{p}\left(a\int_{\mathbb{R}^3} |\nabla w_c|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla w_c|^2 dx\right)^2 + 5\int_{\mathbb{R}^3} |\nabla w_c|^2 w_c^2 dx\right).$$

By Lemma 2.2, $I(w_c) = \sup_{t\in\mathbb{R}} I((v_c)^t)$. To establish $m_c < 0$ for sufficiently large c, it suffices to show that the negative Hardy term asymptotically dominates the positive terms. We compare the scaling exponents of v_c with respect to c. Note that $\|\nabla v_c\|_2^2 = \frac{c}{\mathcal{M}}\mathcal{A}$ and $\int_{\mathbb{R}^3} |x|^{-s}|v_c|^p dx = (\frac{c}{\mathcal{M}})^{p/2}\mathcal{B}$. We obtain

$$\frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v_c|^2 dx\right)^2 = \frac{b\mathcal{A}^2}{4\mathcal{M}^2}c^2,$$
$$\frac{1}{p}\int_{\mathbb{R}^3} |x|^{-s}|v_c|^p dx = \frac{\mathcal{B}}{p\mathcal{M}^{p/2}}c^{p/2}.$$

For the energy to be negative, we require the Hardy term to be strictly larger than the Kirchhoff term (since $p/2 > 2$), which is

$$\frac{\mathcal{B}}{p\mathcal{M}^{p/2}}c^{p/2} > \frac{b\mathcal{A}^2}{4\mathcal{M}^4}c^2$$

Simplify and we have

$$c > \left(\frac{b(6-2s)\mathcal{A}^2}{4\mathcal{B}}\right)^{\frac{1}{1-s}} \mathcal{M}$$

Thus, for $c > c^*$, the negative Hardy term dominates the Kirchhoff term sufficiently to drive the energy $I(w_c)$ negative. Since $m_\infty > 0$, we conclude $m_c < 0 < m_\infty$. □

Lemma 2.6. Let $p = 6 - 2s$ with $s \in [0, \frac{1}{3})$. Then for any $c > c^*$, the infimum $m_c = \inf_{\mathbb{P}_c} I(u)$ is achieved; that is, there exists some $u_c \in \mathbb{P}_c$ such that $I(u_c) = m_c$.
Proof. Let $\{u_n\} \subset \mathbb{P}_c$ be a minimizing sequence for m_c, so we have

$$\lim_{n\to\infty} I(u_n) = m_c \quad \text{and} \quad P(u_n) = 0 \quad \forall n \in \mathbb{N}^*.$$

By Lemma 2.1, $I(u)$ is coercive on $\mathbb{P}_c$, therefore $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. For each u_n, we consider its Schwarz symmetric rearrangement u_n^*. Since $I(u_n^*) \le I(u_n)$ and the mass is preserved, we can assume without loss of generality that $\{u_n\}$ consists of radially symmetric non-negative functions. Since $\{u_n\}$ is bounded, up to a subsequence, there exists $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3); \\ u_n \to u \text{ in } L^q(\mathbb{R}^3) \text{ for } q \in (2, 2^*); \\ u_n \to u \text{ a.e. in } \mathbb{R}^3. \end{cases}$$

We now suppose $u = 0$ for a contradiction, then $u_n \to 0$ in L^p, therefore, $\int_{\mathbb{R}^3} |x|^{-s}|u_n|^p dx \to 0$. Since $P(u_n) = 0$, we have

$$a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 + 5\int_{\mathbb{R}^3} |\nabla u_n|^2 u_n^2 dx = \int_{\mathbb{R}^3} |x|^{-s}|u_n|^p dx \to 0.$$

Since each term on the left-hand side is non-negative, they must all converge to 0, which implies $I(u_n) \to 0$. This contradicts $I(u_n) \ge C_1 > 0$ from Lemma 2.1. Thus, $u \neq 0$.

Since we assume $c > c^*$, by Lemma 2.5 we have $m_c < m_\infty$. According to the Concentration-Compactness Principle, if energy loss occurs, we would have

$$m_c = \lim_{n\to\infty} I(u_n) \ge I(u) + m_\infty \ge m_\infty,$$

which is a contradiction. Thus, we have strong convergence of the Hardy term, which is

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |x|^{-s}|u_n|^p dx = \int_{\mathbb{R}^3} |x|^{-s}|u|^p dx.$$

By weak lower semicontinuity and $P(u_n) = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |x|^{-s}|u|^p dx &= \lim_{n\to\infty}\left[a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 + 5\int_{\mathbb{R}^3} |\nabla u_n|^2 u_n^2 dx\right] \\ &\ge a\int_{\mathbb{R}^3} |\nabla u|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + 5\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx,\end{aligned}$$

which implies $P(u) \le 0$. Now, consider the reduced energy functional which consists only of positive terms. We denote

$$J(u) := I(u) - \frac{1}{p}P(u) = K_1 a\int_{\mathbb{R}^3} |\nabla u|^2 dx + K_2 b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + K_3\int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx,$$

where $K_i > 0$. Since $P(u_n) = 0$, we have $I(u_n) = J(u_n)$. By Fatou's Lemma,

$$J(u) \leq \liminf_{n\to\infty} J(u_n) = \liminf_{n\to\infty} I(u_n) = m_c.$$

We claim $P(u) = 0$. Suppose for contradiction that $P(u) < 0$. By Lemma 2.2, there exists a unique $t < 0$ such that $u^t \in \mathbb{P}_c$. Using the scaling properties established in Lemma 2.2, we express $J(u^t)$ as

$$J(u^t) = K_1 a e^{2t} \int_{\mathbb{R}^3} |\nabla u|^2 dx + K_2 b e^{4t} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + K_3 e^{5t} \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx.$$

Since $u \neq 0$, the integral terms are strictly positive. Since $t < 0$, the exponential scaling factors satisfy $e^{2t}, e^{4t}, e^{5t} < 1$. Consequently, we strictly have

$$J(u^t) < K_1 a \int_{\mathbb{R}^3} |\nabla u|^2 dx + K_2 b \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + K_3 \int_{\mathbb{R}^3} |\nabla u|^2 u^2 dx = J(u).$$

Since $u^t \in \mathbb{P}_c$, we must have $I(u^t) \geq m_c$. Also, on $\mathbb{P}_c$, $I(u^t) = J(u^t)$. Combining these inequalities we get

$$m_c \leq I(u^t) = J(u^t) < J(u) \leq m_c,$$

which is a contradiction. Therefore, we must have $P(u) = 0$, which implies $u \in \mathbb{P}_c$. Since $u \in \mathbb{P}_c$, $I(u) \geq m_c$. Combined with $J(u) \leq m_c$ and $I(u) = J(u)$, we conclude $I(u) = m_c$. Thus, u is a ground state solution. □

Lemma 2.7. Let $p = 6 - 2s$ with $s \in [0, \frac{1}{3})$. Suppose $u_c \in \mathbb{P}_c$ achieves the minimum m_c. Then u_c is a weak solution of equation (1.1) for some Lagrange multiplier $\lambda_c \in \mathbb{R}$.
Proof. Following the strategy developed in Jeanjean **?** and Soave Soave (2020), we employ the Implicit Function Theorem to show that the Lagrange multiplier associated with the Pohozaev constraint vanishes.

Since u_c is a minimizer of I constrained to $\mathbb{P}_c$, strictly speaking, there exist two Lagrange multipliers $\lambda \in \mathbb{R}$ (for the norm constraint $\|u\|_2^2 = c$) and $\mu \in \mathbb{R}$ (for the Pohozaev constraint $P(u) = 0$) such that

$$I'(u_c) - \lambda u_c - \mu P'(u_c) = 0 \quad \text{in } (H^1)^*.$$

Our goal is to show that $\mu = 0$. Consider the fiber map $\Psi(t) = P(u_c^t)$, from Lemma 2.2, we know that $t = 0$ is the unique solution to $\Psi(t) = 0$ (since $u_c \in \mathbb{P}_c$) and that

$$\Psi'(0) = \frac{d}{dt} P(u_c^t)\Big|_{t=0} = \frac{d^2}{dt^2} I(u_c^t)\Big|_{t=0} < 0.$$

Consider an arbitrary variation $\varphi \in C_0^\infty(\mathbb{R}^3)$. For ε small enough, let $u_\varepsilon = \frac{u_c + \varepsilon\varphi}{\|u_c + \varepsilon\varphi\|_2}\sqrt{c}$. Clearly $u_\varepsilon \in S_c$. By the Implicit Function Theorem applied to the function $F(\varepsilon, t) = P((u_\varepsilon)^t)$, since $\frac{\partial F}{\partial t}(0,0) = \Psi'(0) \neq 0$, there exists a C^1 function $t(\varepsilon)$ defined for small ε such that $t(0) = 0$ and

$$P((u_\varepsilon)^{t(\varepsilon)}) = 0 \quad \Longrightarrow \quad (u_\varepsilon)^{t(\varepsilon)} \in \mathbb{P}_c.$$

Since u_c is a minimizer of I on $\mathbb{P}_c$, the function $\gamma(\varepsilon) := I((u_\varepsilon)^{t(\varepsilon)})$ achieves a local minimum at $\varepsilon = 0$. Thus, $\gamma'(0) = 0$.

Computing the derivative

$$\gamma'(0) = \langle I'((u_c)^{t(0)}), \frac{d}{d\varepsilon}(u_\varepsilon)^{t(\varepsilon)}|_{\varepsilon=0} \rangle.$$

Note that $(u_\varepsilon)^{t(\varepsilon)} = e^{\frac{3}{2}t(\varepsilon)} u_\varepsilon(e^{t(\varepsilon)}x)$, by calculation we have

$$\frac{d}{d\varepsilon}(u_\varepsilon)^{t(\varepsilon)}\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}(u_\varepsilon)\Big|_{\varepsilon=0} + t'(0) \cdot \frac{\partial}{\partial t}(u_c^t)\Big|_{t=0}.$$

Thus,

$$0 = \gamma'(0) = \langle I'(u_c), v\rangle + t'(0) \cdot \langle I'(u_c), \frac{\partial}{\partial t}(u_c^t)\big|_{t=0}\rangle.$$

where $v = \frac{d}{d\varepsilon} u_\varepsilon|_{\varepsilon=0}$. The second term vanishes because $P(u_c) = 0$. Therefore, we simply have $\langle I'(u_c), v\rangle = 0$ for any direction v tangent to the sphere S_c. This implies that $I'(u_c)$ is parallel to the normal vector of the sphere S_c at u_c, which is

$$I'(u_c) = \lambda_c u_c$$

for some Lagrange multiplier λ_c.

Thus, the multiplier μ associated with the Pohozaev constraint is 0, and u_c is a weak solution to the equation. □

Proof of Theorem 1.1. By Lemma 2.7, we know that the minimizer of m_c is a weak solution of equation (1.1). Furthermore, since the weak solution of equation (1.1) must belong to the Pohozaev set $\mathbb{P}_c$, this minimizer is consequently a ground state solution of equation (1.1) in S_c. □

3 CONCLUSIONS

In this paper, we have investigated the existence of normalized ground state solutions for a modified Kirchhoff equation involving the quasilinear term $u\Delta(u^2)$ and the Sobolev-Hardy critical exponent in $\mathbb{R}^3$. The presence of the singular Hardy potential and the critical exponent introduces significant difficulties regarding the compactness of the embedding. To overcome these challenges, we employed a constrained variational approach on the Pohozaev manifold $\mathbb{P}_c$.

By constructing a specific test function based on the truncated extremal of the Hardy-Sobolev inequality, we identified a critical mass threshold c^*, which is explicitly defined by structural constants derived from the reference profile. We successfully proved that for any prescribed mass $c > c^*$, the ground state energy satisfies the strict inequality $m_c < m_\infty$, thereby restoring the compactness of the minimizing sequence. Consequently, the existence of a ground state solution was established. This work fills a gap in the literature by addressing the complex interaction between the nonlocal Kirchhoff term, the quasilinear nonlinearity, and the singular Hardy potential in the critical setting.

Beyond the theoretical existence results, our findings have potential implications for computational mathematics and physics. The existence of ground state solutions provides a vital benchmark for testing the convergence and stability of numerical algorithms designed for singular quasilinear problems. Furthermore, the identified mass threshold c^* offers a precise criterion for parameter selection in numerical simulations of superfluid films and elastic materials, bridging the gap between abstract variational theory and practical computational applications.

Declarations

Availability of data and material

Not applicable.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Competing interests

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