η-RICCI–YAMABE SOLITONS ON SASAKIAN MANIFOLDS ADMITTING NON-LEVI-CIVITA CONNECTIONS

ABSTRACT. In this paper, we investigate η-Ricci–Yamabe solitons on Sasakian manifolds by considering a general affine connection rather than the Levi-Civita connection. The study focuses on the geometric properties of Sasakian manifolds admitting η-Ricci solitons with respect to a general connection under the conditions of Ricci pseudosymmetry and Ricci semisymmetry. Within this framework, we derive characterization results for Ricci pseudosymmetric and Ricci semisymmetric Sasakian manifolds corresponding to several important classes of connections, including the quarter-symmetric connection, the Schouten–Van Kampen connection, the Tanaka–Webster connection, and the Zamkovoy connection. Furthermore, we establish necessary and sufficient conditions for the existence of η-Ricci soliton, η-Yamabe soliton, and η-Einstein soliton structures on Sasakian manifolds endowed with a general connection. The results presented in this work extend and unify several known characterizations in Sasakian geometry by incorporating a broader class of connections.

1. Introduction

Geometric flow theory has played a central role in modern differential geometry, particularly since Hamilton's pioneering work on the Ricci flow [1, 2]. As self-similar solutions of geometric flows, Ricci solitons arise naturally in the study of singularity formation and geometric evolution equations. These structures generalize Einstein metrics and have been extensively investigated in various geometric settings (see, for instance, [3, 4]). Closely related to Ricci solitons are Yamabe solitons, which correspond to self-similar solutions of the Yamabe flow and have attracted significant attention in recent years [5, 6].

In order to unify and generalize these notions, Güler and Crasmareanu [7] introduced the concept of Ricci–Yamabe maps, which interpolate between Ricci and Yamabe flows. Motivated by this idea, several authors have studied Ricci–Yamabe solitons and their variants on different geometric structures. In particular, the notion of η-Ricci–Yamabe solitons has emerged as a natural extension in the framework of almost contact and paracontact geometry, where the presence of the structure vector field plays a fundamental role [8, 9].

Almost contact metric manifolds, and Sasakian manifolds in particular, constitute an important class of odd-dimensional manifolds with rich geometric structures. Since the seminal work of Tanno [10], Sasakian geometry has been extensively studied due to its close connections with Kähler geometry, CR-geometry, and mathematical physics. The study of Ricci solitons and related soliton structures

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on Sasakian manifolds has therefore become a natural and active area of research [11, 12].

Beyond the classical Levi-Civita connection, the use of non-metric or torsion-involved connections has proven to be fruitful in understanding deeper geometric phenomena. In this direction, Golab [13] introduced semi-symmetric and quarter-symmetric linear connections, which were later investigated in various geometric contexts. Schouten and Van Kampen [14] developed a canonical connection adapted to nonholonomic structures, while Zamkovoy [15] introduced canonical connections in paracontact geometry. These connections provide natural generalizations of the Levi-Civita connection and allow the study of curvature properties under broader geometric settings.

Sasakian manifolds admitting general connections have been studied by several authors. In particular, Biswas and Baishya [16, 17] initiated the study of Sasakian manifolds endowed with general connections and investigated generalized pseudo Ricci symmetric and almost pseudo symmetric structures. Further curvature restrictions with respect to quarter-symmetric metric connections were examined in [18]. More recently, conformal Ricci solitons on Sasakian manifolds admitting general connections were studied in [19], highlighting the relevance of soliton theory beyond the Levi-Civita framework.

On the other hand, curvature conditions such as Ricci pseudosymmetry and Ricci semisymmetry play an important role in the classification of Riemannian manifolds. These notions generalize local symmetry and have been widely investigated in contact and Lorentzian geometries. In particular, Ricci-pseudosymmetric manifolds admitting η-Ricci solitons were studied in various geometric settings [20]. The interaction between soliton structures and curvature restrictions continues to be an active research topic.

Motivated by the above developments, the present paper aims to study η-Ricci–Yamabe solitons on Sasakian manifolds admitting a general connection. Instead of restricting ourselves to the Levi-Civita connection, we consider a general affine connection and investigate the geometric consequences under the assumptions of Ricci pseudosymmetry and Ricci semisymmetry. Special attention is given to important classes of connections, including the quarter-symmetric connection, the Schouten–Van Kampen connection, the Tanaka–Webster connection, and the Zamkovoy connection.

The main purpose of this work is twofold. First, we obtain characterization results for Ricci pseudosymmetric and Ricci semisymmetric Sasakian manifolds with respect to a general connection. Second, we derive necessary and sufficient conditions for the existence of η-Ricci soliton, η-Yamabe soliton, and η-Einstein soliton structures on Sasakian manifolds endowed with such connections. Our results generalize and unify several known results in the literature and contribute to a deeper understanding of soliton geometry in the setting of Sasakian manifolds equipped with non-Levi-Civita connections.

2. Preliminary

An almost contact structure on a smooth manifold M of dimension $n = (2m + 1)$ is a triplet (ϕ, ξ, η), where ϕ is a $(1, 1)$-tensor field, ξ is a vector field, and η is a 1-form on M satisfying

(1) $$\phi^2\Theta_1 = -\Theta_1 + \eta(\Theta_1)\xi, \eta(\xi) = 1,$$

(2) $$\eta(\phi\Theta_1) = 0, \phi\xi = 0, rank\phi = 2n.$$

A smooth manifold M endowed with an almost contact structure is called an almost contact manifold. A Riemannian metric g on M is said to be compatible with an almost contact structure (ϕ, ξ, η), if

(3) $$g(\phi\Theta_1, \phi\Theta_2) = g(\Theta_1, \Theta_2) - \eta(\Theta_1)\eta(\Theta_2),$$

for all $\Theta_1, \Theta_2 \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all vector fields on M. An almost contact manifold endowed with a compatible Riemannian metric is said to be an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g). The fundamental 2-form Φ on $M(, \phi, \xi, \eta, g)$ is defined by

(4) $$\Phi(\Theta_1, \Theta_2) = g(\Theta_1, \phi\Theta_2)$$

for all $\Theta_1, \Theta_2 \in \chi(M)$. An almost contact metric manifold is said to be Sasakian manifold if

(5) $$(D_{\Theta_1}\phi)\Theta_2 = g(\Theta_1, \Theta_2)\xi - \eta(\Theta_2)\Theta_1,$$

where D denotes Levi_civita connection admitting the Riemannian connection of g. From the above equation, we conclude that for a Sasakian structure

(6) $$D_{\Theta_1}\xi = -\phi\Theta_1.$$

Lemma 1. *n-dimansional Sasakian manifold the following relation holds:*

(7) $$R(\Theta_1, \Theta_2)\xi = \eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2,$$

(8) $$R(\xi, \Theta_2)\Theta_3 = g(\Theta_2, \Theta_3)\xi - \eta(\Theta_3)\Theta_2,$$

(9) $$R(\Theta_1, \xi)\Theta_3 = -g(\Theta_1, \Theta_3)\xi + \eta(\Theta_3)\Theta_1,$$

(10) $$S(\Theta_1, \xi) = (n-1)\eta(\Theta_1),$$

(11) $$Q\xi = (n-1)\xi,$$

(12) $$(D_{\Theta_1}\eta)\Theta_2 = g(\Theta_1, \phi\Theta_2),$$

for all vector fields Θ_1, Θ_2 and Θ_3 on M, where R, S and Q are Riemann Curvature tensor,Ricci tensor and Ricci operator respectively.

It is also defined as

$$S(\Theta_1,\Theta_2)=g(Q\Theta_1,\Theta_2).$$

In this paper, the symboles D^G, D, D^q, D^Z, D^S and D^T are, respectively, denoted for general connection, Levi-Civita connection, quarter-symmetric metric connection, Zamkovoy connection, Schouten-Van Kampen connection and generalized Tanaka-Webster connection.

Recently, Biswas and Baishya introduced and studied a new connection, named general connection in Sasakian geometry[16, 17]. The general connection D^G is defined as

$$D^G_{\Theta_1}\Theta_2=D^{\Theta_2}_{\Theta_1}+\kappa_1\left[(D_{\Theta_1}\eta)(\Theta_2)\xi-\eta(\Theta_2)D_{\Theta_1}\xi\right]+\kappa_2\eta(\Theta_1)\phi\Theta_2, \tag{13}$$

the pair (κ_1,κ_2) being real constants. The beauty of such connection D^G lies in the fact that it has the flavour of

· quarter symmetric metric connection for $(\kappa_1,\kappa_2)=(0,-1)$ in [13, 18],

· Schouten-Van Kampen connection for $(\kappa_1,\kappa_2)=(1,0)$ in [14],

· Tanaka Webster connection for $(\kappa_1,\kappa_2)=(1,-1)$ in [10],

· Zamkovoy connection for $(\kappa_1,\kappa_2)=(1,1)$ in [15].

The torsion tensor T of the connection D^G satisfies

$$\begin{aligned} T^G(\Theta_1,\Theta_2) &= 2\kappa_1 g(\Theta_1,\phi\Theta_2)\xi+\kappa_1\left[\eta(\Theta_2)\phi\Theta_1-\eta(\Theta_1)\phi\Theta_2\right] \\ &\quad +\kappa_2\left[\eta(\Theta_1)\phi\Theta_2-\eta(\Theta_2)\phi\Theta_1\right]. \end{aligned} \tag{14}$$

If we choose $\Theta_2=\xi$ in (13) and by useing (6), we have

$$D^G_{\Theta_1}\xi=(\kappa_1-1)\phi\Theta_1. \tag{15}$$

Lemma 2. *For an n-dimensional Sasakian manifold admitting general connection and if R^G, S^G, r^G, Q^G are Riemannian curvature tensor, Ricci tensor,scalar curvature and Ricci operator in general connection, then following results ([16, 17]) hold:*

$$\begin{aligned} R^G(\Theta_1,\Theta_2)\Theta_3 &= R(\Theta_1,\Theta_2)\Theta_3+\left(\kappa_1^2-2\kappa_1\right)\left[g(\Theta_3,\phi\Theta_1)\phi\Theta_2+g(\Theta_2,\phi\Theta_3)\phi\Theta_1\right] \\ &\quad +(\kappa_1-\kappa_1\kappa_2+\kappa_2)\left[g(\Theta_1,\Theta_3)\eta(\Theta_2)\xi-g(\Theta_2,\Theta_3)\eta(\Theta_1)\xi\right. \\ &\quad \left.+\eta(\Theta_1)\eta(\Theta_3)\Theta_2-\eta(\Theta_2)\eta(\Theta_3)\Theta_1\right]-2\kappa_2 g(\Theta_2,\phi\Theta_1)\phi\Theta_3, \end{aligned} \tag{16}$$

$$S^G(\Theta_2,\Theta_3)=S(\Theta_2,\Theta_3)-\Lambda_1 g(\Theta_2,\Theta_3)+\Lambda_2\eta(\Theta_2)\eta(\Theta_3), \tag{17}$$

$$S^G(\Theta_2,\xi)=-S^G(\xi,\Theta_2)=-(n-1)\Lambda_3\eta(\Theta_2), \tag{18}$$

$$Q^G\Theta_2=Q\Theta_2-\Lambda_1\Theta_2+\Lambda_2\eta(\Theta_2)\xi, \tag{19}$$

$$Q^G\xi = -(n-1)\Lambda_3\xi, \tag{20}$$

$$r^G = r - \Lambda_1 n + \Lambda_2, \tag{21}$$

$$R^G(\Theta_1, \Theta_2)\xi = \Lambda_3[\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1], \tag{22}$$

$$R^G(\xi, \Theta_2)\Theta_3 = \Lambda_3[-g(\Theta_2, \Theta_3)\xi + \eta(\Theta_3)\Theta_2], \tag{23}$$

$$R^G(\Theta_1, \xi)\Theta_3 = \Lambda_3[g(\Theta_1, \Theta_3)\xi - \eta(\Theta_3)\Theta_1], \tag{24}$$

and where

$$\Lambda_1 = \kappa_1^2 - \kappa_1 - \kappa_2 - \kappa_1\kappa_2, \tag{25}$$

$$\Lambda_2 = \kappa_1^2 + (n-2)\kappa_1\kappa_2 - n(\kappa_1 + \kappa_2), \tag{26}$$

$$\Lambda_3 = \kappa_1 - \kappa_1\kappa_2 + \kappa_2 - 1. \tag{27}$$

In the calculations made according to the general connection, some other connections can be expressed as follows with the help of some special choices of $\Lambda_1, \Lambda_2, \Lambda_3$:

∘ For quarter-symmetric metric connection

$$\Lambda_1 = 1, \Lambda_2 = n, \Lambda_3 = -2, \tag{28}$$

∘ For generalized Tanaka-Webster connection

$$\Lambda_1 = 2, \Lambda_2 = 3-n, \Lambda_3 = 0, \tag{29}$$

∘ For Zamkovoy connection

$$\Lambda_1 = -2, \Lambda_2 = -1-n, \Lambda_3 = 0, \tag{30}$$

∘ For Schouten-Van Kampen connection

$$\Lambda_1 = 0, \Lambda_2 = 1-n, \Lambda_3 = 0. \tag{31}$$

In recently years, the theoretical concept of geometric flows, such as Ricci flow and Yamabe flow, has been the focus of numerous studies in geometry. In 2019, Güler and Crasmareanu introduced another geometric flow under the Ricci-Yamabe transformation [7]. This transformation is a scalar combination of the Ricci and Yamabe flows.

The Ricci-Yamabe flow is an evolution equation for the metrics on the Riemannian or semi-Riemannian manifolds defined as

$$\frac{\partial}{\partial t} g(t) = -2\alpha S(t) + \beta r(t) g(t), g_0 = g(0),$$

where S and r are Ricci tensor and scalar curvature of manifold and $\alpha, \beta, \lambda \in \mathbb{R}$. The Ricci-Yamabe soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.

The Ricci-Yamabe flow can be considered as a Riemannian, singular Riemannian, or semi-Riemannian flow depending on the signs of α and β. Such different choices are valuable in certain mathematical and physical models, such as relativistic theories.

A Ricci-Yamabe soliton on (M, g) is a data $(g, V, \lambda, \alpha, \beta)$ fulfilling

$$L_V g + 2\alpha S + (2\lambda - \beta r) g = 0, \tag{32}$$

where L denote the Lie-derivative, S and r are Ricci tensor and scalar curvatures, respectively, and λ, α, β are real constants.

The Ricci-Yamabe soliton is said to be expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$. If λ, β and α are smooth functions on M , then a Ricci-Yamabe soliton is called an almost Ricci-Yamabe soliton. If $\beta = 0, \alpha = 1$, then Ricci-Yamabe soliton induces Ricci soliton [1]. Similarly, it turns into Yamabe soliton if $\beta = 1, \alpha = 0$ [2] . Also, if $\beta = -1$, $\alpha = 1$, it reduces to an Einstein soliton [3]. The Ricci-Yamabe soliton is said to be proper if $\alpha \neq 0, 1$.

On the other hand, Sıddıkı and Akyol described Ricci-Yamabe solitons in 2020[9]. A η-Ricci-Yamabe soliton on (M, g) is a data $(g, V, \lambda, \mu, \alpha, \beta)$ fulfilling

$$L_V g + 2\alpha S + (2\lambda - \beta r) g + 2\mu\eta \otimes \eta = 0, \tag{33}$$

where L being the Lie-derivative, S indicates the Ricci tensor, r denotes the scalar curvature and $\lambda, \mu, \alpha, \beta \in \mathbb{R}$. If λ, μ, β and α are smooth functions on M , then a η-Ricci-Yamabe soliton is called an almost η-Ricci-Yamabe soliton. In particular, if $\mu = 0$, then the notion of η-Ricci-Yamabe soliton $(g, V, \lambda, \mu, \alpha, \beta)$ reduces to the notion of Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta)$.

3. Almost η-Ricci-Yamabe Solitons on Sasaki Manifolds Admitting General Connection

We consider a Sasaki manifold admitting general connection admitting an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, \alpha, \beta)$. Then from (33), it obvious that

$$\left(L_\xi^G g\right)(\Theta_1, \Theta_2) + 2\alpha S^G(\Theta_1, \Theta_2) + \left(2\lambda - \beta r^G\right) g(\Theta_1, \Theta_2) + 2\mu\eta(\Theta_1)\eta(\Theta_2) = 0. \tag{34}$$

Next, we will express the Lie derivative along ξ on M admitting general connection as follows:

$$\begin{aligned} \left(L_\xi^G g\right)(\Theta_1, \Theta_2) &= L_\xi^G g(\Theta_1, \Theta_2) - g\left(L_\xi^G \Theta_1, \Theta_2\right) - g\left(\Theta_1, L_\xi^G \Theta_2\right) \\ &= L_\xi^G g(\Theta_1, \Theta_2) - g\left([\xi, \Theta_1]_G, \Theta_2\right) - g\left(\Theta_1, [\xi, \Theta_2]_G\right). \end{aligned}$$

By means of (4) and (15) , the last equation reduces to

$$\left(L_\xi^G g\right)(\Theta_1, \Theta_2) = 0. \tag{35}$$

By virtue of (35), the equation (34) takes the following form

$$2\alpha S^{G}(\Theta_1,\Theta_2)+\left(2\lambda-\beta r^{G}\right)g(\Theta_1,\Theta_2)+2\mu\eta(\Theta_1)\eta(\Theta_2)=0. \tag{36}$$

Thus, we can state the following theorem.

Theorem 1. *If $(g,\xi,\lambda,\mu,\alpha,\beta)$ is an almost η-Ricci-Yamabe soliton on Sasaki manifold admitting general connection M, then M is an η-Einstein manifold provided $2\lambda\neq\beta r^{G}$.*

Specifically, if $2\lambda\neq\beta r^{G}$ and $\mu=0$, Sasakian manifold admitting general connection M admitting η-Ricci-Yamabe soliton reduces to Einstein manifold.

If we choose $\Theta_2=\xi$ in (36), we have

$$2\alpha S^{G}(\Theta_1,\xi)=\left[\beta r^{G}-2(\lambda+\mu)\right]\eta(\Theta_1). \tag{37}$$

If we use (18) in (37), we have

$$\lambda+\mu=\frac{1}{2}\beta r^{G}+\alpha(n-1)\Lambda_3. \tag{38}$$

Thus, we can give an important result of Theorem-1 as follows.

Corollary 1. *Let M be a Sasakian manifold admitting almost η-Ricci-Yamabe soliton admitting general connection D^{G}, then λ and μ are related by (38).*

Using (38), we will characterize λ and μ for the Sasakian manifold admitting the almost η-Ricci-Yamabe soliton according to different connections as follows.

Theorem 2. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be an almost η-Ricci-Yamabe soliton on M. Then the following holds:*

i. For quarter-symmetric metric connection D^{q}

$$\lambda+\mu=\frac{1}{2}\beta r^{G}-2\alpha(n-1),$$

ii. For Schouten-Van Kampen connection D^{S}

$$\lambda+\mu=\frac{1}{2}\beta r^{G},$$

iii. For Zamkovoy connection D^{Z}

$$\lambda+\mu=\frac{1}{2}\beta r^{G},$$

iv. For generalized Tanaka Webster connection D^{T}

$$\lambda+\mu=\frac{1}{2}\beta r^{G}.$$

Again, by using (38) , we have characterizations λ and μ for different solitons for an n-dimensional Sasakian manifold admitting general connection.

Theorem 3. *Let M be an $n=(2m+1)-$dimensional Sasakian manifold admitting general connection D^{G}. Then the following holds:*

i. For Sasakian manifold admitting η-Ricci soliton

$$\lambda+\mu=(n-1)\Lambda_3,$$

ii. For Sasakian manifold admitting η-Yamabe soliton

$$\lambda+\mu=\frac{1}{2}r^{G},$$

iii. For Sasakian manifold admitting η-Einstein manifold

$$\lambda+\mu=(n-1)\,\Lambda_3-\frac{1}{2}r^{G}.$$

Finally, we can classify the n-dimensional Sasakian manifold as follows, depending on the choice of connection on it and the type of soliton it accepts.

Theorem 4. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold and (g,ξ,λ,μ) be an almost η-Ricci soliton on M. Then the following holds:*
i. For quarter-symmetric metric connection D^q

$$\lambda+\mu=-2\,(n-1)\,,$$

ii. For Schouten-Van Kampen connection D^S

$$\lambda+\mu=0,$$

iii. For Zamkovoy connection D^Z

$$\lambda+\mu=0,$$

iv. For generalized Tanaka Webster connection D^T

$$\lambda+\mu=0.$$

Theorem 5. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold and (g,ξ,λ,μ) be an almost η-Yamabe soliton on M admitting by any of connections the quarter-symmetric metric connection, generalized Tanaka Webster, Zamkovoy or Schouten-Van Kampen. Then*

$$\lambda+\mu=\frac{1}{2}r^{G}.$$

Theorem 6. *Let M be an $n=(2m+1)$-dimensional Sasakian manifold and (g,ξ,λ,μ) be an almost η-Einstein soliton on M. Then the following holds:*
i. For quarter-symmetric metric connection D^q

$$\lambda+\mu=-2\,(n-1)-\frac{1}{2}r^{G},$$

ii. For Schouten-Van Kampen connection D^S

$$\lambda+\mu=-\frac{1}{2}r^{G},$$

iii. For Zamkovoy connection D^Z

$$\lambda+\mu=-\frac{1}{2}r^{G},$$

iv. For generalized Tanaka Webster connection D^T

$$\lambda+\mu=-\frac{1}{2}r^G.$$

Definition 1. *Let* M *be an* $n=(2m+1)-$*dimensional Sasakian manifold admitting general connection* D^G. *If* $R^G\cdot S^G$ *and* $Q^G\left(g,S^G\right)$ *are linearly dependent, then the* M *is said to be* ***Ricci pseudosymmetric***.

In this case, there exists a function L_R on M such that

$$R^G\cdot S^G=\mathcal{L}_RQ^G\left(g,S^G\right).$$

In particular, if $\mathcal{L}_R=0$, the M is said to be **Ricci semisymmetric**.

Theorem 7. *Let* M *be an* $n=(2m+1)-$*dimensional Sasakian manifold admitting general connection* D^g *and* $(g,\xi,\lambda,\mu,\alpha,\beta)$ *be an* η*-Ricci-Yamabe soliton on* M. *If* M *is a Ricci pseudosymmetric, then at least one of the following is true:*
i. $\mathcal{L}_R=-\Lambda_3$,
ii. $\lambda=\alpha(n-1)\Lambda_3-\frac{1}{2}\beta r^G$ *and* $\mu=\beta r^G$,
iii. M *is an expanding if* $\alpha(n-1)\Lambda_3>\frac{1}{2}\beta r^G$,
iv. M *is a steady if* $\alpha(n-1)\Lambda_3=\frac{1}{2}\beta r^G$,
v. M *is a shrinking if* $\alpha(n-1)\Lambda_3<\frac{1}{2}\beta r^G$.

Proof. Let's assume that M is a Ricci pseudosymmetric and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be almost $\eta-$Ricci-Yamabe soliton on M admitting general connection. That's mean

$$\left(R^G(\Theta_1,\Theta_2)\cdot S^G\right)(\Theta_4,\Theta_5)=\mathcal{L}_RQ^G\left(g,S^G\right)(\Theta_4,\Theta_5;\Theta_1,\Theta_2),$$

for all $\Theta_1,\Theta_2,\Theta_4,\Theta_5\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned}&S^G\left(R^G(\Theta_1,\Theta_2)\Theta_4,\Theta_5\right)+S^G\left(\Theta_4,R^G(\Theta_1,\Theta_2)\Theta_5\right)\\&=\mathcal{L}_R\left\{S^G((\Theta_1\wedge_g\Theta_2)\Theta_4,\Theta_5)+S^G(\Theta_4,(\Theta_1\wedge_g\Theta_2)\Theta_5)\right\}.\end{aligned}\tag{39}$$

If we choose $\Theta_5=\xi$ in (34), we get

$$\begin{aligned}&S^G\left(R^G(\Theta_1,\Theta_2)\Theta_4,\xi\right)+S^G\left(\Theta_4,R^G(\Theta_1,\Theta_2)\xi\right)\\&=\mathcal{L}_R\left\{S^G((\Theta_1\wedge_g\Theta_2)\Theta_4,\xi)+S^G(\Theta_4,(\Theta_1\wedge_g\Theta_2)\xi)\right\}.\end{aligned}$$

If we make use of $(1),(18),(22)$ in the last equality, we have

$$\begin{aligned}&-(n-1)\Lambda_3g(\eta(\Theta_2)\Theta_1-\eta(\Theta_1)\Theta_2,\Theta_4)\\&+\Lambda_3S^G(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4)\\&=\mathcal{L}_R\{-(n-1)\Lambda_3g(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4)\\&+S^G(\eta(\Theta_2)\Theta_1-\eta(\Theta_1)\Theta_2,\Theta_4)\}.\end{aligned}\tag{40}$$

If we use (36) in (40), we get

$$\left[2\alpha(n-1)\Lambda_3-\left(2\lambda-\beta r^G\right)\right][\Lambda_3+\mathcal{L}_R]\,g(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4)=0.$$

This completes the proof. ∎

We can give some important results of this theorem as follows.

Corollary 2. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be an η-Ricci-Yamabe soliton on M. If M is a Ricci semisymmetric, then the general connection D^G reduces to Zamkovoy connection D^Z.*

Corollary 3. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting quarter-symmetric metric connection and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be an η-Ricci-Yamabe soliton on M. If M is a Ricci pseudosymmetric, then the following holds:*
i. $\mathcal{L}_R = 2$,
ii. $\lambda = 2\alpha(1-n) - \frac{1}{2}\beta r^G$ *and* $\mu = \beta r^G$,
iii. *M is an expanding if* $2\alpha(1-n) > \frac{1}{2}\beta r^G$,
iv. *M is a steady if* $2\alpha(1-n) = \frac{1}{2}\beta r^G$,
v. *M is a shrinking if* $2\alpha(1-n) < \frac{1}{2}\beta r^G$.

Corollary 4. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be an η-Ricci-Yamabe soliton on M admitting by any of the connections generalized Tanaka Webster, Zamkovoy or Schouten-Van Kampen. If M is a Ricci pseudosymmetric, then the following statements holds:*
i. M is a Ricci semisymmetric,
ii. $\lambda = -\frac{1}{2}\beta r^G$ *and* $\mu = \beta r^G$,
iii. *M is an expanding if* $\beta r^G < 0$,
iv. *M is a steady if* $\beta r^G = 0$,
v. *M is a shrinking if* $\beta r^G > 0$.

Corollary 5. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a Ricci pseudosymmetric, then the following holds:*
i. The η-Ricci soliton reduces to Ricci soliton,
ii. $\lambda = (n-1)\Lambda_3$ *and* $\mu = 0$,
iii. *M is an expanding if* $\Lambda_3 > 0$,
iv. *M is a steady if* $\Lambda_3 = 0$,
v. *M is a shrinking if* $\Lambda_3 < 0$.

Corollary 6. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and (g, ξ, λ, μ) be an η-Yamabe soliton on M. If M is a Ricci pseudosymmetric, then the following holds:*
i. $\lambda = -\frac{1}{2}r^G$ *and* $\mu = r^G$,
ii. *M is an expanding if* $r^G < 0$,
iii. *M is a steady if* $r^G = 0$,
iv. *M is a shrinking if* $r^G > 0$.

Corollary 7. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and (g, ξ, λ, μ) be an η-Einstein soliton on M. If M is a Ricci pseudosymmetric, then the following holds:*
i. $\lambda = (n-1)\Lambda_3 + \frac{1}{2}r^G$ *and* $\mu = -r^G$,
ii. *M is an expanding if* $(n-1)\Lambda_3 + \frac{1}{2}r^G > 0$,
iii. *M is a steady if* $(n-1)\Lambda_3 + \frac{1}{2}r^G = 0$,
iv. *M is a shrinking if* $(n-1)\Lambda_3 + \frac{1}{2}r^G < 0$.

For an $n = (2m+1)$-dimensional semi-Riemann manifold M, the projective curvature tensor is defined as

$$P(\Theta_1, \Theta_2)\Theta_3 = R(\Theta_1, \Theta_2)\Theta_3 - \frac{1}{n-1}\left[S(\Theta_2, \Theta_3)\Theta_1 - S(\Theta_1, \Theta_3)\Theta_2\right].$$

Then, for an $n = (2m+1)$-dimensional Sasakian manifold admitting general connection, the projective curvature tensor is defined as

(41)
$$P^G(\Theta_1, \Theta_2)\Theta_3 = R^G(\Theta_1, \Theta_2)\Theta_3 - \frac{1}{n-1}\left[S^G(\Theta_2, \Theta_3)\Theta_1 - S^G(\Theta_1, \Theta_3)\Theta_2\right].$$

If we choose $\Theta_3 = \xi$ in (41), we can write

$$P^G(\Theta_1, \Theta_2)\xi = 0, \tag{42}$$

and similarly if we take the inner product of both sides of (41) by ξ, we get

$$\eta\left(P^G(\Theta_1, \Theta_2)\Theta_3\right) = 0. \tag{43}$$

Theorem 8. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be an η-Ricci-Yamabe soliton on M. If M is a projectively Ricci pseudosymmetric, then the following holds:*
i) M is a projectively Ricci semi symmetrik.
ii) $\lambda = \alpha(n-1)\Lambda_3 + \frac{1}{2}\beta r^G$ and $\mu = 0$,
iii) η-Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton.
iii) M is an expanding if $\alpha(n-1)\Lambda_3 + \frac{1}{2}\beta r^G > 0$,
iv) M is a steady if $\alpha(n-1)\Lambda_3 + \frac{1}{2}\beta r^G = 0$,
v) M is a shrinking$\alpha(n-1)\Lambda_3 + \frac{1}{2}\beta r^G < 0$.

Proof. Let's assume that $n = (2m+1)$-dimensional Sasakian manifold M be a projectively Ricci pseudosymmetric and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be almost $\eta-$Ricci-Yamabe soliton on M admitting general connection. That's mean

$$\left(P^G(\Theta_1, \Theta_2)\cdot S^G\right)(\Theta_4, \Theta_5) = \mathcal{L}_P Q^G\left(g, S^G\right)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(TM)$. From the last equation, we can easily write

(44)
$$\begin{aligned} &S^G\left(P^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5\right) + S^G\left(\Theta_4, P^G(\Theta_1, \Theta_2)\Theta_5\right) \\ &= \mathcal{L}_P\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\right\}. \end{aligned}$$

If we choose $\Theta_5 = \xi$ in (44), we get

$$\begin{aligned} &S^G\left(P^G(\Theta_1, \Theta_2)\Theta_4, \xi\right) + S^G\left(\Theta_4, P^G(\Theta_1, \Theta_2)\xi\right) \\ &= \mathcal{L}_P\left\{S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \xi) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\xi)\right\}. \end{aligned}$$

If we make use of (18), (42) and (43) in the last equality, we have

(45)
$$\begin{aligned} &\mathcal{L}_P\{-(n-1)\Lambda_3 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ &+ S^G(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\} = 0. \end{aligned}$$

If we use (36) in (45), we have

$$\left[-2\alpha(n-1)\Lambda_3 - \left(\beta r^G - 2\lambda\right)\right]\mathcal{L}_P g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) = 0.$$

This completes the proof. □

We can give the following results as follows.

Corollary 8. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting quarter-symmetric metric connection and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be an η-Ricci-Yamabe soliton on M. If M is a projectively Ricci pseudosymmetric, then the following holds:*
i. M is a projectively Ricci semisymmetrik.,
ii. $\lambda = \frac{1}{2}\beta r^G - 2\alpha(n-1)$ and $\mu = 0$,
iii. η-Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton.
iv. M is an expanding if $\frac{1}{2}\beta r^G > 2\alpha(n-1)$,
v. M is a steady if $\frac{1}{2}\beta r^G = 2\alpha(n-1)$,
vi. M is a shrinking if $\frac{1}{2}\beta r^G < 2\alpha(n-1)$.

Corollary 9. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be an η-Ricci-Yamabe soliton on M admitting by any of the connections generalized Tanaka Webster, Zamkovoy or Schouten-Van Kampen. If M is a projectively Ricci pseudosymmetric, then the following holds:*
i. M is a projectively Ricci semisymmetric,
ii. η-Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton,
iii. $\lambda = \frac{1}{2}\beta r^G$ and $\mu = 0$,
iv. M is an expanding if $\beta r^G > 0$,
v. M is a steady if $\beta r^G = 0$,
vi. M is a shrinking if $\beta r^G < 0$.

Corollary 10. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a projectively Ricci pseudosymmetric, then the following holds:*
i. The η-Ricci soliton reduces to Ricci soliton,
ii. M is a projectively Ricci semisymmetric,
ii. $\lambda = (n-1)\Lambda_3$ and $\mu = 0$,
iii. M is an expanding if $\Lambda_3 > 0$,
iv. M is a steady if $\Lambda_3 = 0$,
v. M is a shrinking if $\Lambda_3 < 0$,
vi. $\mathcal{L}_P = 2\Lambda_3$.

Corollary 11. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and (g, ξ, λ, μ) be an η-Yamabe soliton on M. If M is a projectively Ricci pseudosymmetric, then the following holds:*
i. M is a projectively Ricci semisymmetric,
ii. η-Yamabe soliton reduces to Yamabe soliton,
iii. $\lambda = \frac{1}{2}r^G$ and $\mu = 0$,
iv. M is an expanding if $r^G > 0$,
v. M is a steady if $r^G = 0$,
vi. M is a shrinking if $r^G < 0$.

Corollary 12. *Let M be an $n = (2m+1)-$dimensional Sasakian manifold admitting general connection and (g, ξ, λ, μ) be an η-Einstein soliton on M. If M is a projectively Ricci pseudosymmetric, then the following holds:*
i. M is a projectively Ricci semisymmetric,
ii. η-Einstein soliton reduces to Einstein soliton,
i. $\lambda = (n-1)\Lambda_3 - \frac{1}{2}r^G$ and $\mu = 0$,
ii. M is an expanding if $(n-1)\Lambda_3 > r^G$,
iii. M is a steady if $(n-1)\Lambda_3 = \frac{1}{2}r^G$,
iv. M is a shrinking if $(n-1)\Lambda_3 < r^G$.

For an $n=(2m+1)$-dimensional semi-Riemann manifold M, the W_1-curvature tensor is defined as

$$W_1(\Theta_1,\Theta_2)\Theta_3 = R(\Theta_1,\Theta_2)\Theta_3 + \frac{1}{n-1}\left[S(\Theta_2,\Theta_3)\Theta_1 - S(\Theta_1,\Theta_3)\Theta_2\right].$$

Then, for an $n=(2m+1)$-dimensional Sasakian manifold admitting general connection, the W_1-curvature tensor is defined as

(46)
$$W_1^G(\Theta_1,\Theta_2)\Theta_3 = R^G(\Theta_1,\Theta_2)\Theta_3 + \frac{1}{n-1}\left[S^G(\Theta_2,\Theta_3)\Theta_1 - S^G(\Theta_1,\Theta_3)\Theta_2\right].$$

If we choose $\Theta_3=\xi$ in (46), we can write

$$W_1^G(\Theta_1,\Theta_2)\xi = 2\Lambda_3\left[\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1\right], \tag{47}$$

and similarly if we take the inner product of both sides of (46) by ξ, we get

$$\eta\left(W_1^G(\Theta_1,\Theta_2)\Theta_3\right) = 2\Lambda_3 g\left(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2,\Theta_3\right). \tag{48}$$

Theorem 9. *Let M be an $n=(2m+1)-$dimensional Sasakian manifold admitting general connection and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be an η-Ricci-Yamabe soliton on M. If M is a W_1-Ricci pseudosymmetric, then the following holds:*
i) $\lambda = \alpha(n-1)\Lambda_3 - \frac{1}{2}\beta r^G$ and $\mu = \beta r^G$,
ii) $\mathcal{L}_{W_1} = -2\Lambda_3$,
iii) M is an expanding if $\alpha(n-1)\Lambda_3 > \frac{1}{2}\beta r^G$,
iv) M is a steady if $\alpha(n-1)\Lambda_3 = \frac{1}{2}\beta r^G$,
v) M is a shrinking $\alpha(n-1)\Lambda_3 < \frac{1}{2}\beta r^G$.

Proof. Let's assume that $n=(2m+1)$-dimensional Sasakian manifold M be a W_1-Ricci pseudosymmetric and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be almost $\eta-$Ricci-Yamabe soliton on M admitting general connection. That's mean

$$\left(W_1^G(\Theta_1,\Theta_2)\cdot S^G\right)(\Theta_4,\Theta_5) = \mathcal{L}_{W_1}Q^G\left(g,S^G\right)(\Theta_4,\Theta_5;\Theta_1,\Theta_2),$$

for all $\Theta_1,\Theta_2,\Theta_4,\Theta_5\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} &S^G\left(W_1^G(\Theta_1,\Theta_2)\Theta_4,\Theta_5\right) + S^G\left(\Theta_4,W_1^G(\Theta_1,\Theta_2)\Theta_5\right) \\ &= \mathcal{L}_{W_1}\left\{S^G\left((\Theta_1\wedge_g\Theta_2)\Theta_4,\Theta_5\right) + S^G\left(\Theta_4,(\Theta_1\wedge_g\Theta_2)\Theta_5\right)\right\}. \end{aligned} \tag{49}$$

If we choose $\Theta_5=\xi$ in (49), we get

$$\begin{aligned} &S^G\left(W_1^G(\Theta_1,\Theta_2)\Theta_4,\xi\right) + S^G\left(\Theta_4,W_1^G(\Theta_1,\Theta_2)\xi\right) \\ &= \mathcal{L}_{W_1}\left\{S^G\left((\Theta_1\wedge_g\Theta_2)\Theta_4,\xi\right) + S^G\left(\Theta_4,(\Theta_1\wedge_g\Theta_2)\xi\right)\right\}. \end{aligned}$$

If we make use of (18) and (47) in the last equality, we have

$$\begin{aligned} &-(n-1)\Lambda_3\eta\left(W_1^G(\Theta_1,\Theta_2)\Theta_4\right) \\ &+2\Lambda_3 S^G\left(\Theta_4,\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1\right) \\ &= \mathcal{L}_{W_1}\{-(n-1)\Lambda_3 g\left(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4\right) \\ &+ S^G\left(\eta(\Theta_2)\Theta_1-\eta(\Theta_1)\Theta_2,\Theta_4\right)\}. \end{aligned} \tag{50}$$

If we use (48) in (50), we get

$$
\begin{aligned}
&-2(n-1)\Lambda_3^2 g(\eta(\Theta_2)\Theta_1-\eta(\Theta_1)\Theta_2,\Theta_4)\\
&+2\Lambda_3 S^G(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4)\\
&=\mathcal{L}_{W_1}\{-(n-1)\Lambda_3 g(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4)\\
&+S^G(\eta(\Theta_2)\Theta_1-\eta(\Theta_1)\Theta_2,\Theta_4)\}.
\end{aligned} \tag{51}
$$

If we use (36) in (51), we have

$$
\left[2\alpha(n-1)\Lambda_3+\left(\beta r^G-2\lambda\right)\right]\left[2\Lambda_3+\mathcal{L}_{W_1}\right]g(\eta(\Theta_1)\Theta_2-\eta(\Theta_2)\Theta_1,\Theta_4)=0.
$$

This proves our assertions. □

We can give some important results of this theorem as follows.

Corollary 13. *Let M be an $n=(2m+1)-$dimensional Sasakian manifold admitting general connection and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be an η-Ricci-Yamabe soliton on M. If M is a W_1-Ricci pseudosymmetric, then the following holds:*
i. General connection D^G reduces to Zamkovoy connection D^z,
ii. $\lambda=\alpha(1-n)\Lambda_3-\frac{1}{2}\beta r^G$ and $\mu=\beta r^G$,
iii. M is an expanding if $\alpha(1-n)\Lambda_3>\frac{1}{2}\beta r^G$,
iv. M is a steady if $\alpha(1-n)\Lambda_3=\frac{1}{2}\beta r^G$,
v. M is a shrinking if $\alpha(1-n)\Lambda_3<\frac{1}{2}\beta r^G$.

Corollary 14. *Let M be an $n=(2m+1)-$dimensional Sasakian manifold admitting quarter-symmetric metric connection and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be an η-Ricci-Yamabe soliton on M. If M is a W_1-Ricci pseudosymmetric, then the following holds:*
i. $\mathcal{L}_{W_1}=4$,
ii. $\lambda=\frac{1}{2}\beta r^G-2\alpha(n-1)$ and $\mu=0$,
iii. η-Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton,
iv. M is an expanding if $\frac{1}{2}\beta r^G>2\alpha(n-1)$,
v. M is a steady if $\frac{1}{2}\beta r^G=2\alpha(n-1)$,
vi. M is a shrinking if $\frac{1}{2}\beta r^G<2\alpha(n-1)$.

Corollary 15. *Let M be an $n=(2m+1)-$dimensional Sasakian manifold and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be an η-Ricci-Yamabe soliton on M admitting by any of the connections generalized Tanaka Webster, Zamkovoy or Schouten-Van Kampen. If M is a W_1-Ricci pseudosymmetric, then the following holds:*
i. M is a W_1-Ricci semisymmetric,
ii. $\lambda=\frac{1}{2}\beta r^G$ and $\mu=0$,
iii. η-Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton.
iii. M is an expanding if $\beta r^G>0$,
iv. M is a steady if $\beta r^G=0$,
v. M is a shrinking if $\beta r^G<0$.

Corollary 16. *Let M be an $n=(2m+1)-$dimensional Sasakian manifold admitting general connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a W_1-Ricci pseudosymmetric, then the following holds:*
i. The η-Ricci soliton reduces to Ricci soliton,
ii. $\lambda=(n-1)\Lambda_3$ and $\mu=0$,

iii. *M is an expanding if* $\Lambda_3 > 0$,
iv. *M is a steady if* $\Lambda_3 = 0$,
v. *M is a shrinking if* $\Lambda_3 < 0$,
vi. $\mathcal{L}_{W_1} = -2\Lambda_3$.

Corollary 17. *Let* M *be an* $n = (2m+1)-$*dimensional Sasakian manifold admitting general connection and* (g, ξ, λ, μ) *be an* η*-Yamabe soliton on* M*. If* M *is a* W_1*-Ricci pseudosymmetric, then the following holds:*
i. $\lambda = \frac{1}{2}r^G$ *and* $\mu = 0$,
ii. η*-Yamabe soliton reduces to Yamabe soliton,*
iii. *M is an expanding if* $r^G > 0$,
iv. *M is a steady if* $r^G = 0$,
v. *M is a shrinking if* $r^G < 0$,
vi. $\mathcal{L}_{W_1} = -2\Lambda_3$.

Corollary 18. *Let* M *be an* $n = (2m+1)-$*dimensional Sasakian manifold admitting general connection and* (g, ξ, λ, μ) *be an* η*-Einstein soliton on* M*. If* M *is a* W_1*-Ricci pseudosymmetric, then the following holds:*
i. $\lambda = (n-1)\Lambda_3 - \frac{1}{2}r^G$ *and* $\mu = 0$,
ii. $\mathcal{L}_{W_1} = -2\Lambda_3$,
iii. η*-Einstein soliton reduces to Einstein soliton,*
iv. *M is an expanding if* $(n-1)\Lambda_3 > \frac{1}{2}r^G$,
v. *M is a steady if* $(n-1)\Lambda_3 = \frac{1}{2}r^G$,
vi. *M is a shrinking if* $(n-1)\Lambda_3 < \frac{1}{2}r^G$.

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