On the Action of W_i Curvature Tensors on Para-Sasakian Manifolds

Original Research Article

Abstract

In this study, Riemann, Ricci, W_6-, W_7-, W_8-, and W_9- curvature tensors on para-Sasakian manifolds are examined. Our goal is to ascertain the connections among these tensors and the specific conditions in which being Einstein and $\eta-$ Einstein manifolds emerge. The semisymmetric condition of W_6 and the tensor products of W_8 and W_9 and W_7 and W_9 satisfy the condition of being an $\eta-$ Einstein manifold, while the semisymmetric conditions of W_7 and W_9, the curvature conditions $W_6 \cdot R$ and $W_7 \cdot R$, and the tensor product of W_7 and W_8 satisfy the condition of being an Einstein manifold.

Keywords: Para-Sasakian manifold; Einstein manifold; $\eta-$ Einstein manifold; curvature tensors

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1 Introduction

One of the main areas of study in modern differential geometry has been the geometry of differentiable manifolds with unique tensorial structures. Especially, generalizations of Riemannian structures to the semi-Riemannian constructions are made possible by contact and paracontact metric manifolds. Numerous more investigations have been motivated by the methodical construction of such structures introduced by Yano and Kon (1985). As the paracomplex counterparts of Sasakian manifolds with a Lorentzian or semi-Riemannian metric, para-Sasakian manifolds have an important field among them. Curvature conditions and special tensor fields on para-Sasakian and related manifolds have been studied by several authors. For example, the research of curvature tensors in Lorentzian para-Sasakian manifolds was introduced by Pokhariyal (1996), and it was expanded by Murathan et. al (2006) by examining a class of Lorentzian para-Sasakian manifold with certain geometric characteristics. Furthermore, by examining particular curvature conditions on para-Sasakian manifold with the canonical paracontact connection, Acet et. al (2012) enhanced the theoretical foundation of curvature properties in this setting. Moreover, Gray (1966) studied curvature operators. Semi-Riemannian geometry has more advancements in curvature tensors. The τ-curvature tensor was first defined by Tripathi and Gupta (2011), who also examined its effects on manifold geometry. Subsequently, Jain et. al (2024) investigated the generalization of the M-projective curvature tensor on para-Sasakian manifolds, while Singh and Kishor (2018, 2022) investigated Ricci solitons and para-Kenmotsu manifolds that satisfied pseudo-symmetry or M-projective curvature property. Further insights into curvature constraints under normalcy conditions were obtained by Yıldırım and Atçeken (2019), who investigated the quasi-conformal curvature tensor on nearly paracontact metric manifolds. Deszcz et. al (2023) examined some properties for several generalized curvature tensors. Al-Qashbari (2020) studied some identities for generalized curvature tensors in Finsler space. Para-Sasakian and Lorentzian para-Sasakian manifolds satisfy certain curvature or soliton requirements have received more attention in recent years. Erken (2018) studied three-dimensional quasi-para-Sasakian manifolds that satisfied certain curvature relations, while Sarı and Ünal (2023) investigated the curvatures of semi-invariant submanifolds in Lorentzian para-Sasakian situations. Uygun and

Atçeken (2024) studied P-Sasakian manifolds for similar conditions, whereas Mert and Atçeken (2024) studied para-Sasakian manifolds with certain curvature tensors that accept almost η-Ricci solitons. A thorough PhD work by Mutinda (2021) introduced novel curvature tensors in Lorentzian para-Sasakian manifolds and discussed their geometric ramifications. Additionally, Gupta (2014) contributed to the classification of lower-dimensional situations by using three-dimensional ε-para-Sasakian manifolds. Jha et. al (2022) studied geometric concepts of generic lightlike manifolds. Moreover, Jha et. al (2023) examined similar concepts on product manifolds. Shaikh (2025) examined the generalizations of semisymmetric and pseudosymmetric type manifolds.

This research emphasizes the need for a comprehensive analysis of curvature tensors that extend classical ones in para-Sasakian geometry. Inspired by these advances, the goal of this work is to study the Riemann, Ricci, and numerous W-curvature tensors on para-Sasakian manifolds, namely W_6-, W_7-, W_8-, and W_9-. Establishing the relationships between these tensors and identifying the geometric circumstances under which the manifold displays unique characteristics, such as being Einstein or η-Einstein, are our main objectives. This research provides new geometric interpretations and reveals intrinsic connections between higher-order curvature tensors in para-Sasakian geometry by expanding the frameworks put forth in earlier studies. By doing this, it gets the connection between curvature symmetries and the metric features of these manifolds and offers a more thorough explanation of curvature condition. Moreover, future research could look at other geometric structures on para-Sasakian manifolds and how different curvature tensors interact to affect their global concepts. Additional research could concentrate on expanding the results to higher-dimensional analogues or more general paracontact metric manifolds. Moreover, examining stability conditions and the existence of canonical metrics subject to the derived curvature constraints may lead to deeper insights into the underlying geometry.

2 Preliminaries

In this section, some curvature tensor concepts are introduced with some basic properties of para-Sasakian manifolds.

If a $(2n+1)-$ dimensional smooth manifold M^{2n+1} admits a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η that meet the following prerequisites, then it has an essentially paracontact structure (ϕ, ξ, η).

$$\phi^2 \varsigma_1 = \varsigma_1 - \eta(\varsigma_1)\xi, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0, \;\; \eta(\xi) = 1, \tag{2.1}$$

and

$$g(\phi\varsigma_1, \phi\varsigma_2) = g(\varsigma_1, \varsigma_2) - \eta(\varsigma_1)\eta(\varsigma_2) \tag{2.2}$$

for all vector fields ς_1, ς_2 on M^{2n+1}, then $(M^{2n+1}, \phi, \xi, \eta, g)$ is defined as almost paracontact manifold. It is evident that

$$g(\xi, \varsigma_1) = \eta(\varsigma_1). \tag{2.3}$$

An almost paracontact metic manifold's fundamental 2-form Φ is determined by

$$\Phi(\varsigma_1, \varsigma_2) = g(\varsigma_1, \Phi\varsigma_2). \tag{2.4}$$

An almost paracontact metric manifold is referred to as a paracontact metric manifold if $d\eta = \Phi$. A paracontact metric structure is referred to be para-Sasakian if it is normal. Similarly, if the equations are satisfied by the structure (ϕ, ξ, η, g)

$$\begin{aligned} d\eta &= 0, \;\; \nabla_{\varsigma_1}\xi = \phi\varsigma_1, \\ (\nabla_{\varsigma_1}\phi)\varsigma_2 &= -g(\phi\varsigma_1, \varsigma_2)\xi - \eta(\varsigma_2)\phi\varsigma_1 + 2\eta(\varsigma_1)\eta(\varsigma_2)\xi. \end{aligned} \tag{2.5}$$

The manifold M^{2n+1} is referred to as a para-Sasakian manifold or P-Sasakian manifold, and the Levi-Civita connection on M is denoted as ∇.

Lemma 2.1. *The following relations are provided by a para-Sasakian manifold:*

$$
\begin{aligned}
S(\varsigma_1,\xi) &= -(n-1)\eta(\varsigma_1), && (2.6)\\
Q\xi &= -(n-1)\xi, && (2.7)\\
R(\varsigma_1,\varsigma_2)\xi &= \eta(\varsigma_1)\varsigma_2-\eta(\varsigma_2)\varsigma_1, && (2.8)\\
R(\xi,\varsigma_2)\varsigma_3 &= \eta(\varsigma_3)\varsigma_2-g(\varsigma_2,\varsigma_3)\xi, && (2.9)\\
R(\varsigma_1,\xi)\varsigma_3 &= -g(\varsigma_1,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_1, && (2.10)\\
\eta(R(\varsigma_1,\varsigma_2)\varsigma_3) &= g(\varsigma_1,\varsigma_3)\eta(\varsigma_2)-g(\varsigma_2,\varsigma_3)\eta(\varsigma_1), && (2.11)\\
S(\phi\varsigma_1,\phi\varsigma_2) &= S(\varsigma_1,\varsigma_2)+(n-1)\eta(\varsigma_1)\eta(\varsigma_2), && (2.12)
\end{aligned}
$$

where R and S stand for Ricci tensors and Riemannian curvature, respectively.

Definition 2.1. Assume that M^{2n+1} represents a para-Sasakian manifold. The curvature tensor W_6 is given by

$$W_6(\varsigma_1,\varsigma_2)\varsigma_3=R(\varsigma_1,\varsigma_2)\varsigma_3-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_1-g(\varsigma_2,\varsigma_3)Q\varsigma_1 \tag{2.13}$$

for all vector fields $\varsigma_1,\varsigma_2,\varsigma_3$ on M, Tripathi and Gupta (2011).

Lemma 2.2. *Assume that M^{2n+1} is a para-Sasakian manifold. The following relations hold:*

$$
\begin{aligned}
W_6(\xi,\varsigma_2)\varsigma_3 &= \eta(\varsigma_3)\varsigma_2-2g(\varsigma_2,\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\xi, && (2.14)\\
W_6(\varsigma_1,\xi)\varsigma_3 &= -g(\varsigma_1,\varsigma_3)\xi+2\eta(\varsigma_3)\varsigma_1+\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_1, && (2.15)\\
W_6(\varsigma_1,\varsigma_2)\xi &= \eta(\varsigma_1)\varsigma_2+\frac{1}{n-1}\eta(\varsigma_2)Q\varsigma_1. && (2.16)
\end{aligned}
$$

Definition 2.2. Let M^{2n+1} be a para-Sasakian manifold. The curvature tensor W_7 is given by

$$W_7(\varsigma_1,\varsigma_2)\varsigma_3=R(\varsigma_1,\varsigma_2)\varsigma_3-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_1+\frac{1}{n-1}g(\varsigma_2,\varsigma_3)Q\varsigma_1 \tag{2.17}$$

for all vector fields $\varsigma_1,\varsigma_2,\varsigma_3$ on M, Tripathi and Gupta (2011).

Lemma 2.3. *Assume that M^{2n+1} is a para-Sasakian manifold. The curvature tensor W_7 satisifes*

$$
\begin{aligned}
W_7(\xi,\varsigma_2)\varsigma_3 &= \eta(\varsigma_3)Q\varsigma_2-2g(\varsigma_2,\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\xi, && (2.18)\\
W_7(\varsigma_1,\xi)\varsigma_3 &= -g(\varsigma_1,\varsigma_3)\xi+2\eta(\varsigma_3)\varsigma_1+\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_1, && (2.19)\\
W_7(\varsigma_1,\varsigma_2)\xi &= \eta(\varsigma_1)\varsigma_2+\frac{1}{n-1}\eta(\varsigma_2)Q\varsigma_1. && (2.20)
\end{aligned}
$$

Definition 2.3. Let M^{2n+1} be a para-Sasakian manifold. The curvature tensor W_8 is given by

$$W_8(\varsigma_1,\varsigma_2)\varsigma_3=R(\varsigma_1,\varsigma_2)\varsigma_3-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_1+\frac{1}{n-1}g(\varsigma_1,\varsigma_2)\varsigma_3 \tag{2.21}$$

for all vector fields $\varsigma_1,\varsigma_2,\varsigma_3$ on M, Tripathi and Gupta (2011).

Lemma 2.4. *Assume that M^{2n+1} is a para-Sasakian manifold. Then, the following relations hold:*

$$
\begin{aligned}
W_8(\xi,\varsigma_2)\varsigma_3 &= \eta(\varsigma_3)\varsigma_2-2g(\varsigma_2,\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\xi-\eta(\varsigma_2)\varsigma_3, && (2.22)\\
W_8(\varsigma_1,\xi)\varsigma_3 &= -g(\varsigma_1,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_1-\eta(\varsigma_1)\varsigma_3, && (2.23)\\
W_8(\varsigma_1,\varsigma_2)\xi &= \eta(\varsigma_1)\varsigma_2+\frac{1}{n-1}S(\varsigma_1,\varsigma_2)\xi. && (2.24)
\end{aligned}
$$

Definition 2.4. Let M^{2n+1} be a para-Sasakian manifold. The curvature tensor W_9 is given by

$$W_9(\varsigma_1,\varsigma_2)\varsigma_3 = R(\varsigma_1,\varsigma_2)\varsigma_3 + \frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_1 - \frac{1}{n-1}g(\varsigma_2,\varsigma_3)Q\varsigma_1 \tag{2.25}$$

for all vector fields $\varsigma_1, \varsigma_2, \varsigma_3$ on M, Tripathi and Gupta (2011).

Lemma 2.5. *Assume that M^{2n+1} is a para-Sasakian manifold. Then, the following relations hold:*

$$W_9(\xi,\varsigma_2)\varsigma_3 = \eta(\varsigma_3)\varsigma_2 - \eta(\varsigma_2)\varsigma_3, \tag{2.26}$$

$$W_9(\varsigma_1,\xi)\varsigma_3 = -g(\varsigma_1,\varsigma_3)\xi + \eta(\varsigma_3)\varsigma_1 - \eta(\varsigma_1)\varsigma_3 - \eta(\varsigma_3)\varsigma_1 - \frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_1, \tag{2.27}$$

$$W_9(\varsigma_1,\varsigma_2)\xi = \eta(\varsigma_1)\varsigma_2 - \eta(\varsigma_2)\varsigma_1 + \frac{1}{n-1}S(\varsigma_1,\varsigma_2)\xi - \frac{1}{n-1}\eta(\varsigma_2)Q\varsigma_1. \tag{2.28}$$

3 On geometric properties for para-Sasakian manifolds

In this section, we provide some geometric characterizations for curvature tensors in para-Sasakian manifolds that are W_6-, W_7-, W_8-, and W_9-:

Theorem 3.1. *Assume that M^{2n+1} denotes the $2n+1-$ dimensional para-Sasakian manifold. If M^{2n+1} is W_6- semisymmetric, then M^{2n+1} is an $\eta-$ Einstein manifold.*

Proof. Let M^{2n+1} be the para-Sasakian manifold that provides W_6- semisymmetric:

$$(R(\varsigma_1,\varsigma_2)\cdot W_6)(\varsigma_4,\varsigma_5,\varsigma_3) = 0$$

for all $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5 \in \chi(M)$. Hence, we have

$$R(\varsigma_1,\varsigma_2)W_6(\varsigma_4,\varsigma_5)\varsigma_3 - W_6(R(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3 - W_0(\varsigma_4,R(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3 - W_6(\varsigma_4,\varsigma_5)R(\varsigma_1,\varsigma_2)\varsigma_3 = 0. \tag{3.1}$$

Choosing $\varsigma_1 = \xi$ in (3.1) and exploiting (2.9), and (2.14)-(2.16), we obtain

$$\begin{aligned}
&\eta(W_6(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2 - g(\varsigma_2,W_6(\varsigma_4,\varsigma_5)\varsigma_3)\xi - \eta(\varsigma_4)W_6(\varsigma_2,\varsigma_5)\varsigma_3\\
&+g(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5 - 2g(\varsigma_5,\varsigma_3)\xi - \frac{1}{n-1}S(\varsigma_5,\varsigma_3)\xi] - \eta(\varsigma_5)W_6(\varsigma_4,\varsigma_2)\varsigma_3\\
&+g(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi + 2\eta(\varsigma_3)\varsigma_4 + \frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4] - \eta(\varsigma_3)W_6(\varsigma_4,\varsigma_5)\varsigma_2\\
&+g(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5 + \frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_4] = 0.
\end{aligned} \tag{3.2}$$

If we write $\varsigma_4 = \xi$ in (3.2) and make necessary simplifications by considering (2.1), (2.3), (2.7), and (2.14), we calculate

$$\begin{aligned}
&-2g(\varsigma_5,\varsigma_3)\varsigma_2 - \frac{1}{n-1}S(\varsigma_5,\varsigma_3)\varsigma_2 - \eta(\varsigma_3)g(\varsigma_2,\varsigma_5)\xi - W_6(\varsigma_2,\varsigma_5)\varsigma_3\\
&+g(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi + \frac{1}{n-1}S(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi + 2\eta(\varsigma_3)g(\varsigma_2,\varsigma_5)\xi\\
&+\frac{1}{n-1}S(\varsigma_5,\varsigma_2)\eta(\varsigma_3)\xi + g(\varsigma_2,\varsigma_3)\varsigma_5 = 0.
\end{aligned} \tag{3.3}$$

If we choose $\varsigma_3 = \xi$ in (3.3), we briefly compute

$$-2\eta(\varsigma_5)\varsigma_2 + 3g(\varsigma_2,\varsigma_5)\xi - \frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_2 + \frac{1}{n-1}S(\varsigma_5,\varsigma_2)\xi + \eta(\varsigma_2)\varsigma_5 = 0. \tag{3.4}$$

Taking inner product both sides of the equation by $\varsigma_6 \in \chi(M)$ and choosing $\varsigma_5 = \xi$, we acquire

$$S(\varsigma_2,\varsigma_6) = -2(n-1)g(\varsigma_2,\varsigma_6) + 3(n-1)\eta(\varsigma_2)\eta(\varsigma_6).$$

Hence, M^{2n+1} is an η Einstein manifold. □

Theorem 3.2. *Assume that M^{2n+1} denotes the $2n+1-$ dimensional para-Sasakian manifold. If M^{2n+1} is W_7- semisymmetric, then M^{2n+1} is an Einstein manifold.*

Proof. Let M^{2n+1} be the para-Sasakian manifold that provides W_7- semisymmetric:

$$(R(\varsigma_1,\varsigma_2)\cdot W_7)(\varsigma_4,\varsigma_5,\varsigma_3)=0$$

for all $\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5\in\chi(M)$. Hence, we have

$$R(\varsigma_1,\varsigma_2)W_7(\varsigma_4,\varsigma_5)\varsigma_3-W_7(R(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-W_7(\varsigma_4,R(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-W_7(\varsigma_4,\varsigma_5)R(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.5}$$

Substituting $\varsigma_1=\xi$ in (3.5) and exploiting (2.9), and (2.18)-(2.20), we obtain

$$\begin{aligned}&\eta(W_7(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2-g(\varsigma_2,W_7(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\eta(\varsigma_4)W_7(\varsigma_2,\varsigma_5)\varsigma_3\\&+g(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-2g(\varsigma_5,\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_5,\varsigma_3)\xi]-\eta(\varsigma_5)W_7(\varsigma_4,\varsigma_2)\varsigma_3\\&+g(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+2\eta(\varsigma_3)\varsigma_4+\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4]-\eta(\varsigma_3)W_7(\varsigma_4,\varsigma_5)\varsigma_2\\&g(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5+\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_4]=0.\end{aligned} \tag{3.6}$$

Writing $\varsigma_4=\xi$ in (3.6) and make necessary simplifications by considering (2.1), (2.7), and (2.18), we obtain

$$\begin{aligned}&-2g(\varsigma_5,\varsigma_3)\varsigma_2-\frac{1}{n-1}S(\varsigma_5,\varsigma_3)\varsigma_2-\eta(\varsigma_3)g(\varsigma_2,\varsigma_5)\xi-W_7(\varsigma_2,\varsigma_5)\varsigma_3\\&+2g(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi+2\eta(\varsigma_3)g(\varsigma_2,\varsigma_5)\xi\\&+\frac{1}{n-1}S(\varsigma_5,\varsigma_2)\eta(\varsigma_3)\xi+g(\varsigma_2,\varsigma_3)\varsigma_5-\eta(\varsigma_5)g(\varsigma_2,\varsigma_3)\xi=0.\end{aligned} \tag{3.7}$$

If we choose $\varsigma_3=\xi$ in (3.7), one can easily see

$$-\eta(\varsigma_5)\varsigma_2+g(\varsigma_2,\varsigma_5)\xi-\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_2+\frac{1}{n-1}S(\varsigma_5,\varsigma_2)\xi=0. \tag{3.8}$$

Taking inner product both sides of the equation by $\varsigma_6\in\chi(M)$ and choosing $\varsigma_5=\xi$, we acquire

$$S(\varsigma_2,\varsigma_6)=-(n-1)g(\varsigma_2,\varsigma_6).$$

Hence, M^{2n+1} is an Einstein manifold. □

Theorem 3.3. *Assume that M^{2n+1} denotes the $2n+1-$ dimensional para-Sasakian manifold. If M^{2n+1} is W_9- semisymmetric, then M^{2n+1} is an Einstein manifold.*

Proof. Let M^{2n+1} be the para-Sasakian manifold that provides W_9- semisymmetric:

$$(R(\varsigma_1,\varsigma_2)\cdot W_9)(\varsigma_4,\varsigma_5,\varsigma_3)=0$$

for all $\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5\in\chi(M)$. Hence, we have

$$R(\varsigma_1,\varsigma_2)W_9(\varsigma_4,\varsigma_5)\varsigma_3-W_9(R(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-W_9(\varsigma_4,R(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-W_9(\varsigma_4,\varsigma_5)R(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.9}$$

Substituting $\varsigma_1=\xi$ in (3.9) and exploiting (2.9), and (2.26)-(2.28), we obtain

$$\begin{aligned}&\eta(W_9(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2-g(\varsigma_2,W_9(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\eta(\varsigma_4)W_9(\varsigma_2,\varsigma_5)\varsigma_3\\&+g(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3]-\eta(\varsigma_5)W_9(\varsigma_4,\varsigma_2)\varsigma_3-g(\varsigma_2,\varsigma_5)[g(\varsigma_4,\varsigma_3)\xi\\&+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3-\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4]-\eta(\varsigma_3)W_9(\varsigma_4,\varsigma_5)\varsigma_2\\&+g(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5-\eta(\varsigma_5)\varsigma_4+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi-\frac{1}{n-1}\eta(Q\varsigma_4)]=0.\end{aligned} \tag{3.10}$$

Writing $\varsigma_4 = \xi$ in (3.10) and make necessary simplifications by considering (2.3), (2.6), (2.7), (2.26), and (2.27), we calculate

$$-W_9(\varsigma_2, \varsigma_5)\varsigma_3 - g(\varsigma_2, \varsigma_5)\varsigma_3 + g(\varsigma_2, \varsigma_3)\varsigma_5 = 0. \quad (3.11)$$

If we choose $\varsigma_3 = \xi$ in (3.11), we briefly compute

$$\eta(\varsigma_5)\varsigma_2 + g(\varsigma_2, \varsigma_5)\xi + \frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_2 - \frac{1}{n-1}S(\varsigma_5, \varsigma_2)\xi - g(\varsigma_2, \varsigma_5)\xi = 0. \quad (3.12)$$

Taking inner product both sides of the equation by $\varsigma_6 \in \chi(M)$ and choosing $\varsigma_5 = \xi$, we acquire

$$S(\varsigma_2, \varsigma_6) = -(n-1)g(\varsigma_2, \varsigma_6).$$

Hence, M^{2n+1} is an Einstein manifold. □

Theorem 3.4. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If M provides the curvature condition $W_6 \cdot R = 0$, then M^{2n+1} is an Einstein manifold.*

Proof. Let M^{2n+1} be the $2n+1-$ dimensional para-Sasakian manifold satisfying the curvature condition $W_6(\varsigma_1, \varsigma_2) \cdot R = 0$. Thus, we have

$$W_6(\varsigma_1, \varsigma_2)R(\varsigma_4, \varsigma_5)\varsigma_3 - R(W_6(\varsigma_1, \varsigma_2)\varsigma_4, \varsigma_5)\varsigma_3 - R(\varsigma_4, W_6(\varsigma_1, \varsigma_2)\varsigma_5)\varsigma_3 - R(\varsigma_4, \varsigma_5)W_6(\varsigma_1, \varsigma_2)\varsigma_3 = 0. \quad (3.13)$$

As we write $\varsigma_1 = \xi$ in (3.13) and use 2.6 and (2.14), we acquire

$$2\eta(\varsigma_4)S(\varsigma_2, \varsigma_5) + 2\eta(\varsigma_5)S(\varsigma_2, \varsigma_4) + 2(n-1)g(\varsigma_2, \varsigma_4)\eta(\varsigma_5) + 2(n-1)\eta(\varsigma_4)g(\varsigma_2, \varsigma_5) = 0. \quad (3.14)$$

As we choose $\varsigma_4 = \xi$ in (3.14) and by using (2.1) and (2.6), we obtain as follows:

$$S(\varsigma_2, \varsigma_5) = -(n-1)g(\varsigma_2, \varsigma_5).$$

This proves our assertion. □

Theorem 3.5. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If M provides the curvature condition $W_7 \cdot R = 0$, M^{2n+1} is an Einstein manifold.*

Proof. Let M^{2n+1} be the $2n+1-$ dimensional para-Sasakian manifold. Let M^{2n+1} also satisfy the curvature condition $W_7(\varsigma_1, \varsigma_2) \cdot R = 0$. Thus, we have

$$W_7(\varsigma_1, \varsigma_2)R(\varsigma_4, \varsigma_5)\varsigma_3 - R(W_7(\varsigma_1, \varsigma_2)\varsigma_4, \varsigma_5)\varsigma_3 - R(\varsigma_4, W_7(\varsigma_1, \varsigma_2)\varsigma_5)\varsigma_3 - R(\varsigma_4, \varsigma_5)W_7(\varsigma_1, \varsigma_2)\varsigma_3 = 0. \quad (3.15)$$

As we write $\varsigma_1 = \xi$ in (3.15) and use (2.6) and (2.18), we acquire

$$2\eta(\varsigma_4)S(\varsigma_2, \varsigma_5) + 2\eta(\varsigma_5)S(\varsigma_2, \varsigma_4) + 2g(\varsigma_2, \varsigma_4)\eta(\varsigma_5) + 2(n-1)g(\varsigma_2, \varsigma_5)\xi = 0. \quad (3.16)$$

As we choose $\varsigma_4 = \xi$ in (3.16) and by using 2.1, 2.3, and (2.6), we obtain as follows:

$$S(\varsigma_2, \varsigma_5) = -(n-1)g(\varsigma_2, \varsigma_5) + (n-2)\eta(\varsigma_5)\eta(\varsigma_2).$$

Hence, M^{2n+1} denotes an $\eta-$ Einstein manifold. □

Theorem 3.6. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If M provides the curvature condition $W_8 \cdot R = 0$, then M^{2n+1} is an Einstein manifold.*

Proof. Let M^{2n+1} be the $2n+1-$ dimensional para-Sasakian manifold. Let M^{2n+1} also satisfy the curvature condition $W_8(\varsigma_1,\varsigma_2)\cdot R=0$. Thus, we have

$$W_8(\varsigma_1,\varsigma_2)R(\varsigma_4,\varsigma_5)\varsigma_3-R(W_8(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-R(\varsigma_4,W_8(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-R(\varsigma_4,\varsigma_5)W_8(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.17}$$

As we write $\varsigma_1=\xi$ in (3.17) and use (2.6) and (2.22), we acquire

$$2\eta(\varsigma_4)S(\varsigma_2,\varsigma_5)-2\eta(\varsigma_2)S(\varsigma_4,\varsigma_5)+2S(\varsigma_2,\varsigma_4)\eta(\varsigma_5)+(n-1)g(\varsigma_2,\varsigma_4)\eta(\varsigma_5)+(n-1)\eta(\varsigma_4)g(\varsigma_2,\varsigma_5)=0. \tag{3.18}$$

If we choose $\varsigma_4=\xi$ in (3.18) and by using 2.1, 2.3, and (2.6), we obtain as follows:

$$S(\varsigma_2,\varsigma_5)=-\frac{(n-1)}{2}g(\varsigma_2,\varsigma_5)-\frac{(n-1)}{2}\eta(\varsigma_5)\eta(\varsigma_2)$$

which proves our assertion □

Theorem 3.7. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If M provides the curvature condition $W_9\cdot R=0$, then the Ricci tensor of M^{2n+1} satisfies the following condition:*

$$S(\varsigma_2,\varsigma_5)=-(n-1)\eta(\varsigma_2)\eta(\varsigma_5).$$

Proof. Let M^{2n+1} be the $2n+1-$ dimensional para-Sasakian manifold satisfying the curvature condition $W_9(\varsigma_1,\varsigma_2)\cdot R=0$. Then, we have

$$W_9(\varsigma_1,\varsigma_2)R(\varsigma_4,\varsigma_5)\varsigma_3-R(W_9(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-R(\varsigma_4,W_9(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-R(\varsigma_4,\varsigma_5)W_9(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.19}$$

Taking $\varsigma_1=\xi$ in (3.19) and use (2.6) and (2.26), we reach at

$$\eta(\varsigma_4)S(\varsigma_2,\varsigma_5)-\eta(\varsigma_2)S(\varsigma_4,\varsigma_5)+S(\varsigma_2,\varsigma_4)\eta(\varsigma_5)-\eta(\varsigma_2)S(\varsigma_4,\varsigma_5)=0. \tag{3.20}$$

As we choose $\varsigma_4=\xi$ in (3.20) and by using 2.1 and (2.6), we obtain

$$S(\varsigma_2,\varsigma_5)=-(n-1)\eta(\varsigma_2)\eta(\varsigma_5).$$

□

Theorem 3.8. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If $W_6\cdot W_7=0$, then the Ricci tensor S is of form*

$$S(\varsigma_2,\varsigma_6)=-(n-1)\eta(\varsigma_2)\eta(\varsigma_6).$$

Proof. Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. Let M^{2n+1} also satisfy the curvature condition $W_6(\varsigma_1,\varsigma_2)\cdot W_7=0$. Thus, we have

$$W_6(\varsigma_1,\varsigma_2)W_7(\varsigma_4,\varsigma_5)\varsigma_3-W_7(W_6(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-W_7(\varsigma_4,W_6(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-W_7(\varsigma_4,\varsigma_5)W_6(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.21}$$

Writing $\varsigma_1=\xi$ in (3.21) and using (2.14), (2.18), and (2.20), we acquire

$$\begin{aligned}
&\eta(W_7(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2-2g(\varsigma_2,W_7(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,W_7(\varsigma_4,\varsigma_5)\varsigma_3)\xi\\
&\eta(\varsigma_4)W_7(\varsigma_2,\varsigma_5)\varsigma_3+2g(\varsigma_2,\varsigma_4)\eta(\varsigma_3)\varsigma_5-4g(\varsigma_2,\varsigma_4)g(\varsigma_5,\varsigma_3)\xi\\
&-\frac{2}{n-1}g(\varsigma_2,\varsigma_4)S(\varsigma_5,\varsigma_3)\xi+\frac{1}{n-1}S(\varsigma_2,\varsigma_4)\eta(\varsigma_3)\varsigma_5-\frac{2}{n-1}S(\varsigma_2,\varsigma_4)g(\varsigma_5,\varsigma_3)\xi\\
&\frac{1}{(n-1)^2}S(\varsigma_2,\varsigma_4)S(\varsigma_5,\varsigma_3)\xi-\eta(\varsigma_3)W_7(\varsigma_4,\varsigma_5)\varsigma_2+2g(\varsigma_2,\varsigma_3)\eta(\varsigma_4)\varsigma_5\\
&+\frac{2}{n-1}g(\varsigma_2,\varsigma_3)\eta(\varsigma_5)Q\varsigma_4+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)W_7(\varsigma_4,\varsigma_5)\varsigma_3=0.
\end{aligned} \tag{3.22}$$

Setting $\varsigma_4=\xi$ in (3.22) and using (2.1), (2.3), (2.6), (2.7), and (2.18), we obtain as follows:

$$\eta(\varsigma_3)\eta(\varsigma_5)\varsigma_2-2g(\varsigma_5,\varsigma_3)\varsigma_2-\frac{1}{n-1}S(\varsigma_5,\varsigma_3)\varsigma_2-W_7(\varsigma_2,\varsigma_5)\varsigma_3+2g(\varsigma_2,\varsigma_3)\varsigma_5$$
$$-2g(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_5-\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi=0. \tag{3.23}$$

Writing $\varsigma_3=\xi$ in (3.23) and using (2.1), (2.3), (2.6), and (2.20), we obtain

$$-\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_2-\eta(\varsigma_2)\eta(\varsigma_5)\xi=0. \tag{3.24}$$

After substituting $\varsigma_5=\xi$ and taking the inner product of both sides of (3.24) with $\varsigma_6\in\chi(M)$, we calculate

$$S(\varsigma_2,\varsigma_6)=-(n-1)\eta(\varsigma_2)\eta(\varsigma_6).$$

□

Theorem 3.9. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If $W_8\cdot W_9=0$, then M^{2n+1} is an $\eta-$ Einstein manifold.*

Proof. Assume that M^{2n+1} is the $(2n+1)-$ dimensional para-Sasakian manifold. Let M^{2n+1} also satisfy the curvature condition $W_8(\varsigma_1,\varsigma_2)\cdot W_9=0$. Thus, we have

$$W_8(\varsigma_1,\varsigma_2)W_9(\varsigma_4,\varsigma_5)\varsigma_3-W_9(W_8(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-W_9(\varsigma_4,W_8(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-W_9(\varsigma_4,\varsigma_5)W_8(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.25}$$

Writing $\varsigma_1=\xi$ in (3.25) and using (2.26)- (2.28), we acquire

$$\eta(W_9(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2-g(\varsigma_2,W_9(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,W_9(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\eta(\varsigma_2)W_9(\varsigma_4,\varsigma_5)\varsigma_3$$
$$\eta(\varsigma_4)W_9(\varsigma_2,\varsigma_5)\varsigma_3+g(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3]+\frac{1}{n-1}S(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3]$$
$$+\eta(\varsigma_2)W_9(\varsigma_4,\varsigma_5)\varsigma_3-\eta(\varsigma_5)W_9(\varsigma_4,\varsigma_2)\varsigma_3+g(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3-\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4]$$
$$+\frac{1}{n-1}S(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3-\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4]$$
$$-\eta(\varsigma_3)W_9(\varsigma_4,\varsigma_5)\varsigma_2+g(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5-\eta(\varsigma_5)\varsigma_4+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi-\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_4]$$
$$+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5-\eta(\varsigma_5)\varsigma_4+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi-\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_4]+\eta(\varsigma_2)W_9(\varsigma_4,\varsigma_5)\varsigma_3=0. \tag{3.26}$$

Setting $\varsigma_4=\xi$ in (3.26) and using (2.1), (2.3), (2.6), and (2.7), we obtain as follows:

$$-W_9(\varsigma_2,\varsigma_5)\varsigma_3+\eta(\varsigma_2)\eta(\varsigma_3)\varsigma_5-3\eta(\varsigma_2)\eta(\varsigma_5)\varsigma_3-g(\varsigma_2,\varsigma_5)\varsigma_3$$
$$-\frac{1}{n-1}S(\varsigma_2,\varsigma_5)\varsigma_3+g(\varsigma_2,\varsigma_3)\varsigma_5+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_5=0. \tag{3.27}$$

Writing $\varsigma_3=\xi$ in (3.27) and making some simplifications, we obtain

$$\eta(\varsigma_5)\varsigma_2-\frac{1}{n-1}S(\varsigma_2,\varsigma_5)\xi+\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_2-3\eta(\varsigma_2)\eta(\varsigma_5)\xi-g(\varsigma_2,\varsigma_5)\xi-\frac{1}{n-1}S(\varsigma_2,\varsigma_5)\xi=0. \tag{3.28}$$

After substituting $\varsigma_5=\xi$ and taking the inner product of both sides of (3.28) with $\varsigma_6\in\chi(M)$, we calculate

$$S(\varsigma_2,\varsigma_6)=-(n-1)g(\varsigma_2,\varsigma_6)+2(n-1)\eta(\varsigma_2)\eta(\varsigma_6).$$

Hence, M^{2n+1} denotes an η Einstein manifold. □

Theorem 3.10. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If $W_7\cdot W_9=0$, then M^{2n+1} is an $\eta-$ Einstein manifold.*

Proof. Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold and M^{2n+1} also satisfy the curvature condition $W_7(\varsigma_1,\varsigma_2)\cdot W_9=0$. Thus, we have

$$W_7(\varsigma_1,\varsigma_2)W_9(\varsigma_4,\varsigma_5)\varsigma_3-W_9(W_7(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-W_9(\varsigma_4,W_7(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-W_9(\varsigma_4,\varsigma_5)W_7(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.29}$$

Writing $\varsigma_1=\xi$ in (3.29) and using (2.18), (2.26)- (2.28), we infer

$$\begin{aligned}
&\eta(W_9(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2-2g(\varsigma_2,W_9(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,W_9(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\eta(\varsigma_4)W_9(\varsigma_2,\varsigma_5)\varsigma_3\\
&+2g(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3]+\frac{1}{n-1}S(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3]-\eta(\varsigma_5)W_9(\varsigma_4,\varsigma_5)\varsigma_3\\
&+2g(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3-\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4]\\
&+\frac{1}{n-1}S(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3-\frac{1}{n-1}\eta(\varsigma_3)Q\varsigma_4]\\
&-\eta(\varsigma_3)W_9(\varsigma_4,\varsigma_5)\varsigma_2+2g(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5-\eta(\varsigma_5)\varsigma_4+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi-\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_4]\\
&+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5-\eta(\varsigma_5)\varsigma_4+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi-\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_4]=0.
\end{aligned} \tag{3.30}$$

Setting $\varsigma_4=\xi$ in (3.30) and using (2.1), (2.3), (2.6), (2.7), and (2.26), we obtain as follows:

$$\begin{aligned}
&2\eta(\varsigma_3)g(\varsigma_2,\varsigma_5)\xi-2\eta(\varsigma_5)g(\varsigma_2,\varsigma_3)\xi-W_9(\varsigma_2,\varsigma_5)\varsigma_3-2g(\varsigma_2,\varsigma_5)\varsigma_3\\
&+2\eta(\varsigma_3)g(\varsigma_2,\varsigma_5)\xi+2g(\varsigma_2,\varsigma_3)\varsigma_5-2g(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_5=0.
\end{aligned} \tag{3.31}$$

Writing $\varsigma_3=\xi$ in (3.31) and making some simplifications, we obtain

$$-\frac{1}{n-1}S(\varsigma_2,\varsigma_5)\xi+\frac{1}{n-1}\eta(\varsigma_5)Q\varsigma_2-2\eta(\varsigma_2)\eta(\varsigma_5)\xi+2g(\varsigma_2,\varsigma_5)\xi=0. \tag{3.32}$$

Selecting $\varsigma_5=\xi$ and taking the inner product of both sides of (3.32) with $\varsigma_6\in\chi(M)$, we calculate

$$S(\varsigma_2,\varsigma_6)=-(n-1)g(\varsigma_2,\varsigma_6)+(n-1)\eta(\varsigma_2)\eta(\varsigma_6).$$

Hence, M^{2n+1} denotes an η Einstein manifold. □

Theorem 3.11. *Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. If $W_7\cdot W_8=0$, then M^{2n+1} is an Einstein manifold.*

Proof. Assume that M^{2n+1} is the $2n+1-$ dimensional para-Sasakian manifold. M^{2n+1} satisfies the curvature condition $W_7(\varsigma_1,\varsigma_2)\cdot W_8=0$. Thus, we have

$$W_7(\varsigma_1,\varsigma_2)W_8(\varsigma_4,\varsigma_5)\varsigma_3-W_8(W_7(\varsigma_1,\varsigma_2)\varsigma_4,\varsigma_5)\varsigma_3-W_8(\varsigma_4,W_7(\varsigma_1,\varsigma_2)\varsigma_5)\varsigma_3-W_8(\varsigma_4,\varsigma_5)W_7(\varsigma_1,\varsigma_2)\varsigma_3=0. \tag{3.33}$$

Writing $\varsigma_1=\xi$ in (3.33) and using (2.18), (2.22)- (2.24), we acquire

$$\begin{aligned}
&\eta(W_8(\varsigma_4,\varsigma_5)\varsigma_3)\varsigma_2-2g(\varsigma_2,W_8(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_2,W_8(\varsigma_4,\varsigma_5)\varsigma_3)\xi-\eta(\varsigma_4)W_8(\varsigma_2,\varsigma_5)\varsigma_3\\
&+2g(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3-g(\varsigma_5,\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_5,\varsigma_3)\xi]\\
&+\frac{1}{n-1}S(\varsigma_2,\varsigma_4)[\eta(\varsigma_3)\varsigma_5-\eta(\varsigma_5)\varsigma_3-g(\varsigma_5,\varsigma_3)\xi-\frac{1}{n-1}S(\varsigma_5,\varsigma_3)\xi]\\
&-\eta(\varsigma_5)W_8(\varsigma_4,\varsigma_2)\varsigma_3+2g(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3]\\
&+\frac{1}{n-1}S(\varsigma_2,\varsigma_5)[-g(\varsigma_4,\varsigma_3)\xi+\eta(\varsigma_3)\varsigma_4-\eta(\varsigma_4)\varsigma_3]\\
&-\eta(\varsigma_3)W_8(\varsigma_4,\varsigma_5)\varsigma_2+2g(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi]\\
&+\frac{1}{n-1}S(\varsigma_2,\varsigma_3)[\eta(\varsigma_4)\varsigma_5+\frac{1}{n-1}S(\varsigma_4,\varsigma_5)\xi]=0.
\end{aligned} \tag{3.34}$$

Setting $\varsigma_4 = \xi$ in (3.34) and making some simplifications, we obtain as follows:

$$
\begin{aligned}
&-g(\varsigma_5,\varsigma_3)\varsigma_2 - \frac{1}{n-1}S(\varsigma_5,\varsigma_3)\varsigma_2 + \frac{1}{n-1}\eta(\varsigma_5)S(\varsigma_2,\varsigma_3)\xi \\
&-W_8(\varsigma_2,\varsigma_5)\varsigma_3 + \eta(\varsigma_2)g(\varsigma_5,\varsigma_3)\xi + g(\varsigma_2,\varsigma_3)\eta(\varsigma_5)\xi - 2g(\varsigma_2,\varsigma_5)\varsigma_3 \\
&-\frac{1}{n-1}S(\varsigma_2,\varsigma_5)\varsigma_3 + 2g(\varsigma_2,\varsigma_3)\varsigma_5 + \frac{1}{n-1}S(\varsigma_2,\varsigma_3)\varsigma_5 = 0.
\end{aligned} \tag{3.35}
$$

Writing $\varsigma_3 = \xi$ in (3.35) and making some simplifications, we obtain

$$
S(\varsigma_2,\varsigma_5) = -\frac{3(n-1)}{2}g(\varsigma_2,\varsigma_5).
$$

Hence, M^{2n+1} is an Einstein manifold. □

4 Conclusion

In this study, Riemann, Ricci, W_6-, W_7-, W_8-, and W_9- curvature tensors on para-Sasakian manifolds are explored. Our goal is to determine the links among these tensors and the exact conditions in which being Einstein and $\eta-$ Einstein manifolds occur. Although the semisymmetric conditions of W_7 and W_9, the curvature conditions $W_6 \cdot R$ and $W_7 \cdot R$, and the tensor product of W_7 and W_8 satisfy the condition of being an Einstein manifold, the semisymmetric conditions of W_6 and the tensor products of W_8 and W_9 and W_7 and W_9 satisfy the condition of being a $\eta-$ Einstein manifold.

Disclaimer (Artificial Intelligence)

Authors hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT,COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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