ON COMPLEX INTERVAL VECTORS AND BOUNDED INTERVAL SEQUENCES

ABSTRACT. In this work, we will primarily focus on the topological properties of the n-dimensional complex interval space $\mathbb{I}_{\mathbb{C}}^n$ and demonstrate that it is an $\Omega - space$. Such spaces are a special class of normed quasilinear spaces and are well-suited for applications. Furthermore, since bounded interval sequences play an important role in interval arithmetic and in some scientific applications, we will investigate the properties of the set $\mathbb{I}\ell_\infty$, which is the class of bounded complex interval sequences.

1. INTRODUCTION

In cite [1], quasilinear spaces and normed quasilinear spaces were introduced as a generalization of linear spaces and normed linear spaces, respectively. While defining quasilinear spaces in 1985, Aseev utilized a partial ordering relation, thereby enabling the derivation of consistent counterparts to results from linear functional analysis. This article will address the fundamental definitions, theorems, and significant results pertaining to quasilinear spaces, normed quasilinear spaces, and inner product quasilinear spaces. Aseev's definition is founded upon a partial ordering relation "$\preccurlyeq$". This ordering relation facilitates the definition of the norm concept within quasilinear spaces. The most characteristic feature of quasilinear spaces is the absence of an inverse for every element in the space. If an inverse exists for all elements in a quasilinear space, then the partial ordering relation reduces to the equality relation, and the space becomes a linear space. Every linear space simultaneously constitutes a quasilinear space; however, the converse is not valid. Subsequent studies such as [8], [7, 3] and [5] have defined the inner product concept on quasilinear spaces and have also introduced the concepts of quasilinear dependence and quasilinear independence. Furthermore, after the development of the Hilbert quasilinear space concept in this study, it was shown in [9], [4], and [10] that these spaces could be applied in fields such as signal processing.

The topological structure of quasilinear spaces also has been examined in [4] and [11]. The complex interval space $\mathbb{I}_{\mathbb{R}}^n$ will be important in this study and it is introduced before and it will be shown that $\mathbb{I}_{\mathbb{C}}^n$ is a quasilinear space [12, 8]. Furthermore, $\mathbb{I}_{\mathbb{C}}^n$ will be established as a consolidate space, upon which a norm will be defined. It will then be demonstrated that $\mathbb{I}_{\mathbb{C}}^n$, together with the norm, constitutes a normed quasilinear space. Finally, equipped with the function $\langle \cdot, \cdot \rangle$, it will be shown that $\mathbb{I}_{\mathbb{C}}$ is an inner product quasilinear space. The set of complex

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intervals consists of intervals that are defined in a special way. Any element of $\mathbb{I}^n_{\mathbb{C}}$ is called a complex interval and it is written as

$$z=\underbrace{\left(\left[\underline{z^1_r},\overline{z^1_r}\right],\left[\underline{z^2_r},\overline{z^2_r}\right],\cdots,\left[\underline{z^n_r},\overline{z^n_r}\right]\right)}_{I^n_{\mathbb{R}}}+i\underbrace{\left(\left[\underline{z^1_s},\overline{z^1_s}\right],\left[\underline{z^2_s},\overline{z^2_s}\right],\cdots,\left[\underline{z^n_s},\overline{z^n_s}\right]\right)}_{I^n_{\mathbb{R}}}\in\mathbb{I}^n_{\mathbb{C}}.$$

Further

$$\left(\left[\underline{z^1_r},\overline{z^1_r}\right],\left[\underline{z^2_r},\overline{z^2_r}\right],\cdots,\left[\underline{z^n_r},\overline{z^n_r}\right]\right),\text{ is called the real part of } z,$$

and

$$\left(\left[\underline{z^1_s},\overline{z^1_s}\right],\left[\underline{z^2_s},\overline{z^2_s}\right],\cdots,\left[\underline{z^n_s},\overline{z^n_s}\right]\right),\text{is called the imaginary part of } z,$$

$$i=\sqrt{-1},\text{ is the imaginary unit.}$$

In fact each z is a compact subset of $\mathbb{C}^n$. In this study, we will primarily focus on the topological properties of $\mathbb{I}^n_{\mathbb{C}}$ and demonstrate that $\mathbb{I}^n_{\mathbb{C}}$ is an 3a9-space. Such spaces are a special class of normed quasilinear spaces and are well-suited for applications. Furthermore, since bounded interval sequences play an important role in interval arithmetic, we will investigate the properties of the set $\mathbb{I}\ell_\infty$, which is the class of bounded complex interval sequences.

2. Preliminaries

First let us mention a bit from interval n-tuples [2]. An n-dimensional interval vector is an ordered n-tuple of intervals, expressed as

$$z=(z_1,z_2,...z_n).=\left(\left[\underline{z_1},\overline{z_1}\right],...,\left[\underline{z_n},\overline{z_n}\right]\right)\in\mathbb{I}^n_{\mathbb{R}}.$$

It is important to note that the set of all n-dimensional interval vectors does not form a vector space. The relation $z\preceq w$ holds if and only if $z_k\subseteq w_k$ for each $k=1,2,...,n$, defines a partial order on the set $\mathbb{I}^n_{\mathbb{R}}$ of all n-dimensional interval vectors. The product of two intervals $z=[\underline{z},\overline{z}]$ and $w=[\underline{w},\overline{w}]$ is given by $z\cdot w=[\underline{z},\overline{z}]\,[\underline{w},\overline{w}]=[\min S,\max S]$ where $S=\{\underline{zw},\underline{z}\overline{w},\overline{z}\underline{w},\overline{zw}\}$. The sum of two intervals vector is coordinatewise and the scalar product is easy and can be learn from [2].

Now, let's define a quasilinear space. A set X is called a quasilinear space (QLS), as defined by [1], over the field $\mathbb{K}$ of real or complex numbers, if it satisfies the following conditions, a partial order relation "$\preceq$", along with an algebraic addition operation and a scalar multiplication operation by real numbers, such that for all elements $x,y,z,v\in X$ and all $\alpha,\beta\in\mathbb{K}$, the following properties hold.

$$x\preceq x, \tag{2.1}$$

$$x\preceq z \text{ if } x\preceq y \text{ and } y\preceq z, \tag{2.2}$$

$$x=y \text{ if } x\preceq y \text{ and } y\preceq x, \tag{2.3}$$

$$x+y=y+x, \tag{2.4}$$

$$x+(y+z)=(x+y)+z, \tag{2.5}$$

$$\text{there exists an element (zero) } \theta\in X \text{ such that } x+\theta=x, \tag{2.6}$$

$$\alpha(\beta x)=(\alpha\beta)x, \tag{2.7}$$

$$\alpha(x+y)=\alpha x+\alpha y, \tag{2.8}$$

$$1x=x, \tag{2.9}$$

$$0x=\theta, \tag{2.10a}$$

$$(\alpha+\beta)x\preceq\alpha x+\beta x, \tag{2.11}$$

$$x+z\preceq y+v \text{ if } x\preceq y \text{ and } z\preceq v, \tag{2.12}$$

$$\alpha x\preceq\alpha y \text{ if } x\preceq y. \tag{2.13}$$

Any linear space is a QLS with the partial order relation " $=$ ". One of the most common examples of a nonlinear QLS over the field of real numbers is $\mathbb{I}_{\mathbb{R}}$ with the inclusion relation "$\subseteq$".

Let's note some fundamental results from [1]. In a QLS X, the element θ is minimal, i.e., $x=\theta$ if $x\preceq\theta$. An element x' is called *inverse* of $x\in X$ if $x+x'=\theta$. The inverse is unique whenever it exists and an element x that possesses an inverse is called regular, otherwise, is called singular.

Lemma 1. [1] *Suppose that every element x in a QLS X has an inverse element $x'\in X$. In this case, the partial order in X is defined by equality, the distributivity conditions are satisfied, and as a result X becomes a linear space.*

In a real linear space, equality is the sole way to define a partial order that ensures the conditions (1)-(13) hold. $\mathbb{I}_{\mathbb{R}}^n$ is a quasilinear space with the partial order relation $x\preceq y$ if and only if $\left[\underline{x_k},\overline{x_k}\right]\subseteq\left[\underline{y_k},\overline{y_k}\right]$ for each $k=1,2,...,n$ [12].

A real-valued function $\|\cdot\|$ on a QLS X is called a *norm* if the following conditions hold:

$$\|x\|>0 \text{ if } x\neq 0, \tag{2.14}$$

$$\|x+y\|\leq\|x\|+\|y\|, \tag{2.15}$$

$$\|\alpha x\|=|\alpha|\,\|x\|, \tag{2.16}$$

$$\text{if } x\preceq y, \text{ then } \|x\|\leq\|y\|, \tag{2.17}$$

$$\begin{gathered}\text{if for any } \varepsilon>0 \text{ there exists an element } x_\varepsilon\in X \text{ such that}\\ x\preceq y+x_\varepsilon \text{ and } \|x_\varepsilon\|\leq\varepsilon \text{ then } x\preceq y,\end{gathered} \tag{2.18}$$

where x,y,x_ε are elements of X and α is any scalar. The quasilinear space X with a norm defined on it, is called as a *normed quasilinear space* [1]. From Lemma 2, it follows that if any $x\in X$ has an inverse element $x'\in X$ then the concept of a normed QLS coincides with the concept of a real normed linear space. Hausdorff metric or norm metric on X is defined by the formula

$$h_X(x,y)=\inf\left\{r\geq 0: x\preceq y+a_1^r,\ y\preceq x+a_2^r \text{ and } \|a_i^r\|\leq r, i=1,2\right\}.$$

Since $x\preceq y+(x-y)$ and $y\preceq x+(y-x)$, the quantity $h_X(x,y)$ is well-defined for any elements $x,y\in X$, and the function h_X satisfies all axioms of the metric. Furthermore $h_X(x,y)$ may not equal $\|x-y\|_X$ if X is not a linear space, but it always holds that $h_X(x,y)\leq\|x-y\|_X$ for every $x,y\in X$, [1].

Definition 1. [12] *Let X be a quasilinear space and let $\widehat{X}$ be the consolidated quasilinear space of X. A function $\langle .,. \rangle : X \times X \to \Omega(\mathbb{K})$ is called an inner product if it satisfies the following conditions for every $x, y, z, u, v \in X$ and every $a, b \in \mathbb{K}$,*

$$\text{1) If } x, y \in X_r\ ,\ \langle x, y\rangle \in \Omega(\mathbb{K})_r \equiv \mathbb{K}, \tag{2.19}$$

$$\text{2) } \langle x+y, z\rangle \subseteq \langle x, z\rangle + \langle y, z\rangle\,, \tag{2.20}$$

$$\text{3) } \langle ax, y\rangle = a\ \langle x, y\rangle\,, \tag{2.21}$$

$$\text{4) } \langle x, y\rangle = \langle y, x\rangle\,, \tag{2.22}$$

$$\text{5) If } x \in X_r, \text{ then } \langle x, x\rangle \geq 0 \text{ and } \langle x, x\rangle = 0 \Leftrightarrow x = \theta, \tag{2.23}$$

$$\text{6) } \|\langle x, y\rangle\|_{\Omega(\mathbb{K})} = \left\{ \|\langle a, b\rangle\|_{\Omega(\mathbb{R})}\ :\ a \in F_x^{\widehat{X}}\ ,\ b \in F_y^{\widehat{X}} \right\}, \tag{2.24}$$

$$\text{7) If } x \preceq y \text{ and } u \preceq v \text{ then } \langle x, u\rangle \subseteq \langle y, v\rangle\,,$$

$$\text{8) For every } \epsilon > 0, \text{ there exists at least one } x_\epsilon \in X \text{ such that} \tag{2.25}$$

$$x \preceq y + x_\epsilon \text{ and } \langle x_\epsilon, x_\epsilon\rangle \subseteq s_\epsilon(\theta) \text{ implies } x \preceq y\ . \tag{2.26}$$

If all these conditions are satisfied, the function $\langle .,. \rangle$ is called an inner product, and X, together with this inner product, is called an inner product quasilinear space*. Here, $\mathbb{K}$ denotes the field (representing the real or complex numbers), and $\Omega(\mathbb{K})$, is the quasilinear space consisting of all compact subsets of $\mathbb{K}$. Moreover, $s_\epsilon(\theta)$, denotes the sphere in $\Omega(\mathbb{K})$ centered at θ with radius ϵ. A quasilinear space with an inner product is called an* inner product quasilinear space.

Remark 1. *The regular subspace $\left(\mathbb{I}_{\mathbb{C}}^n\right)_r$ of $\mathbb{I}_{\mathbb{C}}^n$ is just the linear uniter space $\mathbb{C}^n$.*

Definition 2. [1] *A quasilinear normed space X will be called a $\Omega-$space if there exists an element $B_X \neq \theta$, satisfying the condition*

$$\text{if } \|x\|_X \leq \|B_X\|_X \Rightarrow x \preccurlyeq B_X.$$

In this case, from the condition $\|x\|_X \leq r \cdot \|B_X\|_X$, $r \geq 0$, it follows that

$$x \preccurlyeq r \cdot B_X$$

Consequently, one can assume that

$$\|B_X\|_X = 1.$$

3. Some New Results

Let us first give a known result from [8].

Theorem 1. [8]$\mathbb{I}_{\mathbb{C}}$ *is an inner-product QLS by*

$$\langle u, v\rangle_{\mathbb{I}_{\mathbb{C}}} = \{\langle a, b\rangle : a \in u, b \in v\} = \{ab^* : a \in u, b \in v\}\,,$$

for $u = \left[\underline{u_r}, \overline{u_r}\right] + i\left[\underline{u_s}, \overline{u_s}\right]$ and $v = \left[\underline{v_r}, \overline{v_r}\right] + i\left[\underline{v_s}, \overline{v_s}\right] \in \mathbb{I}_{\mathbb{C}}$.

Remark 2. *Inner-product norm on $\mathbb{I}_{\mathbb{C}}$ is derived as*

$$\begin{aligned}\|u\|_{\mathbb{I}_{\mathbb{C}}} &= \sqrt{\left\|\langle u, u\rangle_{\mathbb{I}_{\mathbb{C}}}\right\|} = \left(\sup\{|ab^*| : a, b \in u\}\right)^{1/2} \\ &= \left(\sup\{aa^* : a \in u\}\right)^{1/2} \\ &= \left(\sup\left\{|a|^2 : a \in u\right\}\right)^{1/2} \\ &= \max\{|a| : a \in u\}\end{aligned}$$

.

Remark 3. *If u and v are real intervals in $\mathbb{I}_{\mathbb{C}}$. Then $u=\left[\underline{u_r},\overline{u_r}\right]$ and $v=\left[\underline{v_r},\overline{v_r}\right]$ and*

$$\begin{aligned}\langle u,v\rangle_{\mathbb{I}_{\mathbb{C}}} &= \{\langle a,b\rangle : a\in u, b\in v\}\\ &= \{ab : a\in u, b\in v\}\\ &= [\underline{u},\overline{u}]\cdot[\underline{v},\overline{v}].\end{aligned}$$

But, if u and v are non-real complex intervals in $\mathbb{I}_{\mathbb{C}}$, the equality

$$\begin{aligned}\langle u,v\rangle_{\mathbb{I}_{\mathbb{C}}} &= \left(\left[\underline{u_r},\overline{u_r}\right]+i\left[\underline{u_s},\overline{u_s}\right]\right).\left(\left[\underline{v_r},\overline{v_r}\right]+i\left[\underline{v_s},\overline{v_s}\right]\right)\\ &= \left(\left[\underline{u_r},\overline{u_r}\right]\cdot\left[\underline{v_r},\overline{v_r}\right]+\left[\underline{u_s},\overline{u_s}\right]\cdot\left[\underline{v_s},\overline{v_s}\right]\right)+i(\left[\underline{u_s},\overline{u_s}\right]\cdot\left[\underline{v_r},\overline{v_r}\right]-\left[\underline{u_r},\overline{u_r}\right]\cdot\left[\underline{v_s},\overline{v_s}\right])\end{aligned}$$

is not always true. We can only say

$$\begin{aligned}\langle u,v\rangle_{\mathbb{I}_{\mathbb{C}}} &\subseteq \left(\left[\underline{u_r},\overline{u_r}\right]\cdot\left[\underline{v_r},\overline{v_r}\right]+\left[\underline{u_s},\overline{u_s}\right]\cdot\left[\underline{v_s},\overline{v_s}\right]\right)+i(\left[\underline{u_s},\overline{u_s}\right]\cdot\left[\underline{v_r},\overline{v_r}\right]-\left[\underline{u_r},\overline{u_r}\right]\cdot\left[\underline{v_s},\overline{v_s}\right])\\ &= \left(\left[\underline{u_r},\overline{u_r}\right]+i\left[\underline{u_s},\overline{u_s}\right]\right)\cdot\left(\left[\underline{v_r},\overline{v_r}\right]-i\left[\underline{v_s},\overline{v_s}\right]\right)=u\cdot v^*\end{aligned}$$

This inclusion is obvious. Let us see this inclusion is strict for some u and v. For example, let us take $u=v=[1,3]+i[1,2]\in\mathbb{I}_{\mathbb{C}}$. From the definition

$$\langle u,u\rangle_{\mathbb{I}_{\mathbb{C}}}=\{ab^* : a,b\in u\}.$$

An easy calculation in interval analysis implies

$$\begin{aligned}u\cdot u^* &= ([1,3]+i[1,2])\cdot([1,3]-i[1,2])\\ &= ([1,3]\cdot[1,3]+[1,2]\cdot[1,2])+i\,([1,2]\cdot[1,3]-[1,2]\cdot[1,3])\\ &= ([1,9]+[1,4])+i\,([1,6]-[1,6])\\ &= [2,13]+i[-5,5].\end{aligned}$$

Now let us see that some elements in the set $u\cdot u^*$ are not in the set $\langle u,u\rangle_{\mathbb{I}_{\mathbb{C}}}=\{ab^* : a,b\in u\}$. For example, $13-5i\in u\cdot u^*$ but $13-5i\notin\langle u,u\rangle_{\mathbb{I}_{\mathbb{C}}}$. Please remember that for some $a=a_1+ia_2$ and $b=b_1+ib_2$ in $u=[1,3]+i[1,2]$,

$$ab^*=(a_1+ia_2)(b_1-ib_2)=a_1b_1+a_2b_2+(a_2b_1-a_1b_2)\,i$$

where $a_1,b_1\in[1,3]$ and $a_2,b_2\in[1,2]$. Now, the equation

$$\begin{aligned}a_1b_1+a_2b_2 &= 13\\ a_2b_1-a_1b_2 &= -5\end{aligned}$$

has no solution within the desired intervals. Indeed; for example, the first equation holds only for the choices $a_1=b_1=3$ and $a_2=b_2=2$, meaning that for these elements, $a_1b_1+a_2b_2=13$, but for these elements, $a_2b_1-a_1b_2=0\neq-5$. This implies we cannot find a and b in u satisfying the condition $ab^*=13-5i$.

Remark 4. *The regular subspace $(\mathbb{I}_{\mathbb{C}})_r$ of $\mathbb{I}_{\mathbb{C}}$ is just the linear space $\mathbb{C}$. So, the regular subspace $(\mathbb{I}_{\mathbb{C}}^n)_r$ of $\mathbb{I}_{\mathbb{C}}^n$ is just the uniter space $\mathbb{C}^n$.*

Theorem 2. *$\mathbb{I}_{\mathbb{C}}^n$ is an inner-product quasilinear space with respect to the inner-product*

$$\langle x,y\rangle=\sum_{k=1}^{n}\langle x_k,y_k\rangle_{\mathbb{I}_{\mathbb{C}}}=\sum_k\{ab^* : a\in x_k, b\in y_k\}$$

for $x, y \in \mathbb{I}_{\mathbb{C}}^{n}$*. In particular, if* (x_k) *and* (y_k) *are real interval-valued elements in* $\mathbb{I}_{\mathbb{C}}^{n}$*, then*

$$\begin{aligned}\langle x, y\rangle &= \langle (x_k), (y_k)\rangle = \sum_{k\in\mathbb{Z}} \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}} = \sum_{k} \{ab : a \in x_k, b \in y_k\} \\ &= \sum_{k\in\mathbb{Z}} \left([\underline{x_k}, \overline{x_k}] \cdot [\underline{y_k}, \overline{y_k}] \right)\end{aligned}$$

Proof. First of all we should say that the above function from $\mathbb{I}_{\mathbb{C}}^{n} \times \mathbb{I}_{\mathbb{C}}^{n}$ to $\Omega(\mathbb{C})$ is well-defined. This comes from the following facts;

$$\begin{aligned}\|\langle x, y\rangle\| &= \left\| \sum_{k\in\mathbb{Z}} \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}} \right\| \le \sum_{k\in\mathbb{Z}} \left\| \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}} \right\| \\ &\le \sum_{k\in\mathbb{Z}} \|x_k\|_{\mathbb{I}_{\mathbb{C}}} \|y_k\|_{\mathbb{I}_{\mathbb{C}}} \le \left(\sum_{k\in\mathbb{Z}} \|x_k\|_{\mathbb{I}_{\mathbb{C}}}^2 \right)^{1/2} \left(\sum_{\in k\mathbb{Z}} \|y_k\|_{\mathbb{I}_{\mathbb{C}}}^2 \right)^{1/2} < \infty.\end{aligned}$$

For any $x, y \in (\mathbb{I}_{\mathbb{C}}^{n})_r$ implies $x = x = (x_k)$ and $y = y = (y_k)$ are elements of l_2. Hence

$$\langle x, y\rangle = \sum \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}} = \sum x_k y_k^* \in \Omega(\mathbb{C})_r \equiv \mathbb{C}.$$

For $x, y, z, \in \mathbb{I}_{\mathbb{C}}^{n}$,

$$\begin{aligned}\langle x + y, z\rangle &= \sum \langle x_k + y_k, z_k\rangle_{\mathbb{I}_{\mathbb{C}}} \subseteq \sum \langle x_k, z_k\rangle_{\mathbb{I}_{\mathbb{C}}} + \sum \langle y_k, z_k\rangle_{\mathbb{I}_{\mathbb{C}}} \\ &\subseteq \langle x, z\rangle + \langle y, z\rangle .\end{aligned}$$

Later three conditions are obvious. Now let us verify that

$$\|\langle x, y\rangle\| = \sup \left\{ \|\langle a, b\rangle\| : a \in F_{\widetilde{x}}^{\mathbb{I}_{\mathbb{C}}^{n}}, b \in F_{\widetilde{y}}^{\mathbb{I}_{\mathbb{C}}^{n}} \right\}.$$

By the definition of the norm on $\Omega(\mathbb{C})$,

$$\|\langle x, y\rangle\| = \sup\{|t| : t \in \langle x, y\rangle\} = \sup\{|t| : t \in \sum \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}}\}.$$

Now, $t \in \sum \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}}$ implies that there exist a two-sided complex sequence $(t_k)_{k\in\mathbb{Z}}$ such that $t_k \in \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}}$ and $\sum t_k = t$. Further, for each t_k, there exists $a_k \in x_k$ and $b_k \in y_k$ such that $t_k = \langle a_k, b_k\rangle$. Of course, these representation may not be unique. But this is no problem since we will take supremum of all possible cases. Further, $x = F_x^{\mathbb{I}_{\mathbb{C}}^{n}}$ and $y = F_y^{\mathbb{I}_{\mathbb{C}}^{n}}$. Hence

$$\begin{aligned}\sup\{|t| : t \in \sum \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}}\} &= \sup\{\left|\sum t_k\right| : t \in \sum \langle x_k, y_k\rangle_{\mathbb{I}_{\mathbb{C}}}\} \\ &= \sup\{\left|\sum \langle a_k, b_k\rangle\right| : a_k \in x_k, \quad b_k \in y_k\} \\ &= \sup\{\left|\sum \langle a_k, b_k\rangle\right| : (a_k) \in x, \quad (b_k) \in y\} \\ &= \sup \left\{ \|\langle a, b\rangle\| : a \in F_x^{\mathbb{I}_{\mathbb{C}}^{n}}, b \in F_y^{\mathbb{I}_{\mathbb{C}}^{n}} \right\}.\end{aligned}$$

Easily see that if $x \preceq y$ and $u \preceq v$ then $\langle x, u\rangle \subseteq \langle y, v\rangle$. For the last condition of the inner-product, let us assume that for each $\varepsilon > 0$ there exists an $\widetilde{u}_\varepsilon \in \mathbb{I}_{\mathbb{C}}^{n}$ such that $u \preceq v + \widetilde{u_\varepsilon}$ and $\langle \widetilde{u}_\varepsilon, \widetilde{u}_\varepsilon\rangle \subseteq S_\varepsilon(0)$ where $S_\varepsilon(0)$ is the 0-centered open or closed ball with radius ε in $\Omega(\mathbb{C})$. We will prove that $u \preceq v$ under this condition. The condition $u \preceq v + \widetilde{u_\varepsilon}$ and $\langle \widetilde{u_\varepsilon}, \widetilde{u_\varepsilon}\rangle \subseteq S_\varepsilon(0)$ means that $u_k \preceq v_k + \widetilde{u_{\varepsilon k}}$ for each $k \in \mathbb{Z}$ and $\sum_{k\in\mathbb{Z}} \langle \widetilde{u_{\varepsilon k}}, \widetilde{u_{\varepsilon k}}\rangle_{\mathbb{I}_{\mathbb{C}}} \subseteq S_\varepsilon(0)$. This also implies $\langle \widetilde{u_{\varepsilon k}}, \widetilde{u_{\varepsilon k}}\rangle_{\mathbb{I}_{\mathbb{C}}} \subseteq S_\varepsilon(0)$ for all $k \in \mathbb{Z}$. Hence we get $u_k \preceq v_k$ for all $k \in \mathbb{Z}$ by the Theorem1. This means $u \preceq v$ and the proof is completed. □

Theorem 3. *$\mathbb{I}_{\mathbb{C}}^n$ is an $\Omega -$ space.*

Proof. Consider the element

$$B = ([-1,1] + i\,[-1,1]\,, ..., [-1,-1] + i\,[-1,1])$$

in $\mathbb{I}_{\mathbb{C}}^n$. Let us assume that for any $z = (z_k) \in \mathbb{I}_{\mathbb{C}}^n$,

$$\|z\| \leq \|B\|\,.$$

This implies if and only if

$$\begin{aligned}
&\Leftrightarrow \quad \|z\|^2 \leq \|B\|^2 \\
&\Leftrightarrow \quad \|\langle z, z\rangle\|_{\Omega(\mathbb{C})} \leq \|\langle B, B\rangle\|_{\Omega(\mathbb{C})} \\
&\Leftrightarrow \quad \left\| \sum_{k=1}^{n} \langle z_k, z_k\rangle_{\mathbb{I}_{\mathbb{C}}} \right\|_{\Omega(\mathbb{C})} \leq \left\| \sum_{k=1}^{n} \langle B_k, B_k\rangle_{\mathbb{I}_{\mathbb{C}}} \right\|_{\Omega(\mathbb{C})} \\
&\Leftrightarrow \quad \sum_{k=1}^{n} \left\|\langle z_k, z_k\rangle_{\mathbb{I}_{\mathbb{C}}}\right\|_{\Omega(\mathbb{C})} \leq \sum_{k=1}^{n} \left\|\langle B_k, B_k\rangle_{\mathbb{I}_{\mathbb{C}}}\right\|_{\Omega(\mathbb{C})}.
\end{aligned}$$

Now for each k,

$$\begin{aligned}
\left\|\langle B_k, B_k\rangle_{\mathbb{I}_{\mathbb{C}}}\right\|_{\Omega(\mathbb{C})} &= \|\{ab^* : a, b \in [-1,1] + i\,[-1,1]\}\|_{\Omega(\mathbb{C})} \\
&= \sup\{|ab^*| : a, b \in [-1,1] + i\,[-1,1]\} \\
&= |(-1+i)\,(-1-i)| = 2.
\end{aligned}$$

Hence our assumption implies that

$$\|z\|^2 \leq \|B\|^2 = 2n.$$

On the other hand this gives us $z \subseteq B$, that is for each $k \in \{1, 2, ..., n\}$,

$$\begin{aligned}
\left[\underline{z_r^k}, \overline{z_r^k}\right] + i\left[\underline{z_s^k}, \overline{z_s^k}\right] &\subseteq [-1,1] + i\,[-1,1] \\
&\Leftrightarrow \left[\underline{z_r^k}, \overline{z_r^k}\right] \subseteq [-1,1] \text{ and } \left[\underline{z_s^k}, \overline{z_s^k}\right] \subseteq [-1,1]
\end{aligned}$$

where $\left[\underline{z_r^k}, \overline{z_r^k}\right]$ is the real part and $\left[\underline{z_s^k}, \overline{z_s^k}\right]$ is the imaginary part of the complex interval z_k. Let us see this. For example, if $\left[\underline{z_s^k}, \overline{z_s^k}\right] \subsetneq [-1,1]$ or, $\left[\underline{z_r^k}, \overline{z_r^k}\right] \subsetneq [-1,1]$ this would give us

$$\max\left\{\left|\underline{z_r^k}\right|, \left|\overline{z_r^k}\right|, \left|\underline{z_s^k}\right|, \left|\underline{z_s^k}\right|\right\} > 1.$$

From here, we would arrive at the contradiction

$$\sup\{|ab^*| : a, b \in [-1,1] + i\,[-1,1]\} > 2 \Leftrightarrow \|z\|^2 > 2n.$$

This contradiction completes the proof. □

Now we shall impose a norm and a metric on the previously known finite complex interval sequence space. It is easy to see that this function is a norm, and this norm has also been given in some similar works [6]. What is a little more difficult is to see whether they are complete. Now let us demonstrate this.

Theorem 4. *The quasilinear space $\mathbb{I}\ell_\infty$ of bounded complex interval sequences is a* Banach quasilinear space *together with the norm*

$$\|x\|_{\mathbb{I}\ell_\infty} = \sup_{1\leq k<\infty} \|x_k\|_{\mathbb{I}_\mathbb{C}}\ , \ for\ x \in \mathbb{I}\ell_\infty,$$

where

$$\begin{aligned} \|x_k\|_{\mathbb{I}_\mathbb{C}} &= \sqrt{\left\| \langle x_k, x_k\rangle_{\mathbb{I}_\mathbb{C}} \right\|} = \left(\sup\left\{|ab^*| : a, b \in x_k\right\}\right)^{1/2} \\ &= \left(\max\left\{aa^* : a \in x_k,\ a \in \mathbb{C}\right\}\right)^{1/2} \\ &= \left(\max\left\{|a|^2 : a \in x_k,\ a \in \mathbb{C}\right\}\right)^{1/2} \\ &= \max\left\{|a| : a \in x_k,\ a \in \mathbb{C}\right\} \end{aligned}$$

and remember that

$$x_k = \left[\underline{x_r^k}, \overline{x_r^k}\right] + i\left[\underline{x_s^k}, \overline{x_s^k}\right]$$

as a complex interval.

Proof. For all $x, y \in \mathbb{I}\ell_\infty$, the Hausdorff metric is defined as

$$h(x, y) = \inf\left\{r \geq 0,\ x \preceq y + a_1^r,\ y \preceq x + a_2^r\ ,\ \left\|a_j^r\right\|_{\mathbb{I}\ell_\infty} \leq r : j = 1, 2\right\}.$$

It is easy to see that the space $\mathbb{I}\ell_\infty$ is a normed quasilinear space with the mentioned norm. (See [8], [7], [7] and [5]). The difficult part is proving completeness. Now let's do that. Let (x^n) be a Cauchy sequence in $\mathbb{I}\ell_\infty$. For every $\epsilon > 0$, there exists at least one $n_0 = n_\epsilon \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$x^n \preceq x^m + a_1^r,\ x^m \preceq x^n + a_2^r,\ \text{for}\ \|a_1^r\|_{\mathbb{I}\ell_\infty} \leq r\ \text{and}\ \|a_2^r\|_{\mathbb{I}\ell_\infty} \leq r.$$

In this case, for every $1 \leq k < \infty$,

$$x_k^n \subseteq x_k^m + a_{1,k}^r,\ x_k^m \subseteq x_k^n + a_{2,k}^r,$$

and

$$\|a_{1\ \text{and}\ 2}^r\|_{\mathbb{I}\ell_\infty} = \sup_{1\leq k<\infty} \left\|a_{1,k\ \text{and}\ 2,k}^r\right\|_{\mathbb{I}_\mathbb{C}} \leq r.$$

So for every $1 \leq k < \infty$ above inclusions implies

$$h\left(x_k^n, x_k^m\right) \leq r.$$

Thus, (x_k^n), is a Cauchy sequence in $\mathbb{I}_\mathbb{C}$. Since $\mathbb{I}_\mathbb{C}$ is a complete metric space, for every $1 \leq k < \infty$, there exists $x_k \in \mathbb{I}_\mathbb{C}$ such that $x_k^n \to x_k$, as $n \to \infty$. Letting $m \to \infty$ in above inclusions again, for every $\epsilon > 0$, we have $\exists n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$x_k^n \subseteq x_k + c_k^{1r},\ x_k \subseteq x_k^n + c_i^{2r}\ ,\ \left\|c_i^{jr}\right\|_{\mathbb{I}_\mathbb{C}} \leq r : j = 1, 2.$$

Since $\mathbb{I}_\mathbb{C}$ is a quasilinear space with the partial order $\preceq$, we obtain

$$x^n \preceq x + c^{1r},\ x \preceq x^n + c^{2r}.$$

From this result we get

$$\left\|c_k^{jr}\right\|_{\mathbb{I}_\mathbb{C}} \leq r \Rightarrow \sup_{1\leq k<\infty}\left\{\left\|c_k^{jr}\right\|_{\mathbb{I}_\mathbb{C}}\right\} \leq r,$$

for every $1 \leq k < \infty$. Hence

$$\left\|c^{jr}\right\|_{\mathbb{I}\ell_\infty} \leq r.$$

Thus, we conclude that $(x^n) \subset \mathbb{I}\ell_\infty$, converges to x, i.e.,

$$h(x^n, x) \leq r \text{ as } n \to \infty.$$

Since $x_k \in \mathbb{I}_\mathbb{C}$ for every $1 \leq k < \infty$, we have just proved that $(x_1, x_2, \ldots, x_n, \ldots) = x \in \mathbb{I}\ell_\infty$, and $x^n \to x$ in $\mathbb{I}\ell_\infty$. Since x^n, is an arbitrary Cauchy sequence in $\mathbb{I}\ell_\infty$ this completes the proof. □

Conclusion 1. *Complex interval sequences play a significant role in signals processing theory, especially, in signals with inexact data. Our former works on this topic can be found in references* [14, 13, 8] *and* [9]*. Furthermore, in discrete-time signal processing applications such as image processing, we generally require two-dimensional complex interval sequences. In these processes, we generally deal with a window of these sequences, i.e., a finite-dimensional subspace. For signals with inexact data, this situation requires us to work in the space* $\mathbb{I}_\mathbb{C}^n$*. Therefore, we can say that the* $\mathbb{I}_\mathbb{C}^n$ *provides an infrastructure for processing signals with inexact data in multidimensional signal processing. On the other hand, determining the Bounded Input Bounded Output (BIBO) stability of discrete-time signals with inexact data is an important issue. To determine this, we select the space* $\mathbb{I}\ell_\infty$ *in which this type of signal resides and use the norm on it to reach some conclusions. Such investigations are of considerable importance in digital signal processing. We see that the results presented in this study provide a mathematical foundation for such applications.*

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