The dissipative Bresse-Timoshenko system without a second spectrum is well-posedness and exponential stability

Abstract

This paper investigates the dissipative Bresse-Timoshenko system without second spectrum. By using the theory of C_0-semigroup, We give a detailed calculation process of the well-posedness and we prove that the system is exponential stability for any parameters.

Keywords: Bresse-Timoshenko system; Well-posedness; Exponential stability.

MSC: 35B40, 35B35, 81U30, 65H04.

1 Introduction and main results

In 1921, Tymoshenko [14] optimized the Euler-Bernoulli beam model and the Rayleigh beam model and proposed the following hyperbolic system of two coupled wave equations

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \end{cases} \tag{1.1}$$

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which is called *Timoshenko beam model*, where φ and ψ are the deflection of the beam from its equilibrium position and the rotation of the neutral axis, respectively, $\rho_1 = \rho A, \rho_2 = \rho I, b = EI$ and $k = k'GA$ are positive constants with ρ is the density, A is the cross-sectional area, I is the second moment of area of the cross-sectional area, E is the Young modulus of elasticity, G is the modulus of rigidity, k' is the transverse shear factor. However, it was later discovered that the Timoshenko beam model admits two wave speeds

$$\sqrt{k/\rho_1} \text{ and } \sqrt{b/\rho_2},$$

which contributes to a physical paradox called the *second spectrum* (see, for example, [6, 7, 10]). Based on these reasons, Elishakoff [8] proposed the following truncated version model by combining d'Alembert's principle for dynamic equilibrium from Timoshenko hypothesis,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \end{cases} \tag{1.2}$$

which eliminates the anomaly of the second spectrum since it admits one wave speed

$$\sqrt{b/[\rho_2(1 + \rho_1 b/k\rho_2)]}.$$

The model (1.2) is called *Bresse-Timoshenko system without second spectrum* and has been extensively in recent years (see [1, 2, 5, 9, 13] and references therein).

In this paper, we consider the following dissipative Bresse-Timoshenko system without second spectrum proposed in [4]

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \end{cases} \tag{1.3}$$

where $\mu > 0$ represents the damping coefficient acting on displacement function. Moreover, we consider the boundary conditions of Dirichlet-Neumann type given by

$$\varphi(0, t) = \varphi(L, t) = \psi_x(0, t) = \psi_x(L, t) = 0, \quad t \geq 0. \tag{1.4}$$

and initial conditions given by

$$\varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \quad x \in (0, L). \tag{1.5}$$

When $\rho_2 = 0$, problem (1.3) with boundary conditions of Dirichlet-Neumann type was studied in [3] and the authors showed the exponential decay of the energy. In [4], the authors studied (1.3) with boundary conditions of Dirichlet-Dirichlet or Neumann-Dirichlet type, and the exponential decay of the energy was obtained. However,

1. for boundary conditions of Dirichlet-Neumann type only the case $\rho_2 = 0$ was considered in [3];

2. the well-posedness results were not considered in [4].

Based on the above reasons, we will consider the problem (1.3)-(1.5). The C_0-semigroup theory are applied to study the well-posedness and exponential stability, which is different from [3] and [4], where the multiplying method and energy method were used to study the exponential stability.

From $(1.3)_1$, we get

$$\psi_x = \frac{\rho_1}{k}\varphi_{tt} + \frac{\mu}{k}\varphi_t - \varphi_{xx}.$$

By substituting ψ_x into $(1.3)_2$, we have

$$-\rho_2\varphi_{ttxx} - b\left(\frac{\rho_1}{k}\varphi_{ttxx} + \frac{\mu}{k}\varphi_{txx} - \varphi_{xxxx}\right) + k\varphi_{xx} + k\left(\frac{\rho_1}{k}\varphi_{tt} + \frac{\mu}{k}\varphi_t - \varphi_{xx}\right) = 0,$$

i.e., problem (1.3)-(1.5) can be transformed to

$$\begin{cases} (k\rho_1 I - (b\rho_1 + k\rho_2)\partial_{xx})\,\varphi_{tt} + (k\mu I - b\mu\partial_{xx})\,\varphi_t + bk\varphi_{xxxx} = 0, & \text{in } (0,L)\times(0,\infty), \\ \varphi(0,t) = \varphi(L,t) = \varphi_{xx}(0,t) = \varphi_{xx}(L,t) = 0, & t > 0, \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), & x \in (0,L). \end{cases}$$

Let

$$\mathbf{A} := \partial_{xxxx}, \tag{1.6}$$

$$\mathbf{B} := k\rho_1 I - (b\rho_1 + k\rho_2)\partial_{xx}, \tag{1.7}$$

$$\mathbf{C} := k\mu I - b\mu\partial_{xx}. \tag{1.8}$$

Obviously, $\mathbf{A}$ is a positive self-adjoint operator from $\{\zeta \in H^4(0,L)\cap H_0^1(0,L) : \zeta_{xx} \in H_0^1(0,L)\}$ to $L^2(0,L)$, which can be extended as an isomorphism from $H_*^3(0,L)$ to $H^{-1}(0,L)$; $\mathbf{B}$ and $\mathbf{C}$ are positive self-adjoint operators from $H^2(0,L)\cap H_0^1(0,L)$ to $L^2(0,L)$, which can be extended as an isomorphism from $H_0^1(0,L)$ to $H^{-1}(0,L)$, where

$$H_*^3(0,L) := \left\{\zeta \in H^3(0,L)\cap H_0^1(0,L) : \zeta_{xx} \in H_0^1(0,L)\right\}. \tag{1.9}$$

Then, problem (1.3)-(1.5) can be written as the following abstract form in $H_0^1(0,L)$:

$$\begin{cases} \varphi_{tt} + \mathbf{B}^{-1}\mathbf{C}\varphi_t + bk\mathbf{B}^{-1}\mathbf{A}\varphi = 0, & t > 0, \\ \varphi(0) = \varphi_0 \in H^2(0,L)\cap H_0^1(0,L), \varphi_t(0) = \varphi_1(x) \in H_0^1(0,L). \end{cases} \tag{1.10}$$

Let

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ -bk\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{C} \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} := \begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}, \quad \Phi_0 := \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \tag{1.11}$$

and

$$\mathcal{H} := \left(H^2(0,L) \cap H_0^1(0,L)\right) \times H_0^1(0,L). \tag{1.12}$$

It is obvious that $\mathcal{H}$ is a Hilbert space with scalar product

$$\langle \Phi, \Phi^* \rangle_{\mathcal{H}} = \rho_1 \langle \phi, \phi^* \rangle + \rho_2 \langle \phi_x, \phi_x^* \rangle + b \langle \mathbf{T}\varphi_x, \varphi_x^* \rangle, \quad \forall\, \Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in \mathcal{H},\ \Phi^* = \begin{pmatrix} \varphi^* \\ \phi^* \end{pmatrix} \in \mathcal{H}, \tag{1.13}$$

where

$$\mathbf{T} := -\frac{k}{b\rho_1 + k\rho_2}\left(\rho_2 I + b\rho_1^2 \mathbf{B}^{-1}\right) \circ \partial_{xx} \tag{1.14}$$

and $\langle \cdot, \cdot \rangle$ denotes the standard L^2-scalar product.

With the above preparations, one can see $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, and problem (1.10) can be written as

$$\begin{cases} \dfrac{d}{dt}\Phi = \mathcal{A}\Phi \in \mathcal{H}, \quad t > 0, \\ \Phi(0) = \Phi_0, \end{cases} \tag{1.15}$$

where

$$D(\mathcal{A}) = \left\{ \Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in \mathcal{H} : \varphi \in H_*^3(0,L), \phi \in H^2(0,L) \cap H_0^1(0,L) \right\}. \tag{1.16}$$

Theorem 1.1. *$\mathcal{A}$ generates a C_0-semigroup of contraction $\left(e^{t\mathcal{A}}\right)_{t \ge 0}$ on $\mathcal{H}$, which is exponential stable, i.e., there exist two positive constants M and α such that*

$$\left\| e^{t\mathcal{A}} \right\| \le M e^{-\alpha t}$$

for any $t \ge 0$.

The rest of this paper is devoted to prove the above theorem.

2 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 by using the following two theorems. Let $\theta \in \mathbb{C}$, A be an operator, and f, g be two quantities, Reθ denotes the real part of θ, $\overline{\theta}$ denotes the conjugate complex of θ, $\rho(A)$ denotes the resolvent set of A, the notation $f \lesssim g$ means there exists a constant such that $f \le Cg$.

To show $\mathscr{A}$ generates a C_0-semigroup of contraction, we need the following theorem[11, Theoem 1.2.4], which can be seen as a corollary of the Lumer-Phillips theorem.

Theorem 2.1. *Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and A be a linear operator with dense domain $D(A)$ in H. If A is dissipative, i.e.,*

$$\mathrm{Re}\langle \zeta, A\zeta \rangle_H \leq 0$$

for any $\zeta \in H$ and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ of contraction on H.

To prove the exponential stability of a C_0-semigroup we need the following theorem [11, Theoem 1.3.2].

Theorem 2.2. *Let $S(t)$ be a C_0-semigroup of contractions on a Hilbert space with infinitesimal generator A. Then $S(t)$ is exponentially stable if and only if*

$$\rho(A) \supset i\mathbb{R} := \{i\beta : \beta \in \mathbb{R}\}$$

and

$$\limsup_{|\beta|\to\infty} \left\| (i\beta I - A)^{-1} \right\| < \infty.$$

Proof of Theorem 1.1. It is obvious that $D(\mathscr{A})$ is dense in $\mathscr{H}$. We first show that $\mathscr{A}$ is dissipative. For any

$$\Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in D(\mathscr{A}),$$

by (1.11) and (1.6),

$$\begin{aligned} \mathscr{A}\Phi &= \begin{pmatrix} 0 & I \\ -bk\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{C} \end{pmatrix} \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \\ &= \begin{pmatrix} \phi \\ -\mathbf{B}^{-1}\mathbf{C}\phi - bk\mathbf{B}^{-1}\mathbf{A}\varphi \end{pmatrix} = \begin{pmatrix} \phi \\ -\mathbf{B}^{-1}\mathbf{C}\phi - bk\mathbf{B}^{-1}\varphi_{xxxx} \end{pmatrix}. \end{aligned}$$

Then it follows from (1.13), (1.14), (1.7), and (1.8) that

$$\begin{aligned} \mathrm{Re}\langle \Phi, \mathscr{A}\Phi \rangle_{\mathscr{H}} = \mathrm{Re}\int_0^L \rho_1 \phi \left[-\mathbf{B}^{-1}\mathbf{C}\overline{\phi} - bk\mathbf{B}^{-1}\overline{\varphi}_{xxxx} \right] dx + \mathrm{Re}\int_0^L \rho_2 \phi_x \left[-\mathbf{B}^{-1}\mathbf{C}\overline{\phi} - bk\mathbf{B}^{-1}\overline{\varphi}_{xxxx} \right]_x dx \\ - \mathrm{Re}\int_0^L \frac{bk}{b\rho_1 + k\rho_2} \left(\rho_2 I + b\rho_1^2 \mathbf{B}^{-1} \right) \varphi_{xxx} \overline{\phi}_x dx \end{aligned}$$

$$
\begin{aligned}
=&\mathrm{Re}\int_0^L \mathbf{B}^{-1}\overline{\left[-\rho_1 bk\varphi_{xx}+\rho_2 bk\varphi_{xxxx}+\frac{bk}{b\rho_1+k\rho_2}\left(\rho_2\mathbf{B}+b\rho_1^2 I\right)\varphi_{xx}\right]}\phi_{xx}dx\\
&-\mathrm{Re}\int_0^L \mathbf{B}^{-1}\left[\rho_1(\mathbf{C}\overline{\phi})\phi+\rho_2(\mathbf{C}\overline{\phi}_x)\phi_x\right]dx\\
=&\mathrm{Re}\int_0^L \mathbf{B}^{-1}\overline{\left[\underbrace{-\rho_1 bk\varphi_{xx}+\rho_2 bk\varphi_{xxxx}+\frac{bk^2\rho_1\rho_2}{b\rho_1+k\rho_2}\varphi_{xx}-\rho_2 bk\varphi_{xxxx}+\frac{b^2k\rho_1^2}{b\rho_1+k\rho_2}\varphi_{xx}}_{=0}\right]}\phi_{xx}dx\\
&-\mathrm{Re}\left(\mu k\rho_1\int_0^L \mathbf{B}^{-1}\overline{\phi}\phi dx+(\mu b\rho_1+\mu k\rho_2)\int_0^L \mathbf{B}^{-1}\overline{\phi}_x\phi_x dx+\mu b\rho_2\int_0^L \mathbf{B}^{-1}\overline{\phi}_{xx}\phi_{xx}dx\right)\\
=&-\left(\mu k\rho_1\left\|\mathbf{B}^{-\frac{1}{2}}\phi\right\|_{L^2(0,L)}^2+(\mu b\rho_1+\mu k\rho_2)\left\|\mathbf{B}^{-\frac{1}{2}}\phi_x\right\|_{L^2(0,L)}^2+\mu b\rho_2\left\|\mathbf{B}^{-\frac{1}{2}}\phi_{xx}\right\|_{L^2(0,L)}^2\right)\\
\leq&0,
\end{aligned}
\tag{2.1}
$$

hence $\mathcal{A}$ is dissipative.

Secondly, we prove that $0\in\rho(\mathcal{A})$. At first, we show $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is surjective, i.e., for given $G=\begin{pmatrix} g_1\\ g_2\end{pmatrix}\in\mathcal{H}$, we need to show there exists $\Phi=\begin{pmatrix}\varphi\\ \phi\end{pmatrix}\in D(\mathcal{A})$ satisfying

$$
-\mathcal{A}\Phi=G, \tag{2.2}
$$

this means

$$
\begin{cases}
-\phi=g_1\in H^2(0,L)\cap H_0^1(0,L),\\
\mathbf{C}\phi+bk\varphi_{xxxx}=\mathbf{B}g_2\in H^{-1}(0,L),\\
\varphi(0)=\varphi(L)=\varphi_{xx}(0)=\varphi_{xx}(L)=0.
\end{cases}
$$

Then we get

$$
\phi=-g_1\in H^2(0,L)\cap H_0^1(0,L),
$$

and φ satisfies

$$
\begin{cases}
bk\varphi_{xxxx}=\mathbf{B}g_2+\mathbf{C}g_1\in H^{-1}(0,L),\\
\varphi(0)=\varphi(L)=\varphi_{xx}(0)=\varphi_{xx}(L)=0.
\end{cases}
\tag{2.3}
$$

The standard theory of elliptic equations shows that (2.3) admits a unique solution $\varphi\in H_*^3(0,L)$ and

$$
\|\varphi\|_{H_*^3(0,L)}\lesssim\|\mathbf{B}g_2+\mathbf{C}g_1\|_{H^{-1}(0,L)}\lesssim\|G\|_{\mathcal{H}}.
$$

So the above analysis shows that (2.2) admits a unique solution $\Phi\in D(A)$ and

$$
\|\Phi\|_{H_*^3(0,L)\times\left(H^2(0,L)\cap H_0^1(0,L)\right)}\lesssim\|G\|_{\mathcal{H}}. \tag{2.4}
$$

Following (2.4), we get, $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is injective. So $\mathcal{A}^{-1} : \mathcal{H} \to D(\mathcal{A})$ exists, and by (2.4) again $\mathcal{A}^{-1}$ is a bounded linear operator on $\mathcal{H}$. Therefore, we get $0 \in \rho(\mathcal{A})$.

So by Theorem 2.1, $\mathcal{A}$ generates a C_0-semigroup of contraction $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ on $\mathcal{H}$.

Next we show $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is exponentially stable by using Theorem 2.2. We first show

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{2.5}$$

by contradiction argument. If (2.5) in not true, since we have shown $0 \in \rho(\mathcal{A})$, by the proof of [11, Theoem 2.2.1], there is a constant $\omega \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\| \leq |\omega| < \infty$ such that $\{i\beta : |\beta| < |\omega|\} \subset \rho(\mathcal{A})$ and

$$\sup_{|\beta|<|\omega|} \left\|(i\beta - \mathcal{A})^{-1}\right\| = \infty.$$

Then there exists a sequence $\{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$ with $\beta_n \to \omega$ ($n \to \infty$), $|\beta_n| < |\omega|$ and a sequence

$$\{\Phi_n\}_{n=1}^{\infty} = \left\{ \begin{pmatrix} \varphi_n \\ \phi_n \end{pmatrix} \right\}_{n=1}^{\infty} \subset D(\mathcal{A})$$

with

$$\begin{aligned}
\|\Phi_n\|_{\mathcal{H}}^2 &= \rho_1\|\phi_n\|_{L^2(0,L)}^2 + \rho_2\|\phi_{nx}\|_{L^2(0,L)}^2 + b\langle \mathbf{T}\varphi_{nx}, \varphi_{nx}\rangle \\
&= \rho_1\|\phi_n\|_{L^2(0,L)}^2 + \rho_2\|\phi_{nx}\|_{L^2(0,L)}^2 + \frac{kb}{b\rho_1 + k\rho_2}\left(\rho_2 \left\|\varphi_{nxx}\right\|_{L^2(0,L)}^2 + b\rho_1^2 \left\|\mathbf{B}^{-\frac{1}{2}}\varphi_{nxx}\right\|_{L^2(0,L)}^2\right) \\
&= 1
\end{aligned} \tag{2.6}$$

such that

$$\|(i\beta_n - \mathcal{A})\Phi_n\|_{\mathcal{H}} \to 0 \tag{2.7}$$

as $n \to \infty$, i.e.,

$$i\beta_n\varphi_n - \phi_n \to 0 \qquad \text{in } H^2(0,L) \cap H_0^1(0,L), \tag{2.8}$$

$$i\beta_n\phi_n + \mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx} \to 0 \qquad \text{in } H_0^1(0,L). \tag{2.9}$$

Similar to the proof of (2.1), we get

$$\begin{aligned}
&\mathrm{Re}\langle (i\beta_n I - \mathcal{A})\Phi_n, \Phi_n\rangle_{\mathcal{H}} \\
&\quad = \mu k\rho_1 \left\|\mathbf{B}^{-\frac{1}{2}}\phi_n\right\|_{L^2(0,L)}^2 + (\mu b\rho_1 + \mu k\rho_2)\left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nx}\right\|_{L^2(0,L)}^2 + \mu b\rho_2 \left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nxx}\right\|_{L^2(0,L)}^2,
\end{aligned}$$

which, together with (2.6) and (2.7), implies

$$\lim_{n\to\infty} \|\phi_n\|_{L^2(0,L)} = 0 \text{ and } \lim_{n\to\infty} \|\phi_{nx}\|_{L^2(0,L)} = 0, \tag{2.10}$$

where we have used the facts that

$$\|\phi_n\|_{L^2(0,L)} \lesssim \left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nx}\right\|_{L^2(0,L)} \text{ and } \|\phi_{nx}\|_{L^2(0,L)} \leq \left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nxx}\right\|_{L^2(0,L)}.$$

Then, it follows from (2.9) that

$$\begin{aligned}\limsup_{n\to\infty}\|\varphi_{nxx}\|_{L^2(0,L)} &\lesssim \limsup_{n\to\infty}\left\|\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} \\ &\lesssim \lim_{n\to\infty}\left\|i\beta_n\phi_n + \mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} + \lim_{n\to\infty}\|\phi_n\|_{L^2(0,L)} = 0,\end{aligned} \tag{2.11}$$

where we have used the facts that

$$\|\varphi_{nxx}\|_{L^2(0,L)} \lesssim \left\|\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)},$$

$|\beta_n| \leq \omega + 1 < \infty$ for n large enough since $\beta_n \to \omega$ $(n \to \infty)$ and $|\omega| < \infty$, and

$$\left\|\mathbf{B}^{-1}\mathbf{C}\phi_n\right\|_{L^2(0,L)} \lesssim \|\phi_n\|_{L^2(0,L)}.$$

By (2.8) and (2.10) we get

$$\begin{aligned}\limsup_{n\to\infty}\left\|\mathbf{B}^{-\frac{1}{2}}\varphi_{nxx}\right\|_{L^2(0,L)} &\lesssim \limsup_{n\to\infty}\|\varphi_{nx}\|_{L^2(0,L)} \leq \frac{2}{|\omega|}\limsup_{n\to\infty}\|i\beta_n\varphi_{nx}\|_{L^2(0,L)} \\ &\lesssim \lim_{n\to\infty}\|i\beta_n\varphi_{nx} - \phi_{nx}\| + \lim_{n\to\infty}\|\phi_{nx}\|_{L^2(0,L)} = 0,\end{aligned} \tag{2.12}$$

where we have used the facts that

$$\left\|\mathbf{B}^{-\frac{1}{2}}\varphi_{nxx}\right\|_{L^2(0,L)} \lesssim \|\varphi_{nx}\|_{L^2(0,L)}$$

and $|\beta_n| \geq \frac{|\omega|}{2}$ for n large since $\beta_n \to \omega$ $(n \to \infty)$ and $|\omega| \geq \|\mathcal{A}^{-1}\| > 0$.

By (2.10), (2.11) and (2.12), we get $\lim_{n\to\infty}\|\Phi_n\|_{\mathcal{H}} = 0$, which contradicts $\|\Phi_n\|_{\mathcal{H}} = 1$. So (2.5) holds.

We now prove

$$\limsup_{|\beta|\to\infty}\left\|(i\beta I - A)^{-1}\right\| < \infty \tag{2.13}$$

by a contradiction argument again. Suppose that (2.13) is not true. Then there exists a sequence $\{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$ with $|\beta_n| \to \infty$ $(n \to \infty)$, and a sequence

$$\{\Phi_n\}_{n=1}^{\infty} = \left\{\begin{pmatrix}\varphi_n \\ \phi_n\end{pmatrix}\right\}_{n=1}^{\infty} \subset D(\mathcal{A})$$

satisfying (2.6) such that (2.7) holds. Again we also have (2.8), (2.9), (2.10) and (2.12) except for (2.11) since in this case $\{\beta_n\}_{n=1}^{\infty}$ is unbounded.

Since

$$\|i\beta_n\phi_n\|_{L^2(0,L)} \lesssim \left\|i\beta_n\phi_n+\mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)}+\left\|\mathbf{B}^{-1}\mathbf{C}\phi_n\right\|_{L^2(0,L)}+\left\|bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)}$$
$$\lesssim \left\|i\beta_n\phi_n+\mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} + \|\phi_n\|_{L^2(0,L)} + \|\varphi_{nxx}\|_{L^2(0,L)}$$

it follows from $\left\{\|\varphi_{nxx}\|_{L^2(0,L)}\right\}_{k=1}^{\infty}$ and $\left\{\|\phi_n\|_{L^2(0,L)}\right\}_{k=1}^{\infty}$ are bounded sequences (see (2.6)), and (2.9) that

$$\left\{\|i\beta_n\phi_n\|_{L^2(0,L)}\right\}_{n=1}^{\infty} \text{ is a bounded sequence.} \tag{2.14}$$

Similar to (2.11), we get

$$\|\varphi_{nxx}\|_{L^2(0,L)} \lesssim \left\|\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} \lesssim \left\|i\beta_n\phi_n+\mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} + \|i\beta_n\phi_n\|_{L^2(0,L)}.$$

Then we get from (2.9) and (2.14) that

$$\left\{\|\varphi_{nxx}\|_{L^2(0,L)}\right\}_{n=1}^{\infty} \text{ is a bounded sequence,}$$

which, together with $|\beta_n| \to \infty$ as $n \to \infty$ that

$$\lim_{n\to\infty}\left\|\frac{\varphi_{nxx}}{\beta_n}\right\|_{L^2(0,L)} = 0. \tag{2.15}$$

Dividing (2.8) by β_n, we get

$$i\varphi_n - \frac{\phi_n}{\beta_n} \to 0 \text{ in } H^2(0,L)\cap H_0^1(0,L).$$

Then it follows from (2.15) that

$$\limsup_{n\to\infty}\|\varphi_{nxx}\|_{L^2(0,L)} \le \lim_{n\to\infty}\left\|i\varphi_{nxx} - \frac{\phi_{nxx}}{\beta_n}\right\|_{L^2(0,L)} + \lim_{n\to\infty}\left\|\frac{\varphi_{nxx}}{\beta_n}\right\|_{L^2(0,L)} = 0,$$

i.e., (2.11) also holds. Then we get $\lim_{n\to\infty}\|\Phi_n\|_{\mathcal{H}} = 0$, which contradicts $\|\Phi_n\|_{\mathcal{H}} = 1$. So (2.13) holds. The desired result follows from Theorem 2.2, (2.5) and (2.13). □

3 Conclusions

In this article, we have investigated a dissipative Bresse-Timoshenko system without second spectrum. The C_0-semigroup theory are applied to study the well-posedness and exponential stability, which is different from others, where the multiplying method and energy method were used to used to study theexponential stability. This result substantially improves earlier results in the literature.

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