KIRCHHOFF BIHARMONIC SYSTEM WITH CHOQUARD NONLINEARITY AND SINGULAR WEIGHTS

ABSTRACT. The aim of this paper is to find the existence of solutions for the following Kirchhoff type biharmonic system with exponential nonlinearity and singular weights

$$\begin{cases} m\left(\|u\|^2+\|v\|^2\right)\Delta^2 u=\left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_1(x,u,v)}{|x|^\alpha} & \text{in} \quad \Omega; \\ m\left(\|u\|^2+\|v\|^2\right)\Delta^2 v=\left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_2(x,u,v)}{|x|^\alpha} & \text{in} \quad \Omega; \\ u=0, \quad v=0, \quad \nabla u=\mathbf{0}, \quad \nabla v=\mathbf{0} & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^4$ containing the origin with smooth boundary, $\mu\in(0,4)$, $0<\alpha<\frac{\mu}{2}$, $I_\mu(x)=\frac{1}{|x|^{4-\mu}}$, m is a Kirchhoff type function, $\|u\|^2=\int_\Omega|\Delta u|^2dx$, f_i behaves like $e^{\beta_0 s^2}$ when $|s|\to\infty$ for some $\beta_0>0$, and there is C^1 function $F:\mathbb{R}^2\to\mathbb{R}$ such that $\left(\frac{\partial F(x,u,v)}{\partial u},\frac{\partial F(x,u,v)}{\partial v}\right)=(f_1(x,u,v),f_2(x,u,v))$. We establish sufficient conditions for the solutions of the above system by using variational methods with Adams inequality.

1. Introduction

In this paper, we are concerned with the existence of solutions for the following biharmonic Kirchhoff system with exponential nonlinearity and singular weights

$$(1.1)\qquad \begin{cases} m\left(\|u\|^2+\|v\|^2\right)\Delta^2 u=\left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_1(x,u,v)}{|x|^\alpha} & \text{in} \quad \Omega; \\ m\left(\|u\|^2+\|v\|^2\right)\Delta^2 v=\left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_2(x,u,v)}{|x|^\alpha} & \text{in} \quad \Omega; \\ u=0, \quad v=0, \quad \nabla u=\mathbf{0}, \quad \nabla v=\mathbf{0} & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^4$ containing the origin with smooth boundary, $\mu\in(0,4)$, $0<\alpha<\frac{\mu}{2}$, and $\|u\|^2=\int_\Omega|\Delta u|^2dx$. I_μ is defined as $I_\mu(x)=\frac{1}{|x|^{4-\mu}}$. $m:\mathbb{R}^+\to\mathbb{R}^+$ is a Kirchhoff type function. F satisfies suitable growth assumptions and $f_1=\frac{\partial F}{\partial u}$, $f_2=\frac{\partial F}{\partial v}$.

Problems involving biharmonic equations have been studied extensively by many researchers until now. For instance, Rani and Goyal in [27] considered the following biharmonic critical Choquard equation:

$$(1.2)\qquad \begin{cases} \Delta^2 u=\lambda f(x)|u|^{q-2}u+g(x)\left(\int_\Omega\frac{g(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u, & \text{in } \Omega; \\ u=0,\ \nabla u=0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N\geq 5)$ with smooth boundary, $1<q<2$, $0<\alpha<N$, $2^*_\alpha=\frac{2N-\alpha}{N-4}$ and $f,g:\bar{\Omega}\to\mathbb{R}$ are continuous sign-changing weight functions. They proved

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the existence of two nontrivial solutions for the problem (1.2) in a suitable range of λ. Specifically the readers expressing an interest in the above part we refer to [10,16,17,22–24] and the references therein for the existence and multiplicity of solutions for biharmonic equations.

Biharmonic equations involving critical exponential nonlinearities have been also investigated recently. In fact, let Ω be a smooth bounded domain in $\mathbb{R}^n$, we know the classical Sobolev space embedding shows that

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\bar{p}}(\Omega) \quad \text{if } n > p, \text{ where } \bar{p} = \frac{np}{n-p}.$$

As for $n = p$, we say $W_0^{1,n}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \le s < \infty$, however $L^\infty(\Omega)$ does not hold. Later, Pohozaev [25] and Trudinger [30] found the function $\phi(t) = e^{|t|^{\frac{n}{n-1}}} - 1$ such that

$$\sup_{\|u\|_{W_0^{1,n}(\Omega)} \le 1} \int_\Omega \phi(u)dx < \infty.$$

Then Moser in [21] further improved the above result and obtained the following inequality

$$\sup_{\|u\|_{W_0^{1,n}(\Omega)} \le 1} \int_\Omega \exp\left(\beta |u|^{\frac{n}{n-1}}\right) dx < \infty, \quad u \in W_0^{1,n}(\Omega), \quad \text{if and only if } \beta \le \beta_n,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\beta_n = n\omega_n^{\frac{1}{n-1}}$ and ω_n is the surface area for unit ball of $\mathbb{R}^n$. After that, Adams [1] extended this result to higher order Sobolev spaces. That is, let Ω be a bounded domain in $\mathbb{R}^n$ and $n, m \in \mathbb{N}$ satisfying $m < n$, then for all $0 \le \zeta \le \zeta_{n,m}$ and $u \in W_0^{m,\frac{n}{m}}(\Omega)$, it follows that

$$\sup_{\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \le 1} \int_\Omega \exp\left(\zeta |u|^{\frac{n}{n-m}}\right) dx < \infty,$$

where $\zeta_{n,m}$ is sharp and given by

$$\zeta_{n,m} = \begin{cases} \frac{n}{\omega_{n-1}} \left(\frac{\pi^{n/2} 2^m \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)} \right)^{\frac{n}{n-m}} & \text{when } m \text{ is odd,} \\ \frac{n}{\omega_{n-1}} \left(\frac{\pi^{n/2} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right)^{\frac{n}{n-m}} & \text{when m is even.} \end{cases}$$

And the symbol $\nabla^m u$ denotes the m^{th}-order gradient of u and is defined as

$$\nabla^m u = \begin{cases} \nabla \Delta^{(m-1)/2} u; & \text{if } m \text{ is odd,} \\ \Delta^{(m)/2} u; & \text{if } m \text{ is even,} \end{cases}$$

where Δ and ∇ denote the usual Laplacian and gradient operators respectively. With regard to the problem (1.1) in this paper, we consider the case $m = 2$, $n = 4$, and we will use the following inequality [1]:

$$\sup_{u \in H_0^2(\Omega), \|\Delta u\|_{L^2(\Omega)} \le 1} \int_\Omega \exp\left(\beta |u|^2\right) dx < \infty, \quad \text{for all } 0 < \beta \le 32\pi^2, \ \Omega \subset \mathbb{R}^4.$$

Moreover, in the case $n = 4, m = 2$, we say that $f(t)$ has critical exponential growth at infinity if there exists $\beta_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all} \quad \beta > \beta_0; \quad \text{and} \quad \lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = \infty, \quad \text{for all} \quad \beta < \beta_0.$$

At this point, if Ω is a smooth bounded domain in $\mathbb{R}^4$, in [26], Robert and Struwe considered the biharmonic equations involving critical exponential growth

$$\begin{cases} \Delta^2 u_\epsilon = \lambda u_\epsilon e^{32\pi^2 u_\epsilon^2}, & \text{in } \Omega; \\ u_\epsilon = \frac{\partial u_\epsilon}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$. They described the asymptotics of u_ϵ as $\epsilon \to 0$, supposing that $u_\epsilon \to 0$ weakly in a suitable space $H^2_{2,0}(\Omega)$ when $\sup_\Omega u_\epsilon \to \infty$. As for the whole Euclidean space $\mathbb{R}^4$, we refer the reader to [29, 32] for the existence results on biharmonic equations with critical exponential nonlinearities.

Due to the nonlocal term $m\left(\|u\|^2 + \|v\|^2\right)$, Kirchhoff problems were firstly mentioned in 1883 by Kirchhoff, see [18], which the typical equation is known as follows

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = g(x, u),$$

where ρ is the mass density, P_0 is the inital tension, h is the area of cross-section, E is the Young modulus of the material and L is the length of the string. Besides this, some problems concerning the nonlocal term are also applied in various fields, mostly in biological and physical domains. The readers interested in these aspects we refer to the articles [2–4, 11]. Actually new problems involved with Kirchhoff type emerged from the above researches and many authors obtained the existence results of solutions for the Kirchhoff type equations involving critical exponential nonlinearities via variational methods. In relation to Kirchhoff problems, the existence and multiplicity of solutions for elliptic equations inolving critical exponential nonlinearity can be found in the literatures [12,31] and a class of elliptic equations with a small nonhomogeneous term was studied in the articles [13–15] in a bounded domain of $\mathbb{R}^2$. For biharmonic equations we refer to [6,20] in the whole Euclidean space $\mathbb{R}^4$.

Another common nonlocal problem is the Choquard nonlinear term. Later, Lü [19] studied the non-degenerate Choquard equation with Kirchhoff function in $\mathbb{R}^3$ and obatined the ground state solution via the method of Nehari manifold. Meanwhile, Arora et al. in [7] established the existence of solutions for the Kirchhoff-Choquard type problem involving critical exponential nonlinearities by using the Mountain-pass theorem and the boundness of the corresponding Palais-Smale sequence. We recall that the existence solutions for the nonlinear Choquard N-Laplacian equation in a bounded domain of $\mathbb{R}^N$ can be found in [7], and for the polyharmonic problem we refer to [9].

Recently, applications of system involving critical exponential nonlinearity had attracted many researchers to join in this field. In [8], Arora et al. infered the relevant Adams-Moser inequality in $W_0^{m,\frac{n}{m}}(\Omega) \times W_0^{m,\frac{n}{m}}(\Omega)$. And they established the existence solutions for the following Kirchhoff system by using the method of Nehari manifold

$$\begin{cases} -m\left(\int_\Omega |\nabla u|^n dx\right)\Delta_n u = \left(\int_\Omega \frac{F(y,u,v)}{|x-y|^\mu}dy\right) f_1(x,u,v), u>0 & \text{in} \quad \Omega; \\ -m\left(\int_\Omega |\nabla v|^n dx\right)\Delta_n v = \left(\int_\Omega \frac{F(y,u,v)}{|x-y|^\mu}dy\right) f_2(x,u,v), v>0 & \text{in} \quad \Omega; \\ u=0, \quad v=0, & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain, $n \geq 2$, $0<\mu<n$, $m:\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $\Delta_n u := \mathrm{div}(|\nabla u|^{n-2}\nabla u)$, F satisfies suitable growth assumptions and $f_1 = \frac{\partial F}{\partial u}$, $f_1 = \frac{\partial F}{\partial v}$. Furthermore, in [14], the authors extended to $k(\in \mathbb{N})$ equations for singular Trudinger-Moser growth in Ω which is a smooth bounded domain in $\mathbb{R}^2$ containing the origin with smooth boundary. And they obtained the multiplicity of solutions of the following elliptic Kirchhoff system

$$\begin{cases} -m\Big(\sum_{i=1}^k(\int_\Omega |\nabla u_j|^2 dx)\Big)\Delta u = \frac{f_i(x,u_1,\ldots,u_k)}{|x|^\beta} + \varepsilon h_i(x) & \text{in} \quad \Omega; i=1\ldots,k, \\ u_1=u_2=\ldots=u_k=0, & \text{on} \quad \partial\Omega, \end{cases}$$

where $\beta \in [0,2)$, m is a continuous function, f_i behaves like $e^{\alpha_0 s^2}$ when $|s| \to \infty$ for some $\alpha_0>0$, and there is C^1 function $F:\Omega\times\mathbb{R}^k \to \mathbb{R}$ such that $\left(\frac{\partial F}{\partial u_1},\ldots,\frac{\partial F}{\partial u_k}\right) = (f_1,\ldots,f_k)$, $h_i \in ((H_0^1(\Omega))^*, \|\cdot\|_*)$, ε is a small positive parameter. We also refer to [31] for Kirchhoff type elliptic system involving critical exponential growth.

Motivated by the above results, in this paper, we consider the existence of solutions for the Kirchhoff type biharmonic system (1.1). Then the Kirchhoff term is a difficulty which implies that the equation in problem (1.1) is no longer a pointwise identity. It means that we need to overcome the lack of compactness due to Choquard nonlinearity involving critical exponential growth as well as the Kirchhoff term. Especially important, we firstly need to obtain the improved Adams-Trudinger inequality involving two variables.

In order to treat the system problem (1.1), now we give some definitions as follows. We introduce $H_0^2(\Omega)$ with the scalar product

$$\langle u,v\rangle = \int_\Omega \Delta u \Delta v dx$$

for each $u,v \in H_0^2(\Omega)$. Then we denote

$$H_0^2(\Omega,\mathbb{R}^2) := H_0^2(\Omega)\times H_0^2(\Omega),$$

endowed with the scalar product

$$\langle U,V\rangle = \int_\Omega \Delta u_1 \Delta v_1 dx + \int_\Omega \Delta u_2 \Delta v_2 dx,$$

where $U=(u_1,u_2)$ and $V=(v_1,v_2)$, to which corresponds the norm $\|U\| = \langle U,U\rangle^{1/2} = (\|u_1\|^2+\|u_2\|^2)^{1/2}$, then $H_0^2(\Omega,\mathbb{R}^2)$ is well defined and also a Hilbert space. That is, for any $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$,

$$\|(u,v)\| = (\|u_1\|^2_{H_0^2(\Omega)} + \|u_2\|^2_{H_0^2(\Omega)})^{\frac{1}{2}},$$

where $\|u\|_{H_0^2(\Omega)} = \left(\int_\Omega |\Delta u|^2 dx\right)^{1/2}$. Moreover, for all $1 \leq p < \infty$ we define $L^p(\Omega, \mathbb{R}^2)$ as

$$L^p(\Omega, \mathbb{R}^2) := L^p(\Omega) \times L^p(\Omega),$$

where $L^p(\Omega)$ is the standard L^p-space, we can know $L^p(\Omega, \mathbb{R}^2)$ is well defined and for any $(u, v) \in L^p(\Omega, \mathbb{R}^2)$, we define

$$\|(u, v)\|_p = \left(\int_\Omega |u|^p dx + \int_\Omega |v|^p dx\right)^{\frac{1}{p}}, \quad \text{and } |(u, v)| = \left(|u|^2 + |v|^2\right)^{\frac{1}{2}}.$$

For any $1 \leq p < \infty$, by Sobolev embedding theorem, we can know that the embedding $H_0^2(\Omega, \mathbb{R}^2) \hookrightarrow L^p(\Omega, \mathbb{R}^2)$ is compact.

Now, let us introduce the precise assumptions under what our problem is studied. For this, we define $M(t) = \int_0^t m(s)ds$, the primitive of m so that $M(0) = 0$. The hypotheses on Kirchhoff function $m : \mathbb{R}^+ \to \mathbb{R}^+$ are the following:

- (m_1) There exists $m_0 > 0$ such that $m(t) \geqslant m_0$ for $t \geqslant 0$ and $m(t)$ is non-decreasing on $[0, +\infty)$;
- (m_2) There exists $\theta > 1$ such that $\frac{m(t)}{t^{\theta-1}}$ is non-increasing for all $t \in (0, +\infty)$.

Remark 1.1. (1) By $m(t)$ is nondecreasing for $t \geq 0$, we have $\int_{t_1}^{t_1+t_2} m(s)ds \geq \int_0^{t_2} m(s)ds$ for all $t_1, t_2 \geq 0$, then it holds that $\int_0^{t_1} m(s)ds + \int_{t_1}^{t_1+t_2} m(s)ds \geq \int_0^{t_1} m(s)ds + \int_0^{t_2} m(s)ds$, i.e. $M(t_1 + t_2) \geq M(t_1) + M(t_2)$.
(2) From (m_2), we can see that

$$\theta M(t) - m(t)t \geq 0, \quad \text{for all } t \geq 0. \tag{1.3}$$

Indeed, for any $0 < t_1 < t_2$, we deduce that

$$\begin{aligned}
\theta M(t_1) - m(t_1)t_1 &= \theta M(t_2) - \theta \int_{t_1}^{t_2} m(t)dt - \frac{m(t_1)}{t_1^{\theta-1}} t_1^\theta \\
&\leqslant \theta M(t_2) - \frac{m(t_2)}{t_2^{\theta-1}}(t_2^\theta - t_1^\theta) - \frac{m(t_2)}{t_2^{\theta-1}} t_1^\theta \\
&= \theta M(t_2) - m(t_2)t_2.
\end{aligned}$$

Therefore, $\theta M(t) - m(t)t$ is nondecreasing for $t \geq 0$. In particular,

$$\theta M(t) - m(t)t \geq 0 \quad \text{for all } t \geq 0.$$

(3) From (m_2) we obtain

$$M(t) \geq M(1)t^\theta \quad \text{for} \quad 0 \leq t \leq 1; \quad M(t) \leq M(1)t^\theta \quad \text{for} \quad t \geq 1.$$

Then there exist $C_1, C_2 > 0$ such that

$$M(t) \leq C_1 t^\theta + C_2, \quad \text{for all } t \geq 0. \tag{1.4}$$

Moreover, by (m_2), for $t_1, t_2 > 0$ one has

$$\begin{aligned}
M(t_1 + t_2) &= \int_0^{t_1+t_2} m(s)ds = \int_0^{t_1} m(s)ds + \int_{t_1}^{t_1+t_2} m(s)ds \\
&= M(t_1) + \int_0^{t_2} m(t_1 + s)ds \leq M(t_1) + \frac{m(t_1)}{t_1^{\theta-1}} \int_0^{t_2} (t_1 + s)^{\theta-1} ds
\end{aligned}$$

$$(1.5) \qquad = M(t_1) + \frac{t_1 m(t_1)}{\theta}\left[\left(1+\frac{t_2}{t_1}\right)^{\theta} - 1\right].$$

Remark 1.2. A typical example of a function m satisfying the conditions $(m_1) - (m_2)$ is given by $m(t) = m_0 + at^{\theta-1}$ with $\theta > 1$ and $a \geq 0$.

Throughout this paper, we assume f_i: $\Omega \times \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ are continuous functions satisfying the following conditions:

(f_1) $f_i(x,t,s) = 0$ when either $t \leq 0$ or $s \leq 0$ and $f_i(x,t,s) > 0$ when $t, s > 0$ for all $x \in \Omega$ and $i = 1, 2$.

(f_2) For $i = 1, 2$, f_i has critical exponential growth at infinity, that is, there exists $\beta_0 > 0$ such that

$$\lim_{|(t,s)|\to+\infty} \frac{|f_i(x,t,s)|}{e^{\beta|(t,s)|^2}} = \begin{cases} 0, & \forall \beta > \beta_0; \\ +\infty, & \forall \beta < \beta_0. \end{cases}$$

(f_3) There exists $l > \theta$ such that the maps $u \mapsto \frac{f_1(x,t,s)}{|t|^l}$, $v \mapsto \frac{f_2(x,t,s)}{|s|^l}$ are increasing functions of t and s respectively.

(f_4) There exist $q \in (0,1], a_0, b_0, M_0 > 0$ such that $0 < |t|^q F(x,t,s) \leq M_0 f_1(x,t,s)$ for all $|t| \geqslant a_0$ and $0 < |s|^q F(x,t,s) \leq M_0 f_2(x,t,s)$ for all $|s| \geqslant b_0$ uniformly in $x \in \Omega$.

(f_5) There exists $r > 0$ such that $\lim_{|(t,s)|\to(0,0)} \frac{f_i(x,t,s)}{|t|^r+|s|^r} = 0$ holds for $i = 1, 2$.

(f_6) There exists κ such that

$$\liminf_{t,s\to\infty} \frac{(|t|+|s|)F(x,t,s)}{e^{\beta_0(|t|^2+|s|^2)}} > \kappa > \left[\frac{4\pi^2(4+\mu-2\alpha)e^{\frac{21(4+\mu-2\alpha)-8}{8}} m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)}{\beta_0^2 C_\mu d^{4+\mu-2\alpha}}\right]^{\frac{1}{2}}$$

uniformly in $x \in \Omega$, where $C_\mu = \frac{24\pi^4}{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}$, and $2d$ is the radius of the largest open ball containing the origin contained in Ω.

Remark 1.3. By (f_3), for any $1 < p \leq l$, it holds that $tf_1(x,t,s) - pF(x,t,s) > 0$, $sf_2(x,t,s) - pF(x,t,s) > 0$, for all $(x,t,s) \in \Omega \times \mathbb{R}^2$.

In fact, for any $1 < p \leq l$, from (f_3) we have $\frac{f_1(x,t,s)}{t^{p-1}}$ is increasing functions of $t > 0$ uniformly of $s > 0$. Let $s > 0$ and $0 < t_1 < t_2$ be fixed, then it holds that

$$t_1 f_1(x,t_1,s) - pF(x,t_1,s) < \frac{f_1(x,t_2,s)}{t_2^{p-1}} t_1^p - pF(x,t_2,s) + p\int_{t_1}^{t_2} f_1(x,t,s)dt.$$

On the other hand,

$$p\int_{t_1}^{t_2} f_1(x,t,s)dt < p\frac{f_1(x,t_2,s)}{t_2^{p-1}} \int_{t_1}^{t_2} t^{p-1}dt = \frac{f_1(x,t_2,s)}{t_2^{p-1}}(t_2^p - t_1^p).$$

From the above inequalities, we derive that

$$t_1 f_1(x,t_1,s) - pF(x,t_1,s) < t_2 f_1(x,t_2,s) - pF(x,t_2,s).$$

Then we obtain $tf_1(x,t,s) - pF(x,t,s) > 0$ for all $(x,t,s) \in \Omega \times \mathbb{R}^2$. Similarly, we also obtain $sf_2(x,t,s) - pF(x,t,s) > 0$ for all $(x,t,s) \in \Omega \times \mathbb{R}^2$.

Remark 1.4. Here in [8], the condition for the estimate on energy level is as follows

$$\lim_{t,s\to\infty}\frac{(f_1(x,t,s)t+f_2(x,t,s)s)\,F(x,t,s)}{\exp(q(|t|^{\frac{n}{n-1}}+|s|^{\frac{n}{n-1}}))}=\infty\quad\text{uniformly}\quad\text{in}\quad x\in\Omega,\tag{1.6}$$

for some $q>2$. For a single equation, we know if the condition for the estimation method is

$$\liminf_{t\to\infty}\frac{f(x,t)}{e^{\beta_0 t^2}}>K>0,$$

then we obtain

$$\liminf_{t\to\infty}\frac{tF(x,t)}{e^{\beta_0 t^2}}\geq\liminf_{t\to\infty}\frac{\int_0^t sf(x,s)ds}{e^{\beta_0 t^2}}=\liminf_{t\to\infty}\frac{f(x,t)}{2\beta_0 e^{\beta_0 t^2}}.$$

It means that we can derive $tF(x,t)\geq\frac{K-\varepsilon}{2\beta_0}e^{\beta_0 t^2}$ when $f(x,t)\geq(K-\epsilon)e^{\beta_0 t^2}$ for t large enough and $\varepsilon>0$ small enough. In view of the above results, we found that

$$\liminf_{t\to\infty}\frac{tf(x,t)F(x,t)}{e^{2\beta_0 t^2}}>\frac{K^2}{2\beta_0}.$$

Then according to the above analysis, for the system (1.1) we know (f_6) is weaker than (1.6).

Theorem 1.5. *Suppose that m satisfies $(m_1)-(m_2)$, f satisfies $(f_1)-(f_6)$. Then (1.1) has a ground state solution $(u,v)\in\mathcal{N}$ such that $\Phi(u,v)=b:=\inf_{\mathcal{N}}\Phi$, where*

$$\mathcal{N}=\{(u,v)\in H_0^2(\Omega,\mathbb{R}^2)\backslash\{(0,0)\}:\langle\Phi'(u,v),(u,v)\rangle=0\}.\tag{1.7}$$

The remainder of this paper is organized as follows. We prove the Adams inequality and the version of Lion's Lemma in the Sobolev spaces, namely $H_0^2(\Omega,\mathbb{R}^2)$ in Section 2. In Section 3, we give the energy functional and prove that the system (1.1) satisfies the geometric conditions of the Mountain-pass theorem and the corresponding Palais-Smale sequence is bounded. In this article, we denote that C, C_i, c_i are some positive constants.

2. Preliminaries and auxiliary results

In this section, we introduce some famous inequalities as follows, and inspired by these we conclude some similar forms of inequalities and give some preliminaries.

Lemma 2.1. *([1]) For each $u\in H_0^2(\Omega)$, $\Omega\subset\mathbb{R}^4$ is a bounded domain, then for any $\beta>0$,*

$$\int_\Omega\exp\left(\beta|u|^2\right)dx<\infty.$$

Moreover, we have

$$\sup_{\|u\|\leqslant 1}\int_\Omega\exp\left(\beta|u|^2\right)dx<\infty,\quad\text{provided}\quad\beta\leq 32\pi^2.$$

Lemma 2.2. *For each $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$, $\Omega\subset\mathbb{R}^4$ is a bounded domain, then for any $\beta>0$,*

$$\int_\Omega\exp\left(\beta\left(|u|^2+|v|^2\right)\right)dx<\infty.$$

Moreover, we have

$$\sup_{\|(u,v)\|=1}\int_\Omega \exp\left(\beta\left(|u|^2+|v|^2\right)\right)dx<\infty, \quad \text{provided} \quad \beta\le 32\pi^2.$$

Proof. From Lemma 2.1 and Young's inequality, for each $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$, and for any $\beta>0$, we obtain

$$\int_\Omega e^{\beta\left(|u|^2+|v|^2\right)}dx=\int_\Omega e^{\beta|u|^2}e^{\beta|v|^2}dx\le\int_\Omega e^{2\beta|u|^2}dx+\int_\Omega e^{2\beta|v|^2}dx<\infty.$$

Now for any $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$ satisfying $\|(u,v)\|=1$, then $\|u\|^2,\|v\|^2\leqslant 1$. Let $r_1=\|u\|^2, r_2=\|v\|^2$, then we know $r_1+r_2=1$. Hence by using Hölder inequality and Lemma 2.1 we obtain

$$\int_\Omega e^{\beta\left(|u|^2+|v|^2\right)}dx\leqslant\left(\int_\Omega e^{\beta|u|^2/r_1}dx\right)^{r_1}\left(\int_\Omega e^{\beta|v|^2/r_2}dx\right)^{r_2}<C,$$

thus the proof is completed. □

Lemma 2.3. *Let $\{(u_n,v_n)\}$ be a sequence in $H_0^2(\Omega,\mathbb{R}^2)$ satisfying $\|(u_n,v_n)\|=1$ such that $(u_n,v_n)\rightharpoonup(u,v)\neq 0$ weakly in $H_0^2(\Omega,\mathbb{R}^2)$. Then for any $0<\beta<\frac{32\pi^2}{1-\|(u,v)\|^2}$, we have*

$$\sup_{n\in\mathbb{N}}\int_\Omega \exp\left(\beta\left(|u_n|^2+|v_n|^2\right)\right)dx<\infty.$$

Proof. For $(u_n,v_n)\rightharpoonup(u,v)\neq 0$ in $H_0^2(\Omega,\mathbb{R}^2)$ satisfying $\|(u_n,v_n)\|=1$, it is easy to see that

(2.1)
$$\lim_{n\to\infty}\|(u_n-u,v_n-v)\|^2=\lim_{n\to\infty}\left(1-2\langle u_n,u\rangle-2\langle v_n,v\rangle+\|(u,v)\|^2\right)=1-\|(u,v)\|^2<\frac{32\pi^2}{\beta},$$

and

$$u_n^2\le(u_n-u)^2+\epsilon u_n^2+C_\epsilon u^2, \tag{2.2}$$

for ϵ small enough, where C_ϵ is a positive constant related to ϵ. Then by using (2.2) we have

$$\int_\Omega e^{\beta\left(|u_n|^2+|v_n|^2\right)}dx\le\int_\Omega e^{\beta\left((u_n-u)^2+(v_n-v)^2\right)}e^{\beta\epsilon\left(u_n^2+v_n^2\right)}e^{\beta C_\epsilon\left(u^2+v^2\right)}dx.$$

Now we take $r_1,r_2,r_3>1$ such that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=1$, and by using Hölder inequality we obtain

$$\int_\Omega e^{\beta\left(|u_n|^2+|v_n|^2\right)}dx\le\left(\int_\Omega e^{\beta r_1\left((u_n-u)^2+(v_n-v)^2\right)}dx\right)^{1/r_1}\left(\int_\Omega e^{\epsilon\beta r_2\left(u_n^2+v_n^2\right)}dx\right)^{1/r_2}$$
$$\times\left(\int_\Omega e^{\beta C_\epsilon r_3\left(u^2+v^2\right)}dx\right)^{1/r_3}.$$

Using Lemma 2.2, then we can choose ϵ small enough such that

$$\int_\Omega e^{\epsilon\beta r_2\left(u_n^2+v_n^2\right)}dx\le C, \quad \int_\Omega e^{\beta C_\epsilon r_3\left(u^2+v^2\right)}dx\le C.$$

Moreover, we choose $r_1 > 1$ close to 1 such that $\beta r_1 \|(u_n-u, v_n-v)\|^2 < 32\pi^2$, then according to (2.1) and Lemma 2.2 we obtain

$$\begin{aligned}&\int_\Omega \exp\left(\beta r_1\left((u_n-u)^2+(v_n-v)^2\right)\right)dx\\&=\int_\Omega \exp\left[\beta r_1\left(\left(\frac{u_n-u}{\|(u_n-u,v_n-v)\|}\right)^2+\left(\frac{v_n-v}{\|(u_n-u,v_n-v)\|}\right)^2\right)\|(u_n-u,v_n-v)\|^2\right]dx\\&< C,\end{aligned}$$

then we obtain $\int_\Omega \exp\left(\beta\left(|u_n|^2+|v_n|^2\right)\right)dx$ is bounded, and this lemma is proved. □

Lemma 2.4. *For any $(u, v) \in H_0^2(\Omega, \mathbb{R}^2)$, if $\beta > 0, s > 0$ satisfying $\|(u, v)\| \leqslant M$ such that $\beta M^2 < 32\pi^2$, then there exists $C = C(\beta, M, s) > 0$ such that*

$$\int_\Omega |(u,v)|^s e^{\beta|(u,v)|^2}dx \leqslant C\|(u,v)\|^s.$$

Proof. For each $(u, v) \in H_0^2(\Omega, \mathbb{R}^2)$ satisfying $\|(u, v)\| \leqslant M$, we choose $r > 1$ close to 1 such that $r\beta M^2 \leqslant 32\pi^2$ and $sq > 1$, where $q = \frac{s}{s-1}$. By using Hölder inequality and Lemma 2.2, we obtain

$$\int_\Omega |(u,v)|^s e^{\alpha|(u,v)|^2}dx \leqslant \left(\int_\Omega e^{r\beta|(u,v)|^2}dx\right)^{1/r}\|(u,v)\|_{qs}^s \leqslant C\|(u,v)\|_{qs}^s.$$

Since $sq > 1$, using the continuous embedding $H_0^2(\Omega, \mathbb{R}^2) \hookrightarrow L^{qs}(\Omega, \mathbb{R}^2)$, we finish the proof. □

Proposition 2.5. *([28]) Let $t, r > 1$ and $0 < \mu < N$ with $m, n \geqslant 0$, $\frac{1}{t} + \frac{\mu+m+n}{N} + \frac{1}{r} = 2$, $\mu + m + n \leqslant N$. Then there exists a constant $C(m, n, t, \mu, r) > 0$ which is dependent of $f \in L^t(\mathbb{R}^N), h \in L^r(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\mu|y|^m|x|^n}dxdy \leqslant C(m,n,t,\mu,r)\|f\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}.$$

3. The variational framework and the minimax estimate

We now consider the energy functional $\Phi(u, v)$ given by

$$\Phi(u,v) = \frac{1}{2}M\left(\|(u,v)\|^2\right) - \frac{1}{2}\int_\Omega\left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{F(x,u,v)}{|x|^\alpha}dx. \tag{3.1}$$

Lemma 3.1. *Assume that (f_2) and (f_3) hold, then we have that $\Phi(u, v)$ is well defined on $H_0^2(\Omega, \mathbb{R}^2)$. Moreover,*

$$\begin{aligned}\langle \Phi'(u,v),(\phi,\psi)\rangle &= m\left(\|(u,v)\|^2\right)\left(\int_\Omega \Delta u\Delta\phi dx+\int_\Omega \Delta v\Delta\psi dx\right)\\&\quad-\int_\Omega\left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{(f_1(x,u,v)\phi+f_2(x,u,v)\psi}{|x|^\alpha}dx,\end{aligned} \tag{3.2}$$

for any $(\phi, \psi) \in H_0^2(\Omega, \mathbb{R}^2)$.

Proof. For $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$, we know f_i has critical exponential growth, then by (f_2) we choose $\beta>\beta_0$ and there exists $C_1>0$ such that

$$|f_i(x,u,v)|\leqslant C_1 e^{\beta|(u,v)|^2},\qquad \text{for all } (x,u,v)\in\Omega\times\mathbb{R}^2, i=1,2. \tag{3.3}$$

Moreover, by (f_3), given $\epsilon>0$ there exist $C_2,C_3>0$ and $\delta>0$ such that

$$f_1(x,u,v)\leq C_2|u|,\quad f_2(x,u,,v)\leq C_3|v|\quad \text{always that } |u|,|v|\leq\delta. \tag{3.4}$$

Then combing (3.3) and (3.4), we have

$$|F(x,u,v)|\leqslant C_4|(u,v)|e^{\beta|(u,v)|^2}+C_5|(u,v)|^2, \tag{3.5}$$

for all $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$. Now using (3.5) and Proposition 2.5 with $N=4$, $t=r=\frac{8}{4+\mu-2\alpha}$ and $m=n=\alpha$, we obtain

$$\begin{aligned}
&\int_\Omega\left[I_\mu*\frac{F(x,u,v)}{|x|^\alpha}\right]\frac{F(x,u,v)}{|x|^\alpha}dx\\
&\leqslant C(\mu,\alpha)\|F(x,u,v)\|^2_{\frac{8}{4+\mu-2\alpha}}\\
&\leqslant C(\mu,\alpha)\left[\int_\Omega\left(|(u,v)|^2+|(u,v)|\exp\left(\beta|(u,v)|^2\right)\right)^{\frac{8}{4+\mu-2\alpha}}dx\right]^{\frac{4+\mu-2\alpha}{4}}\\
&\leqslant C(\mu,\alpha)\left[\int_\Omega|(u,v)|^{\frac{16}{4+\mu-2\alpha}}dx+\int_\Omega|(u,v)|^{\frac{16}{4+\mu-2\alpha}}\exp\left(\frac{8\beta|(u,v)|^2}{4+\mu-2\alpha}\right)dx\right]^{\frac{4+\mu-2\alpha}{4}}.
\end{aligned}$$

According to Lemma 2.4 and the continuous embedding $H_0^2(\Omega,\mathbb{R}^2)\hookrightarrow L^s(\Omega,\mathbb{R}^2)$ with $s\geq 1$, we know $\Phi(u,v)$ is well defined, and we can see $\Phi\in C^1(H_0^2(\Omega,\mathbb{R}^2),\mathbb{R})$. □

From Lemma 3.1, we have that critical points of the functional Φ are precisely weak solutions of problem (1.1). Then we will verify that the functional Φ satisfies the conditions of Mountain-pass theorem.

Lemma 3.2. *Under the assumptions* (m_1), (m_2) *and* (f_1).

(i) there exists $R_0,\Upsilon>0$ *such that* $\Phi(u,v)\geqslant\Upsilon$ *for any* $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$ *satisfying* $\|(u,v)\|=R_0$.

(ii) there exists a $(\widetilde{u},\widetilde{v})\in H_0^2(\Omega,\mathbb{R}^2)$ *with* $\|(\widetilde{u},\widetilde{v})\|>R_0$ *such that* $\Phi(\widetilde{u},\widetilde{v})<0$.

Proof. Let $(u,v)\in H_0^2(\Omega,\mathbb{R}^2)$ such that $\|(u,v)\|=R_0$. Similarly, taking Proposition 2.5 with $N=4$, $t=r=\frac{8}{4+\mu-2\alpha}$ and $m=n=\alpha$, by (3.5) and Hölder inequality we obtain

$$\begin{aligned}
&\int_\Omega\left[I_\mu*\frac{F(x,u,v)}{|x|^\alpha}\right]\frac{F(x,u,v)}{|x|^\alpha}dx\\
&\leqslant C(\mu,\alpha)\|F(x,u,v)\|^2_{\frac{8}{4+\mu-2\alpha}}\\
&\leqslant C(\mu,\alpha)\left[\int_\Omega|(u,v)|^{\frac{16}{4+\mu-2\alpha}}dx+\int_\Omega|(u,v)|^{\frac{16}{4+\mu-2\alpha}}\exp\left(\frac{8\beta|(u,v)|^2}{4+\mu-2\alpha}\right)dx\right]^{\frac{4+\mu-2\alpha}{4}}\\
&\leqslant C(\mu,\alpha)\left\{\int_\Omega|(u,v)|^{\frac{16}{4+\mu-2\alpha}}dx+\left[\int_\Omega|(u,v)|^{\frac{32}{4+\mu-2\alpha}}dx\right]^{\frac{1}{2}}\right.
\end{aligned}$$

$$\times \left[\int_\Omega \exp\left(\frac{16\beta\|(u,v)\|^2}{4+\mu-2\alpha} \left(\frac{|(u,v)|}{\|(u,v)\|} \right)^2 \right) dx \right]^{\frac{1}{2}} \Bigg\}^{\frac{4+\mu-2\alpha}{4}}.$$

Now we choose suitable $R_0 > 0$ such that $\frac{16\beta R_0^2}{4+\mu-2\alpha} \leqslant 32\pi^2$, then by using Sobolev imbedding and Lemma 2.2 we have

$$\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha} \right] \frac{F(x,u,v)}{|x|^\alpha} dx \leqslant C(\mu,\alpha) \left(\|(u,v)\|^{\frac{16}{4+\mu-2\alpha}} + \|(u,v)\|^{\frac{16}{4+\mu-2\alpha}} \right)^{\frac{4+\mu-2\alpha}{4}}$$
$$\leqslant C(\mu,\alpha)\|(u,v)\|^4.$$

Then for any $\|(u,v)\| = R_0 < \sqrt{\frac{2\pi^2(4+\mu-2\alpha)}{\beta}}$, by (m_1) we know

$$\Phi(u,v) \geqslant \frac{m_0}{2}\|(u,v)\|^2 - C(\mu,\alpha)\|(u,v)\|^4.$$

So we choose $\|(u,v)\| = R_0$ small enough so that $\Phi(u,v) \geqslant \Upsilon$ for some $\Upsilon > 0$ and hence (i) follows.

Now we choose $q \in \mathbb{R}$ such that $\theta < q < l$ in Remark 1.3, then we obtain $tf_1(x,t,s) - qF(x,t,s) > 0$ and $sf_2(x,t,s) - qF(x,t,s) > 0$ for all $(x,t,s) \in \Omega \times \mathbb{R}^2$. So we obtain

$$F(x,u,v) \geq c_1|u|^q - c_2, \quad F(x,u,v) \geq c_3|v|^q - c_4,$$

thus we conclude that

$$F(x,u,v) \geq c_1|u|^q + c_3|v|^q - c_5, \quad \text{for all } (x,u,v) \in \Omega \times \mathbb{R}^2. \tag{3.6}$$

Then using (3.6) it follows that

$$\int_\Omega \left[I_\mu * \frac{F(x,tu,tv)}{|x|^\alpha} \right] \frac{F(x,tu,tv)}{|x|^\alpha} dx \geqslant \int_\Omega \int_\Omega \frac{(c_1|tu|^q + c_3|tv|^q - c_4)^2}{|x|^\alpha |y|^\alpha |x-y|^\mu} dxdy$$
$$\geqslant c_6 t^{2q} - c_7 t^q + c_8.$$

By using (1.4) we have $M(t) \leq C_1 t^\theta + C_2$, then using (3.1) we obtain

$$\Phi(tu,tv) \leqslant c_9 t^{2\theta} - c_{10} t^{2q} + c_{11} t^q - c_{12}.$$

Since $q > \theta$ we obtain $\Phi(tu,tv) \to -\infty$ as $t \to \infty$. Then there exists $(\widetilde{u},\widetilde{v}) = (t_0 u, t_0 v) \in H_0^2(\Omega,\mathbb{R}^2)$ with t_0 large enough such that $\|(\widetilde{u},\widetilde{v})\| > R_0$ and $\Phi(\widetilde{u},\widetilde{v}) < 0$, hence (ii) holds. □

Then according to Lemma 3.2, we know $\Phi(u,v)$ satisfies the geometric conditions of the Mountain-pass theorem, let

$$c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) > 0 \tag{3.7}$$

be the minimax level of Φ, where

$$\Gamma = \{\gamma \in C([0,1], H_0^2(\Omega,\mathbb{R}^2)) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Then there exists a Palais-Smale sequence $\{(u_n,v_n)\} \subset H_0^2(\Omega,\mathbb{R}^2)$ satisfying

$$\Phi(u_n,v_n) \to c^*, \quad \Phi'(u_n,v_n) \to 0, \tag{3.8}$$

as $n \to \infty$.

Lemma 3.3. *Assume that* (f_3) *and* (m_1) *hold, then every Palais-Smale sequence of* Φ *is bounded in* $H_0^2(\Omega, \mathbb{R}^2)$.

Proof. Let $\{(u_n, v_n)\}$ be a Palais-Smale sequence of Φ for $c^* \in \mathbb{R}$ in $H_0^2(\Omega, \mathbb{R}^2)$, then it follows that

$$\Phi(u_n, v_n) \to c^*, \quad \Phi'(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore we have

$$\Phi(u_n, v_n) = \frac{1}{2} M\left(\|(u_n, v_n)\|^2\right) - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha}\right] \frac{F(x, u_n, v_n)}{|x|^\alpha} dx = c^* + \delta_n, \tag{3.9}$$

where $\delta_n \to 0$ as $n \to \infty$, and

$$\begin{aligned} \langle \Phi'(u_n, v_n), (\phi, \psi)\rangle = m\left(\|(u_n, v_n)\|^2\right) \int_\Omega \left(\Delta u_n \Delta\phi + \Delta v_n \Delta\psi\right) dx \\ - \int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha}\right] \frac{f_1(x, u_n, v_n)\phi + f_2(x, u_n, v_n)\psi}{|x|^\alpha} dx \\ \leqslant \epsilon_n \|(\phi, \psi)\|, \end{aligned} \tag{3.10}$$

for all $(\phi, \psi) \in H_0^2(\Omega, \mathbb{R}^2)$. By Remark 1.3, we obtain that

$$lF(x,t,s) \leqslant u f_1(x,t,s), \text{ and } lF(x,t,s) \leqslant v f_2(x,t,s) \text{ for all } (x,t,s) \in \Omega \times \mathbb{R}^2. \tag{3.11}$$

Then by (1.3), (3.9), (3.10), (3.11) and (m_1), for large n we have

$$\begin{aligned} & c^* + \epsilon_n \|(u_n, v_n)\| \geq \Phi(u_n, v_n) - \frac{1}{4l}\langle \Phi'(u_n, v_n), (u_n, v_n)\rangle \\ & = \frac{1}{2} M\left(\|(u_n, v_n)\|^2\right) - \frac{1}{4l} m\left(\|(u_n, v_n)\|^2\right)\|(u_n, v_n)\|^2 - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha}\right]\frac{F(x, u_n, v_n)}{|x|^\alpha} dx \\ & \quad + \frac{1}{4l}\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha}\right] \frac{f_1(x, u_n, v_n)u_n + f_2(x, u_n, v_n)v_n}{|x|^\alpha} dx \\ & \geqslant \frac{1}{2} M\left(\|(u_n, v_n)\|^2\right) - \frac{1}{4l} m\left(\|(u_n, v_n)\|^2\right)\|(u_n, v_n)\|^2 \\ & \quad + \frac{1}{4l}\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha}\right] \frac{f_1(x, u_n, v_n)u_n + f_2(x, u_n, v_n)v_n - 2lF(x, u_n, v_n)}{|x|^\alpha} dx \\ & \geqslant \frac{1}{2\theta}\left[\theta M\left(\|(u_n, v_n)\|^2\right) - m\left(\|(u_n, v_n)\|^2\right)\|(u_n, v_n)\|^2\right] \\ & \quad + \left(\frac{1}{2\theta} - \frac{1}{4l}\right) m\left(\|(u_n, v_n)\|^2\right)\|(u_n, v_n)\|^2 \\ & \geqslant \left(\frac{1}{2\theta} - \frac{1}{4l}\right) m_0 \|(u_n, v_n)\|^2, \end{aligned}$$

thus we know $\{(u_n, v_n)\}$ is bounded in $H_0^2(\Omega, \mathbb{R}^2)$ with $l > \theta$. □

Now we give a precise estimation about the Mountain pass level c^* defined by (3.7). Inspired by [**?**] and [32], we give the definite Adams functions $\widetilde{\phi}_n(x)$ supported in $B_{2d}(0) \subset \Omega$

as follows.

$$\widetilde{\phi}_n(x) = \begin{cases} \sqrt{\frac{\ln n}{8\pi^2}} - \frac{n^2}{\delta^2\sqrt{32\pi^2\ln n}}|x|^2 + \frac{1}{\sqrt{32\pi^2\ln n}}, & \text{for } |x| \leqslant \frac{d}{n}; \\ \frac{\ln(d/|x|)}{\sqrt{8\pi^2\ln n}}, & \text{for } \frac{d}{n} < |x| \leqslant d; \\ \eta_n(x), & \text{for } d < |x| \leq 2d; \\ 0, & \text{for } |x| > 2d, \end{cases} \tag{3.12}$$

where $\eta_n(x) = -\frac{1}{2\sqrt{2\pi^2\ln n}d^3}(|x|-d)^3 + \frac{1}{\sqrt{2\pi^2\ln n}d^2}(|x|-d)^2 - \frac{1}{2\sqrt{2\pi^2\ln n}d}(|x|-d)$ and d was given in (f_6). Moreover, $\eta_n(x)$ is a radial function on annulus $B_{2d}\setminus B_d$ satisfying the boundary condition

$$\eta_n(x)|_{\partial B_d} = 0, \quad \frac{\partial\eta_n(x)}{\partial\nu}|_{\partial B_d} = -\frac{4}{d\sqrt{128\pi^2\ln n}},$$

and

$$\eta_n(x)|_{\partial B_{2d}} = 0, \quad \frac{\partial\eta_n(x)}{\partial\nu}|_{\partial B_{2d}} = 0.$$

Then straightforward calculations show that

$$\|\widetilde{\phi}_n\|^2 = 1 + \frac{21}{8\ln n}. \tag{3.13}$$

Now we set

$$\phi_n(x) = \frac{\widetilde{\phi}_n(x)}{\|\widetilde{\phi}_n\|},$$

then it follows that

$$\|\phi_n\| = 1. \tag{3.14}$$

Motivated by [12], we have the following lemma:

Lemma 3.4. *Assume that* $(m_1),(m_2)$ *and* $(f_1)-(f_6)$ *hold. Then there exists* $\overline{n}\in\mathbb{N}$ *such that*

$$c* \leqslant \max_{t\geqslant 0}\Phi\left(\frac{\sqrt{2}}{2}t\phi_{\overline{n}}(x), \frac{\sqrt{2}}{2}t\phi_{\overline{n}}(x)\right) < \frac{1}{2}M\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right). \tag{3.15}$$

Proof. By a straight estimation, we have

$$\begin{aligned}\int_{B_{d/n}}\frac{1}{|y|^{\alpha}}dy\int_{B_{d/n}}\frac{1}{|x|^{\alpha}|x-y|^{4-\mu}}dx &= d^{4+\mu-2\alpha}\int_{B_{1/n}}\frac{1}{|y|^{\alpha}}dy\int_{B_{1/n}}\frac{1}{|x|^{\alpha}|x-y|^{4-\mu}}dx\\
&\geqslant d^{4+\mu-2\alpha}\left(\frac{1}{n}\right)^{-2\alpha}\int_{B_{1/n}}dx\int_{B_{1/n}}\frac{1}{|x-y|^{4-\mu}}dy\\
&\geqslant d^{4+\mu-2\alpha}\left(\frac{1}{n}\right)^{-2\alpha}\int_{B_{1/n}}dx\int_{B_{1/n}}\frac{1}{|z|^{4-\mu}}dz\\
&\geqslant d^{4+\mu-2\alpha}\left(\frac{1}{n}\right)^{-2\alpha}\int_{B_{1/n}}dx\int_{B_{1/n-|x|}}\frac{1}{|z|^{4-\mu}}dz\\
&= d^{4+\mu-2\alpha}\left(\frac{1}{n}\right)^{-2\alpha}\frac{4\pi^4}{\mu}\int_0^{\frac{1}{n}}\left(\frac{1}{n}-r\right)^{\mu}r^3dr\\
&= \frac{24\pi^4}{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}\left(\frac{d}{n}\right)^{4+\mu-2\alpha}.\end{aligned}$$

Briefly, we deduce

$$\int_{B_{d/n}}\frac{1}{|y|^{\alpha}}dy\int_{B_{d/n}}\frac{1}{|x|^{\alpha}|x-y|^{4-\mu}}dx\geqslant C_{\mu}\left(\frac{d}{n}\right)^{4+\mu-2\alpha}, \tag{3.16}$$

where $C_{\mu}=\frac{24\pi^4}{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}$. According to the Remark 1.4 and (f_6), we know

$$\liminf_{t,s\to\infty}\frac{(|t|+|s|)F(x,t,s)}{e^{\beta_0(|t|^2+|s|^2)}}>\kappa,\quad \text{for any } x\in\Omega,$$

where

$$\kappa>\left[\frac{4\pi^2(4+\mu-2\alpha)e^{\frac{21(4+\mu-2\alpha)-8}{8}}m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)}{\beta_0^2C_{\mu}d^{4+\mu-2\alpha}}\right]^{\frac{1}{2}}$$

in (f_6), we choose $\varepsilon>0$ such that

$$\frac{(\kappa-\varepsilon)^2}{(1+\varepsilon)^2}>\frac{4\pi^2(4+\mu-2\alpha)e^{\frac{21(4+\mu-2\alpha)}{8}}}{\beta_0^2C_{\mu}d^{4+\mu-2\alpha}e}m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right) \tag{3.17}$$

and

$$\frac{21(4+\mu-2\alpha)}{8}+\ln\frac{4\pi^2(1+\varepsilon)^2(4+\mu-2\alpha)m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)}{(\kappa-\varepsilon)^2\beta_0^2C_{\mu}d^{4+\mu-2\alpha}}<\frac{1-\varepsilon}{1+\varepsilon}. \tag{3.18}$$

Using (f_6), we know that there exists $t_{\varepsilon}>0$ such that

$$(|t_1|+|t_2|)F(x,t_1,t_2)\geqslant(\kappa-\varepsilon)e^{\beta_0|(t_1,t_2)|^2},\quad \forall x\in\Omega,\ |t_1|,|t_2|\geqslant t_{\varepsilon}. \tag{3.19}$$

Ther are four possible cases as follows. From now on, in the sequel, all inequalities hold for large $n\in\mathbb{N}$.

Case i) $t \in \left[\, 0, \sqrt{\frac{2\pi^2(4+\mu-2\alpha)}{\beta_0}}\,\right]$. Then it follows that

$$\begin{aligned}\Phi\left(\frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n\right) &= \frac{1}{2}M\left(t^2\|\phi_n\|^2\right) - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}\right]\frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}dx \\ &\le \frac{1}{2}M\left(\frac{2\pi^2(4+\mu-2\alpha)}{\beta_0}\right).\end{aligned}$$

Clearly, there exists $\overline{n} \in \mathbb{N}$ such that (3.15) holds.

Case ii) $t \in \left[\, \sqrt{\frac{2\pi^2(4+\mu-2\alpha)}{\beta_0}}, \sqrt{\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}}\,\right]$. Then $\frac{\sqrt{2}}{2}t\phi_n(x) \geqslant t_\varepsilon$ for $x \in B_{d/n}(0)$ and for large $n \in \mathbb{N}$, from (f_6), (3.12), (3.16) and (3.19), it follows that

$$\begin{aligned}&\int_\Omega \left[I_\mu * \frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}\right]\frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}dx \\ &\geqslant \int_{B_{d/n}}\int_{B_{d/n}} \frac{F(x, \frac{\sqrt{2}}{2}t\phi_n(x), \frac{\sqrt{2}}{2}t\phi_n(x))F(y, \frac{\sqrt{2}}{2}t\phi_n(y), \frac{\sqrt{2}}{2}t\phi_n(y))}{|x|^\alpha|x-y|^{4-\mu}|y|^\alpha}dxdy \\ &\geqslant \frac{(\kappa-\varepsilon)^2}{2t^2}\int_{B_{d/n}}\int_{B_{d/n}} \frac{e^{\beta_0 t^2\phi_n^2(x)+\beta_0 t^2\phi_n^2(y)}}{\phi_n(x)\phi_n(y)|x|^\alpha|x-y|^{4-\mu}|y|^\alpha}dxdy \\ &\geqslant \frac{\beta_0(\kappa-\varepsilon)^2}{8\pi^2(4+\mu-2\alpha)}\int_{B_{d/n}}\int_{B_{d/n}} \frac{e^{\beta_0 t^2\phi_n^2(x)+\beta_0 t^2\phi_n^2(y)}}{\phi_n(x)\phi_n(y)|x|^\alpha|x-y|^{4-\mu}|y|^\alpha}dxdy \\ &\geqslant \frac{4C_\mu\beta_0\|\widetilde{\phi}_n\|^2(\kappa-\varepsilon)^2d^{4+\mu-2\alpha}\ln n}{(4+\mu-2\alpha)\left(4(\ln n)^2+4\ln n+1\right)n^{4+\mu-2\alpha}}e^{(4\pi^2\|\widetilde{\phi}_n\|^2)^{-1}\beta_0 t^2\ln n}. \qquad (3.20)\end{aligned}$$

Then from (3.1), (3.14) and (3.20), it holds that

$$\begin{aligned}\Phi\left(\frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n\right) &= \frac{1}{2}M\left(t^2\|\phi_n\|^2\right) - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}\right]\frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}dx \\ &\leqslant \frac{1}{2}M(t^2) - \frac{2C_\mu\beta_0\|\widetilde{\phi}_n\|^2(\kappa-\varepsilon)^2d^{4+\mu-2\alpha}\ln n}{(4+\mu-2\alpha)\left(4(\ln n)^2+4\ln n+1\right)n^{4+\mu-2\alpha}}e^{(4\pi^2\|\widetilde{\phi}_n\|^2)^{-1}\beta_0 t^2\ln n} \\ &=: \varphi(t). \qquad (3.21)\end{aligned}$$

Let $t_n > 0$ such that $\varphi_n'(t_n) = 0$, thus

$$m(t_n^2) = \frac{C_\mu\beta_0^2(\kappa-\varepsilon)^2d^{4+\mu-2\alpha}(\ln n)^2}{\pi^2(4+\mu-2\alpha)\left(4(\ln n)^2+4\ln n+1\right)n^{4+\mu-2\alpha}}e^{(4\pi^2\|\widetilde{\phi}_n\|^2)^{-1}\beta_0 t_n^2\ln n}. \qquad (3.22)$$

Then by using (3.22) we obtain

$$\lim_{n\to\infty} t_n^2 = \frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}. \qquad (3.23)$$

Set

$$A := \ln\frac{4\pi^2(1+\varepsilon)(4+\mu-2\alpha)m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)}{C_\mu\beta_0^2(\kappa-\varepsilon)^2d^{4+\mu-2\alpha}}. \qquad (3.24)$$

Then (3.17) and (3.18) show that

$$(1+\varepsilon)\max\left\{\frac{21(4+\mu-2\alpha)}{8}+A,0\right\}-(1-\varepsilon)<0. \tag{3.25}$$

From (3.22), (3.23) and (3.24), we have

$$\begin{aligned} t_n^2 &= \frac{4\pi^2(4+\mu-2\alpha)\|\widetilde{\phi}_n\|^2}{\beta_0}\left[1+\frac{\ln\left((4+\mu-2\alpha)\pi^2 m(t_n^2)\left(4(\ln n)^2+4\ln n+1\right)\right)}{(4+\mu-2\alpha)\ln n}\right. \\ &\quad \left.-\frac{\ln((\ln n)^2)+\ln\left(C_\mu\beta_0(\kappa-\varepsilon)^2 d^{4+\mu-2\alpha}\right)}{(4+\mu-2\alpha)\ln n}\right] \\ &\leqslant \frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}+\frac{4\pi^2 A}{\beta_0\ln n}+\frac{21(4+\mu-2\alpha)\pi^2}{2\beta_0\ln n}+O\left(\frac{1}{(\ln n)^2}\right) \end{aligned} \tag{3.26}$$

and

$$\varphi_n(t)\leqslant\varphi_n(t_n)=\frac{1}{2}M(t_n^2)-\frac{2\pi^2\|\widetilde{\phi}_n\|^2 m(t_n^2)}{\beta_0\ln n}, \quad \forall\, t\geq 0. \tag{3.27}$$

By (1.5) in Remark 1.1, we know

$$M(t_1+t_2)\leqslant M(t_1)+\frac{t_1}{\theta}\left[\left(1+\frac{t_2}{t_1}\right)^\theta-1\right]m(t_1), \quad \forall\, t_1,t_2>0.$$

Then using (m_1), (1.5), (3.23), (3.26) and (3.27), we obtain

$$\begin{aligned} \varphi_n(t) &\leqslant \varphi_n(t_n)=\frac{1}{2}M(t_n^2)-\frac{2\pi^2\|\widetilde{\phi}_n\|^2 m(t_n^2)}{\beta_0\ln n} \\ &\leqslant \frac{1}{2}M\left[\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}+\frac{4\pi^2 A}{\beta_0\ln n}+\frac{21(4+\mu-2\alpha)\pi^2}{2\beta_0\ln n}+O\left(\frac{1}{(\ln n)^2}\right)\right] \\ &\quad -\frac{2\pi^2(1-\varepsilon)m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\alpha_0}\right)}{\beta_0\ln n} \\ &\leqslant \frac{1}{2}M\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)+\frac{(1+\varepsilon)\max\left\{\frac{21(4+\mu-2\alpha)}{8}+A,0\right\}-(1-\epsilon)}{\beta_0\ln n} \\ &\quad \times 2\pi^2 m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)+O\left(\frac{1}{(\ln n)^2}\right). \end{aligned} \tag{3.28}$$

Hence, combining (3.21) with (3.28), one has

$$\begin{aligned} \Phi\left(\frac{\sqrt{2}}{2}t\phi_n,\frac{\sqrt{2}}{2}t\phi_n\right) &\leqslant \frac{1}{2}M\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)+\frac{(1+\varepsilon)\max\left\{\frac{21(4+\mu-2\alpha)}{8}+A,0\right\}-(1-\varepsilon)}{\beta_0\ln n} \\ &\quad \times 2\pi^2 m\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right)+O\left(\frac{1}{(\ln n)^2}\right). \end{aligned}$$

Clearly, in this case, the above estimate implies that there exists $\overline{n}$ large enough such that (3.15) holds.

Case iii) $t \in \left[\sqrt{\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}}, \sqrt{\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}(1+\varepsilon)}\right]$. Then $\frac{\sqrt{2}}{2}t\phi_n(x) \geqslant t_\varepsilon$ for $x \in B_{d/n}(0)$ and for large $n \in \mathbb{N}$, from (f_6), (3.12), (3.16) and (3.19), the process is similar to case **ii**, we delete it.

Case iv) $t \in \left(\sqrt{\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}(1+\varepsilon)}, +\infty\right)$. Then $\frac{\sqrt{2}}{2}t\phi_n(x) \geqslant t_\varepsilon$ for $x \in B_{d/n}(0)$ and for large $n \in \mathbb{N}$, from (3.1), (3.12), (3.14), (3.16) and (3.19), it follows that

$$\begin{aligned}
&\Phi\left(\frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n\right)\\
&= \frac{1}{2}M(t^2\|\phi_n\|^2) - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}\right]\frac{F(x, \frac{\sqrt{2}}{2}t\phi_n, \frac{\sqrt{2}}{2}t\phi_n)}{|x|^\alpha}dx\\
&\leqslant \frac{1}{2}M(t^2) - \frac{8C_\mu\pi^2\|\widetilde{\phi}_n\|^2(\kappa-\varepsilon)^2 d^{4+\mu-2\alpha}\ln n}{(4(\ln n)^2+4\ln n+1)\, t^2 n^{4+\mu-2\alpha}} e^{(4\pi^2\|\widetilde{\phi}_n\|^2)^{-1}\beta_0 t^2\ln n}\\
&\leqslant \frac{1}{2}M\left(\frac{4\pi^2(4+\mu-2\alpha)(1+\varepsilon)}{\beta_0}\right) - \frac{2C_\mu\beta_0\ln n\|\widetilde{\phi}_n\|^2(\kappa-\varepsilon)^2 d^{4+\mu-2\alpha} e^{\frac{(8\varepsilon\ln n-21)(4+\mu-2\alpha)\ln n}{8\ln n+21}}}{(4+\mu-2\alpha)(1+\varepsilon)\,(4(\ln n)^2+4\ln n+1)}\\
&\leqslant \frac{1}{3}M\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right),
\end{aligned}$$

then there exists $\overline{n} \in \mathbb{N}$ such that (3.15) holds. In the above derivation process, we use the fact that the function $\frac{1}{2}M(t^2) - \frac{8C_\mu\pi^2\|\widetilde{\phi}_n\|^2(\kappa-\varepsilon)^2 d^{4+\mu-2\alpha}\ln n}{(4(\ln n)^2+4\ln n+1)t^2 n^{4+\mu-2\alpha}} e^{(4\pi^2\|\widetilde{\phi}_n\|^2)^{-1}\beta_0 t^2\ln n}$ is decreasing on $t \in \left(\sqrt{\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}(1+\varepsilon)}, +\infty\right)$, since its stagnation points tend to $\sqrt{\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}}$ as $n \to \infty$. □

4. **The proof of Theorem 1.5**

In this section, we will give the proof of Theorem 1.5.

Lemma 4.1. *Assume that* (m_2) *and* (f_3) *hold, then* $c^* \leqslant b$, *where* $b = \inf_{\mathcal{N}} \Phi$ *in* (1.7).

Proof. For each $(u, v) \in \mathcal{N}$, then it follows that $\langle \Phi'(u, v), (u, v)\rangle = 0$. Now we define the continuous map $g : (0, +\infty) \to \mathbb{R}$ such that $g(t) = \Phi(tu, tv)$. By using (3.2) we have

$$g'(t) = m\left(\|(tu, tv)\|^2\right)\|(u, v)\|^2 t - \int_\Omega \left[I_\mu * \frac{F(x, tu, tv)}{|x|^\alpha}\right]\frac{f_1(x, tu, tv)u + f_2(x, tu, tv)v}{|x|^\alpha}dx,$$

for all $t \in (0, +\infty)$ and

$$\begin{aligned}
g'(t) &= g'(t) - t^{2\theta-1}\langle \Phi'(u, v), (u, v)\rangle\\
&= m\left(\|(tu, tv)\|^2\right)\|(u, v)\|^2 t - t^{2\theta-1} m\left(\|(u, v)\|^2\right)\|(u, v)\|^2\\
&\quad + t^{2\theta-1}\int_\Omega \left[I_\mu * \frac{F(x, u, v)}{|x|^\alpha}\right]\frac{f_1(x, u, v)u + f_2(x, u, v)v}{|x|^\alpha}dx\\
&\quad - \int_\Omega \left[I_\mu * \frac{F(x, tu, tv)}{|x|^\alpha}\right]\frac{f_1(x, tu, tv)u + f_2(x, tu, tv)v}{|x|^\alpha}dx
\end{aligned}$$

$$
\begin{aligned}
&= t^{2\theta-1}\left[\frac{m\left(\|(tu,tv)\|^2\right)}{\left(\|(tu,tv)\|^2\right)^{\theta-1}} - \frac{m\left(\|(u,v)\|^2\right)}{\left(\|(u,v)\|^2\right)^{\theta-1}}\right]\left(\|(u,v)\|^2\right)^{\theta} \\
&\quad + t^{2\theta-1}\left\{\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_1(x,u,v)u + f_2(x,u,v)v}{|x|^\alpha}dx \right. \\
&\quad \left. -\frac{1}{t^{2\theta}}\int_\Omega \left[I_\mu * \frac{F(x,tu,tv)}{|x|^\alpha}\right]\frac{f_1(x,tu,tv)tu + f_2(x,tu,tv)tv}{|x|^\alpha}dx\right\}.
\end{aligned}
$$

From Remark 1.3, we choose $p = \theta < l$, then we derive that

$$
tf_1(x,t,s) - \theta F(x,t,s) > 0, \quad sf_2(x,t,s) - \theta F(x,t,s) > 0,
$$

for all $(x,t,s) \in \Omega \times \mathbb{R}^2$, along with (f_3) which shows that

$$
t \mapsto \frac{F(x,tu,tv)}{t^\theta} \quad \text{is nondecreasing for} \quad t > 0. \tag{4.1}
$$

Then for $0 < t \leqslant 1$, $x \in \Omega$ from (f_3) and (4.1) , we obtain

$$
\begin{aligned}
g'(t) &\geqslant t^{2\theta-1}\left(\frac{m\left(\|(tu,tv)\|^2\right)}{\left(\|(tu,tv)\|^2\right)^{\theta-1}} - \frac{m\left(\|(u,v)\|^2\right)}{\left(\|(u,v)\|^2\right)^{\theta-1}}\right)\left(\|(u,v)\|^2\right)^{\theta} \\
&\quad + t^{2\theta-1}\left\{\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\left(\frac{f_1(x,u,v)u}{u^\theta} - \frac{f_1(x,tu,tv)tu}{(tu)^\theta}\right)u^\theta dx \right. \\
&\quad \left. + \int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\left(\frac{f_2(x,u,v)v}{v^\theta} - \frac{f_2(x,tu,tv)tv}{(tv)^\theta}\right)v^\theta dx\right\} \\
&\geqslant 0.
\end{aligned}
$$

This shows that $g'(t) \geqslant 0$ for $0 < t \leqslant 1$ and $g'(t) < 0$ for $t > 1$. So we obtain $g(1) = \max_{t\geqslant 0}\Phi(tu,tv)$, thus $\Phi(u,v) = \max_{t\geqslant 0}\Phi(tu,tv)$. Now we define $h : [0,1] \to H_0^2(\Omega,\mathbb{R}^2)$ as $h(t) = (t_0u, t_0v)t$ where $t_0 > 1$ is such that $\Phi(t_0u,t_0v) < 0$. So, $h \in \Gamma$ which implies that

$$
c^* \leqslant \max_{t\in[0,1]}\Phi(h(t)) \leqslant \max_{t\geqslant 0}\Phi(tu,tv) = \Phi(u,v).
$$

Since $(u,v) \in \mathcal{N}$ is arbitrary, we get $c^* \leqslant b$. □

Lemma 4.2. *Assume that (f_4) holds and let $\{(u_n,v_n)\} \in H_0^2(\Omega,\mathbb{R}^2)$ be a Palais-Smale sequence for Φ, i.e.*

$$
\Phi(u_n,v_n) \to c^*, \Phi'(u_n,v_n) \to 0 \quad as \quad n \to \infty.
$$

Then there exists $(u,v) \in H_0^2(\Omega,\mathbb{R}^2)$ such that, up to subaequence, $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $H_0^2(\Omega,\mathbb{R}^2)$,

$$
\int_\Omega \left[I_\mu * \frac{F(x,u_n,v_n)}{|x|^\alpha}\right]\frac{f_i(x,u_n,v_n)}{|x|^\alpha}\varphi dx \to \int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_i(x,u,v)}{|x|^\alpha}\varphi dx \tag{4.2}
$$

for all $\varphi \in C_0^\infty(\Omega)$ and $i = 1,2,$ and

$$
\int_\Omega \left[I_\mu * \frac{F(x,u_n,v_n)}{|x|^\alpha}\right]\frac{F(x,u_n,v_n)}{|x|^\alpha}dx \to \int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right]\frac{F(x,u,v)}{|x|^\alpha}dx.
$$

Proof. Similar to [8, Lemma 3.3], so we delete the proof. □

Now we are ready to give the proof of our main result.

Proof of Theorem 1.5. Let $\{(u_n, v_n)\}$ be a Palais Smale sequence at the Mountain pass level c^*. By using Lemma 3.3, we know $\{(u_n, v_n)\}$ is bounded, then there exists $u, v \in H_0^2(\Omega)$ such that, up to subsequence, $u_n \rightharpoonup u, v_n \rightharpoonup v$ weakly in $H_0^2(\Omega)$ as $n \to \infty$. Next we will make some claims as follows.
Claim 1: $u, v \not\equiv 0$.
If $u = 0$ or $v = 0$, by using Lemma 4.2 we obtain

$$\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha} \right] \frac{F(x, u_n, v_n)}{|x|^\alpha} dx \to 0 \quad \text{as } n \to \infty. \tag{4.3}$$

From (3.9) and (4.3) we know $\lim_{n\to\infty} \Phi(u_n, v_n) = \frac{1}{2} \lim_{n\to\infty} M\left(\|(u_n, v_n)\|^2\right) = c^*$. Now from (3.15) and $m(t) > 0$ for $t \geq 0$ which implies $M(t)$ is strictly increasing we obtain

$$\|(u_n, v_n)\|^2 < \frac{4\pi^2(4 + \mu - 2\alpha)}{\beta_0},$$

for n large enough, and this shows that $\sup_n \int_\Omega (f_i(u_n, v_n))^q dx < +\infty$ for some $q > \frac{8}{4+\mu-2\alpha}$, $i = 1, 2$. Besides this, from Lemma 2.2, Proposition 2.5, (3.2) and Vitali's convergence theorem we derive that

$$\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha} \right] \frac{f_1(x, u_n, v_n)u_n + f_2(x, u_n, v_n)v_n}{|x|^\alpha} dx \to 0 \quad \text{as } n \to \infty.$$

Then from (3.8), we know $\lim_{n\to\infty} \langle \Phi'(u_n, v_n), (u_n, v_n) \rangle = 0$, hence we obtain

$$\lim_{n\to\infty} m\left(\|(u_n, v_n)\|^2\right) \|(u_n, v_n)\|^2 = 0.$$

From (f_1), we know it is obvious that $\lim_{n\to\infty} \|(u_n, v_n)\|^2 = 0$. Then (3.9) and (4.3) show that $c^* = \lim_{n\to\infty} \Phi(u_n, v_n) = 0$, which contradicts $c^* > 0$. Hence we claim that $u, v \not\equiv 0$, now we assume that $\|(u_n, v_n)\|^2 \to \sigma^2$ as $n \to \infty$. From Lemma 4.2 we obtain

$$\begin{aligned} &\int_\Omega \left[I_\mu * \frac{F(x, u_n, v_n)}{|x|^\alpha} \right] \frac{f_1(x, u_n, v_n)\phi + f_2(x, u_n, v_n)\psi}{|x|^\alpha} dx \\ &\quad \to \int_\Omega \left[I_\mu * \frac{F(x, u, v)}{|x|^\alpha} \right] \frac{f_1(x, u, v)\phi + f_2(x, u, v)\psi}{|x|^\alpha} dx, \quad \text{as } n \to \infty, \end{aligned}$$

and

$$m(\sigma^2) \int_\Omega (\Delta u \Delta\phi + \Delta v \Delta\psi) dx = \int_\Omega \left(I_\mu * \frac{F(x, u, v)}{|x|^\alpha} \right) \frac{f_1(x, u, v)\phi + f_2(x, u, v)\psi}{|x|^\alpha} dx, \tag{4.4}$$

for all $\phi, \psi \in H_0^2(\Omega)$. Indeed, if we take $\phi = u^-$ and $\psi = 0$ in (4.4), we have $m\left(\sigma^2\right) \|u^-\| = 0$. By using (m_1), we get that $u^- = 0$ a.e. in Ω. Therefore $u, v \geq 0$ a.e. in Ω.
Claim 2: $m\left(\|(u, v)\|^2\right) \|(u, v)\|^2 \geqslant \int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha} \right] \frac{f_1(x,u,v)u + f_2(x,u,v)v}{|x|^\alpha} dx$.

Supposing by contradiction that

$$m\left(\|(u, v)\|^2\right) \|(u, v)\|^2 < \int_\Omega \left[I_\mu * \frac{F(x, u, v)}{|x|^\alpha} \right] \frac{f_1(x, u, v)u + f_2(x, u, v)v}{|x|^\alpha} dx, \tag{4.5}$$

let $h(t)=\langle \Phi'(tu,tv),(tu,tv)\rangle$, then from (3.2) and (4.5) we know $h(1)=\langle \Phi'(u,v),(u,v)\rangle < 0$. Then for $t>0$ small enough, using (f_5) and Remark 1.3, we obtain that

$$\begin{aligned}
&\langle \Phi'(tu,tv),(tu,tv)\rangle \\
&\geqslant m_0 t^2\|(u,v)\|^2-\frac{1}{2}\int_\Omega\int_\Omega \frac{(f_1(x,tu,tv)tu+f_2(x,tu,tv)tv)\,(f_1(y,tu,tv)tu+f_2(y,tu,tv)tv)}{|x-y|^{4-\mu}|x|^\alpha|y|^\alpha}dxdy \\
&\geqslant m_0 t^2\|(u,v)\|^2-\frac{t^{2r+2}}{2}\int_\Omega\left(\int_\Omega \frac{(u^r+v^r)u+(u^r+v^r)v}{|x-y|^{4-\mu}|y|^\alpha}dy\right)\frac{(u^r+v^r)u+(u^r+v^r)v}{|x|^\alpha}dx \\
&>0.
\end{aligned}$$

Since $r>0$, then the above inequality shows that $h(t)>0$ for t small enough. So there exists a $t_*\in(0,1)$ such that $h(t_*)=\langle \Phi'(t_*u,t_*v),(t_*u,t_*v)\rangle=0$, i.e. $(t_*u,t_*v)\in\mathcal{N}$. So using (3.1), (3.2), Lemma 4.1 and the lower semicontinuity we get

$$\begin{aligned}
c^* &\leqslant b\leqslant \Phi(t_*u,t_*v)-\frac{1}{2\theta}\langle \Phi'(t_*u,t_*v),(t_*u,t_*v)\rangle \\
&=\frac{1}{2}M\left(\|(t_*u,t_*v)\|^2\right)-\frac{1}{2}\int_\Omega\left[I_\mu*\frac{F(x,t_*u,t_*v)}{|x|^\alpha}\right]\frac{F(x,t_*u,t_*v)}{|x|^\alpha}dx \\
&\quad -\frac{1}{2\theta}m\left(\|(t_*u,t_*v)\|^2\right)\|(t_*u,t_*v)\|^2 \\
&\quad +\frac{1}{2\theta}\int_\Omega\left[I_\mu*\frac{F(x,t_*u,t_*v)}{|x|^\alpha}\right]\frac{f_1(x,t_*u,t_*v)t_*u+f_2(x,t_*u,t_*v)t_*v}{|x|^\alpha}dx \\
&<\frac{1}{2}M\left(\|(u,v)\|^2\right)-\frac{1}{2\theta}m\left(\|(u,v)\|^2\right)\|(u,v)\|^2 \\
&\quad +\frac{1}{2\theta}\int_\Omega\left[I_\mu*\frac{F(x,t_*u,t_*v)}{|x|^\alpha}\right]\frac{f_1(x,t_*u,t_*v)t_*u+f_2(x,t_*u,t_*v)t_*v-\theta F(x,t_*u,t_*v)}{|x|^\alpha}dx \\
&\leqslant\frac{1}{2}M\left(\|(u,v)\|^2\right)-\frac{1}{2\theta}m\left(\|(u,v)\|^2\right)\|(u,v)\|^2 \\
&\quad +\frac{1}{2\theta}\int_\Omega\left[I_\mu*\frac{F(x,u,v)}{|x|^\alpha}\right]\frac{f_1(x,u,v)u+f_2(x,u,v)v-\theta F(x,u,v)}{|x|^\alpha}dx \\
&\leqslant\liminf_{n\to\infty}\left\{\frac{1}{2}M\left(\|(u_n,v_n)\|^2\right)-\frac{1}{2\theta}m\left(\|(u_n,v_n)\|^2\right)\|(u,v)\|^2\right. \\
&\quad \left.+\frac{1}{2\theta}\int_\Omega\left[I_\mu*\frac{F(x,u_n,v_n)}{|x|^\alpha}\right]\frac{f_1(x,u_n,v_n)u+f_2(x,u_n,v_n)v_n-\theta F(x,u_n,v_n)}{|x|^\alpha}dx\right\} \\
&=\liminf_{n\to\infty}\left(\Phi(u_n,v_n)-\frac{1}{2\theta}\langle \Phi'(u_n,v_n),(u_n,v_n)\rangle\right)=c^*.
\end{aligned}$$

This gives a contradiction and completes the proof of Claim 2.

Claim 3: $\Phi(u,v)=c^*$.

By using the weakly lower semicontinuity of norms we know $\Phi(u,v)\leqslant\lim_{n\to+\infty}\Phi(u_n,v_n)=c^*$. If $\Phi(u,v)<c^*$, then from (3.1) and Lemma 4.2 we know $M(\|(u,v)\|^2)<\lim_{n\to+\infty}M(\|(u_n,v_n)\|^2)$. Since $M(t)$ is increasing and continuous and $\lim_{n\to\infty}\|(u_n,v_n)\|^2=\sigma^2$, we obtain $\|(u,v)\|^2<\sigma^2$. Moreover, from (3.1), (3.8) and Lemma 4.2 we derive that

$$M\left(\sigma^2\right)=\lim_{n\to\infty}M\left(\|(u_n,v_n)\|^2\right)=2\left(c^*+\frac{1}{2}\int_\Omega\left[I_\mu*\frac{F(x,u,v)}{|x|^\alpha}\right]\frac{F(x,u,v)}{|x|^\alpha}dx\right). \tag{4.6}$$

Now let

$$(\widetilde{u}_n, \widetilde{v}_n) = \left(\frac{u_n}{\|(u_n, v_n)\|}, \frac{v_n}{\|(u_n, v_n)\|} \right),$$

obviously we obtain $\|(\widetilde{u}_n, \widetilde{v}_n)\| = 1$ and $(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}, \widetilde{v}) = (\frac{u}{\sigma}, \frac{v}{\sigma})$ weakly in $H_0^2(\Omega, \mathbb{R}^2)$. From Lemma 2.3, we have that

$$\sup_{n\in\mathbb{N}} \int_\Omega \exp\left(\beta(|\widetilde{u}_n|^2 + |\widetilde{v}_n|^2)\right) dx < +\infty, \text{ for } 1 < \beta < \frac{32\pi^2}{1 - \|(\widetilde{u}, \widetilde{v})\|^2}.$$

Then from (1.3), (3.1) and Claim 2 we obtain

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}M\left(\|(u,v)\|^2\right) - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right] \frac{F(x,u,v)}{|x|^\alpha} dx \\
&\geq \frac{1}{2}M\left(\|(u,v)\|^2\right) - \frac{1}{2}\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right] \frac{F(x,u,v)}{|x|^\alpha} dx - \frac{1}{2\theta}m\left(\|(u,v)\|^2\right)\|(u,v)\|^2 \\
&+ \frac{1}{2\theta}\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right] \frac{f_1(x,u,v)u + f_2(x,u,v)v}{|x|^\alpha} dx \\
&= \frac{1}{2\theta}\left(\theta M\left(\|(u,v)\|^2\right) - m\left(\|(u,v)\|^2\right)\|(u,v)\|^2\right) \\
&\quad + \frac{1}{2\theta}\int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right] \frac{f_1(x,u,v)u + f_2(x,u,v)v - \theta F(x,u,v)}{|x|^\alpha} dx \\
&\geq 0,
\end{aligned}$$

and from Remak 1.1, (3.1), Lemma 3.4 and (4.6) we get

$$\begin{aligned}
M\left(\sigma^2\right) &= 2c^* + \int_\Omega \left[I_\mu * \frac{F(x,u,v)}{|x|^\alpha}\right] \frac{F(x,u,v)}{|x|^\alpha} dx \\
&= 2c^* - 2\Phi(u,v) + M\left(\|(u,v)\|^2\right) \\
&< M\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}\right) + M\left(\|(u,v)\|^2\right) \\
&\leqslant M\left(\frac{4\pi^2(4+\mu-2\alpha)}{\beta_0} + \|(u,v)\|^2\right).
\end{aligned}$$

For the monotonicity of the function $M(t)$ we know

$$\sigma^2 < \frac{1}{1 - \|(\widetilde{u}, \widetilde{v})\|^2} \frac{4\pi^2(4+\mu-2\alpha)}{\beta_0}.$$

Thus for $n \in \mathbb{N}$ large enough it is possible $p > \beta_0$ and p close to β_0 such that

$$\frac{8}{4+\mu-2\alpha} p\|(u_n, v_n)\|^2 \leqslant \frac{32\pi^2}{1 - \|(\widetilde{u}, \widetilde{v})\|^2}.$$

From Lemma 2.3, there exists $C > 0$ such that

$$\int_\Omega \exp\left(\frac{8}{4+\mu-2\alpha} p\left(|u_n|^2 + |v_n|^2\right)\right) \leqslant C$$

and

$$\int_{\Omega}\left[I_{\mu}*\frac{F(x,u_n,v_n)}{|x|^{\alpha}}\right]\frac{f_1(x,u_n,v_n)u_n+f_2(x,u_n,v_n)v_n}{|x|^{\alpha}}dx$$
$$\rightarrow \int_{\Omega}\left[I_{\mu}*\frac{F(x,u,v)}{|x|^{\alpha}}\right]\frac{f_1(x,u,v)u+f_2(x,u,v)v}{|x|^{\alpha}}dx,$$

as $n\to\infty$, which implies $(u_n,v_n)\to(u,v)$ strongly in $H_0^2(\Omega,\mathbb{R}^2)$. Hence $\Phi(u,v)=c^*$ which gives a contradiction and completes the proof of Claim 3.

Finalizing the proof of Theorem 1.5: By Claim 3 and (4.6) we deduce that $\lim_{n\to\infty}M\left(\|(u_n,v_n)\|^2\right)=M\left(\|(u,v)\|^2\right)$ which indicates $(u_n,v_n)\to(u,v)$ in $H_0^2(\Omega,\mathbb{R}^2)$. Then finally we have

$$m\left(\|(u,v)\|^2\right)\int_{\Omega}(\Delta u\Delta\phi+\Delta v\Delta\psi)dx=\int_{\Omega}\left[I_{\mu}*\frac{F(x,u,v)}{|x|^{\alpha}}\right]\frac{f_1(x,u,v)\phi+f_2(x,u,v)\psi}{|x|^{\alpha}}dx,$$

for all $\phi,\psi\in H_0^2(\Omega)$. That is, (u,v) is a solution of problem (1.1) satisfying $\Phi(u,v)=c^*$ and according to Lemma 4.1, the proof of our main result is completed. □

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