

Convergence and Stability of Multi-Step Iterative Schemes for Ciric-Kannan- α - ψ Contractions in F-Metric Spaces

Abstract

In order to ascertain strong convergence and stability of the sequence $\{i_n\}$, the goal of this work is to investigate convergence and stability analysis of a fixed point in entire F -metric space by utilizing a few multi-step iterative techniques and Ciric-Kannan α - ψ -contraction. Our findings enhance some established findings in the literature.

Keywords: Convergence, Ciric-Kannan α - ψ -contraction, Stability and, F -metric space

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1. Introduction

Researchers studying fixed-point theory have made considerable use of two distance-controlled functions to determine the fixed points of mappings, such as expansive or contractive mappings found in nature. There are numerous uses for metric fixed point theory in functional analysis. In order to solve metric fixed point problems, the contractive conditions with underlying functions are crucial. Many mathematicians have expanded the Banach contraction principle over the years. [1] established a number of fixed point theorems for such mappings in complete metric spaces and introduced the notions of α - ψ contraction and α -admissible mapping. [2], which are generalizations of the findings in [1], later modified his work.

The idea of sequential F -metric spaces, a generalization of the normal metric spaces, b -metric spaces, JS -metric spaces, and primarily F -metric spaces, was examined by [3]. A few topological characteristics of these spaces were examined. They demonstrated fixed-point theorems for certain classes of contractive mappings by taking this idea into account. Examples were provided to support their fixed-point theorems and to investigate the validity of the underlying space. They solved a system of linear algebraic equations by using the fixed-point theorem.

Strong convergence and stability findings for Picard-Mann hybrid iterative schemes evaluated in a real normed linear space were demonstrated by [4] using the general class of contractive-like operators proposed

by [5]. As a result, they demonstrated the Picard iterative scheme's great convergence and stability. A novel idea of Ciric-Kannan- α - ψ -contractions in the context of F-metric space is presented in

[6]. Some fixed point theorems for such mappings in F-complete metric space were established using this concept, supported by appropriate hypotheses. Novel existence conditions for a solution of nonlinear neutral differential equations were examined as an application to demonstrate the applicability of the findings.

A method for approximating the fixed point of generalized Suzuki nonexpansive mappings on hyperbolic spaces was described by [7]. The Noor iterative procedure for generalized Suzuki nonexpansive mappings (GSNM) on uniform convex hyperbolic spaces is used to show the Δ -convergence and strong convergence theorems. The results of this work can be used as an extension and generalization of numerous well-known conclusions in Banach spaces as well as CAT(0) spaces because of the richness of uniform convex hyperbolic spaces. [8] presented new hybrid iterative schemes, specifically Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes, and used certain quasi-contractive operators to demonstrate convergence and stability results for these iterative schemes. Additionally, numerical examples comparing the convergence rate and applications of recently released iterative techniques were shown. The outcomes enhance, expand upon, and generalize Olatinwo's writings.

[9] expands on the idea of metric space by introducing the concept of a (ϕ, ψ) -metric space. The symmetry property is maintained in these regions. They discussed the topological features of such spaces and introduced a natural topology $\tau(\phi, \psi)$. Additionally, they established the Banach contraction principle in the setting of (ϕ, ψ) -metric spaces and provided examples to highlight the importance of their primary theorem. In the end, as applications, it was established that there was a single solution to Fredholm-type integral equations in both one and two dimensions, and an example was shown to support this claim.

Three fixed point theorems for the α - ψ class of operators in full metric spaces were established by [1]. Several findings and well-known fixed point theorems by Banach, Kannan, Chatterjea, Zamfirescu, Berinde, Suzuki, Ciric, Nieto, Lopez, and numerous others were expanded upon in their findings. They demonstrated that the Picard iteration can be used to determine the fixed points of a wide class of contractive type operators, which are unified by α - ψ contractions. Lastly, they explore the existence and uniqueness of solutions to a class of quadratic integral equations using their findings.

In this work, we study the convergence and stability analysis of Ciric-Kannan α - ψ contraction using some multi-step iterative procedures in F -metric space.

2. Preliminary

This section discussed some concepts that will be needed in the sequel. Throughout this work Ω represents the family of monotone functions ψ on $[0, \infty)$ satisfying $\sum_{j=1}^{\infty} \psi^j(t) < \infty \quad \forall \quad 0 < t$.

Lemma 2.1. [10]. *For every self map $\psi \in [0, \infty)$, the following holds: if ψ is increasing, $0 < t, \quad 0 = \lim_{j \rightarrow \infty} \psi^j(t)$ yields $t > \psi(t)$.*

Definition 2.1. [10] Let (G, D) be an F-metric space $T : X \rightarrow X$ be a map. We call T an $\alpha - \psi$ - contraction if $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Omega$ with

$$D(T_i, T_y)\alpha(i, j) \leq \psi(D(i, j)) \quad (1)$$

Definition 2.2. [10]. Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be maps. We call T an α -admissible if $1 \leq \alpha(i, j)$ yields $\alpha(T_i, T_y) \geq 1 \quad \forall i, j \in X$

Example 2.1. [10] Let $G = (0, \infty)$. Define $T : G \rightarrow G$ and $\alpha : G \times G \rightarrow [0, \infty)$ by

$$T_g = \ln(g) \quad \forall g \in G$$

and

$$\alpha(g, h) = \begin{cases} 2 & \text{if } h \leq g \\ 0 & \text{if } h > g \end{cases} \quad \text{Then, } T \text{ is } \alpha\text{-admissible.}$$

Definition 2.3. [11]. Let J denote the closed unit interval, T a selfmap of J .

The Ishikawa iterates of T are defined by

$$x_{n+1} = (1 - \alpha_n)i_n + \alpha_n T j_n \quad (2)$$

$$j_n = (1 - \beta_n)i_n + \beta_n T i_n \quad n \geq 0 \quad (3)$$

were $x_0 \in J$ and α_n, β_n satisfy the conditions

- (i) $0 \leq \alpha_n \leq \beta_n \leq 1 \quad \forall n$
- (ii) $\lim \beta_n = 0$ and
- (iii) $\sum \alpha_n \beta_n = \infty$

Definition 2.4. [12] Let G denote the Banach space, T a selfmap of G for all $x_0 \in X$

$$x_{n+1} = (1 - a_n)i_n + a_n T j_n$$

$$j_n = (1 - b_n)i_n + b_n T z_n$$

$$z_n = (1 - c_n)i_n + c_n T i_n \quad \forall n \geq 0.$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1]$

Definition 2.5. [13] Let G denote the Banach space, T a selfmap of G for all $x_0 \in X$

$$x_{n+1} = T(1 - a_n)i_n + a_n T j_n$$

$$j_n = T z_n$$

$$z_n = (1 - b_n)i_n + b_n T i_n \quad \forall n \geq 0.$$

where $\{a_n\},$ and $\{b_n\}$ are sequences in $(0, 1)$

Definition 2.6. F-Metric space

[14] made use of a certain class of auxiliary functions to coin the idea of F -metric spaces. We begin with the collection of such functions.

Let $f \in F$ and $f : (0, +\infty) \rightarrow \mathbb{R}$ be such that:

(F1) $0 < i < j \implies f(i) \leq f(j)$; and

(F2) for $\{i_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} i_n = 0 \leftrightarrow \lim_{n \rightarrow \infty} f\{i_n\} = -\infty$

Example 2.2. [14] *The following are some examples of the previously discussed kind of auxiliary functions:*

i $-\frac{1}{k}$ where $k \in (0, \infty)$;

ii $-\exp^{\frac{1}{k}}$ for all $k \in (0, \infty)$.

Utilizing such functions, the authors generalized the concept of usual metric spaces and originated the notion of F -metric spaces as follows:

Definition 2.7.

[14] Let M be a nonempty set, and let $d_F : G \times G \rightarrow [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in F \times [0, +\infty)$ such that:

D1 $(i, j) \in M \times G, D(i, j) = 0 \iff i = j$;

D2 $D(i, j) = D(j, i)$ for all $(i, j) \in G \times G$; and

D3 for every $(i, j) \in G \times G, N \in \mathbb{N}, N \geq 2$, and $(i_k)_k^n = 1 \subset G$ with $(i_1, i_N) = (i, j)$, we get

$$D(i, j) > 0 \text{ implies } f(D(i, j)) \leq f\left(\sum_{k=1}^{N-1} D(i_k, j_{k+1})\right) + \alpha$$

Then D is an F -metric on G , and the pair (G, D) is said to be an F -metric space.

It is observed that any metric on G is an F -metric, but the converse is not true.

Lemma 2.2. *Let X be a F -metric space*

i. *For $i, j \in G$ and $t \in [0, 1], \exists$ a unique point $z \in [i, j]$ such that*

$$D(i, k) = tD(i, j) \text{ and } D(j, k) = (1 - t)D(i, j)$$

ii. *For $i, j, K \in G$ and $t \in [0, 1]$, we have*

$$D(1 - t)i \oplus tjk \leq (1 - t)D(i, k) + tD(j, k)$$

Lemma 2.3. Let $\{p_n\}_{n=p}^{\infty}$, $\{q_n\}_{n=p}^{\infty}$, $\{r_n\}_{n=p}^{\infty}$ be sequence of non negative numbers satisfying the following condition:

$$p_{n+1} \leq (1 - S_n)p_n + q_n + r_n, \quad \forall x \geq 0$$

where

$$[S_n]_{n=p}^{\infty} \subset [0, 1]. \text{ if } \sum_{n=0}^{\infty} S_n = \infty; \lim_{n \rightarrow \infty} q_n = 0 \text{ and } \sum_{n=p}^{\infty} r_n < \infty, \text{ Then } \lim_{n \rightarrow \infty} p_n = 0.$$

Definition 2.8. Let (G, D) be a F - metric space, $T : G \rightarrow G$ is a mapping, $i_0 \in i$ and assume that the iteration procedure used is the sequence $\{i_n\}_{n=1}^{\infty}$ provided by the iterative procedure, converges to a fixed point p of T . Let $\{j_n\}_{n=1}^{\infty}$ be an arbitrary sequence in G and $\varepsilon_n = D(j_{n+1}, f(T, j_n))$, in $n = 0, 1, 2, \dots$. Then, the fixed point iteration procedure is T -stable or stable with respect to T if and only if

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 \iff \lim_{n \rightarrow \infty} j_n = p$$

3. Main Result

Definition 3.1. [15]. Let (G, D) be an F -metric space $T : G \rightarrow G$ be a map. T is called a Ciric-Kannan $\alpha - \psi$ contraction, if there exists $l \in [0, \frac{1}{2})$, $\psi \in \Omega$ and $\alpha : G \times G \rightarrow [0, \infty)$ such that

$$\alpha(i, j)D(T_i, T_j) \leq \psi(M(i, j) + l[D(i, T_i) + D(i, T_j)])$$

where

$$M(i, j) = \max[D(i, j), D(i, T_i), D(i, T_j)]$$

if a fixed point $u = T_u = j$ exists for the contraction, then

$$\begin{aligned} M(i, u) &= \max[D(i, u), D(i, T_i) + D(u, T_u)] \\ &= \max[D(i, u), D(i, T_i)] \\ &= \max[D(i, u), D(i, u) + D(u, T_i)] \end{aligned}$$

Hence

$$\begin{aligned} M(i, u) &= D(i, u) + D(u, T_i) \\ &= D(i, T_i) \end{aligned}$$

$$\begin{aligned}
D(T_i, u) &\leq \psi D(i, T_i) + L[(D(i, T_i) + D(u, T_u))]\alpha(i, u) \\
&\leq \psi D(i, T_i) + L[D(i, T_i)]\alpha(i, u) \\
&\leq [\psi + L\alpha(i, u)]D(i, T_i) \\
&\leq [\psi + L\alpha(i, u)][D(i, u) + D(u, T_i)]
\end{aligned}$$

$$D(T_i, u) - [\psi + L\alpha(i, u)]D(T_i, u) \leq [\psi + L\alpha(i, u)]D(i, u)$$

$$[1 - (\psi + L\alpha(i, u))]D(T_i, u) \leq [\psi + L\alpha(i, u)]D(i, u)$$

$$D(T_i, u) \leq \frac{[\psi + L\alpha(i, u)]}{[1 - (\psi + L\alpha(i, u))]}D(i, u)$$

Let

$$k = \frac{[\psi + L\alpha(i, u)]}{[1 - (\psi + L\alpha(i, u))]} < 1$$

\therefore

$$D(T_i, u) \leq kD(i, u) \tag{4}$$

similarly,

$$D(Ti_n, u) \leq kD(i_n, u) \tag{5}$$

$$D(Tj_n, u) \leq kD(j_n, u) \tag{6}$$

$$D(Tz_n, u) \leq kD(z_n, u) \tag{7}$$

Theorem 3.1. *Let (G, D) be a complete F -metric space and $T : X \rightarrow X$ be a Ciric-Kannan $\alpha - \psi$ contraction. Let the sequence $\{i_n\}$ be defined by the Ishikawa iterative scheme. If $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{i_n\}$ converges strongly to the unique fixed point of T .*

Proof. Let p be the fixed point of T using Ciric-Kannan $\alpha - \psi$ contraction and Ishikawa iteration we have

$$\begin{aligned}
D(Tx_{n+1}, Ti_n) &\leq D(Tx_{n+1}, Ti_n)\alpha(x_{n+1}, i_n) \\
D(j_n, p) &\leq D((1 - \beta_n)i_n + \beta_n Ti_n, p) \\
&\leq D((1 - \beta_n)i_n, p) + D(\beta_n Ti_n, p) \\
&= (1 - \beta_n)D(i_n, p) + \beta_n D(Ti_n, p) \\
&\leq (1 - \beta_n)D(i_n, p) + \beta_n kD(i_n, p) \\
&= [(1 - \beta_n) + \beta_n k]D(i_n, p)
\end{aligned}$$

$$= [(1 - \beta_n + \beta_n k)D(i_n, p)] \quad (8)$$

$$\begin{aligned} D(x_{n+1}, p) &= D((1 - \alpha_n)i_n + \alpha_n Tj_n, p) \\ &\leq (1 - \alpha_n)D(i_n, p) + \alpha_n D(Tj_n, p) \\ &\leq (1 - \alpha_n)D(i_n, p) + \alpha_n k D(j_n, p) \\ &\leq (1 - \alpha_n)D(i_n, p) + \alpha_n k [(1 - \beta_n + k\beta_n)]D(i_n, p) \\ &\leq [(1 - \alpha_n) + \alpha_n k(1 - \beta_n + k\beta_n)]D(i_n, p) \\ &= [(1 - \alpha_n + \alpha_n k) - g_n]D(i_n, p) \end{aligned}$$

since $0 < k < 1$ and by the assumption of $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, it shows that

$\sum_{n=1}^{\infty} g_n = \infty$. Hence, $\lim_{n \rightarrow \infty} D(i_n, p) = 0$. Thus $\{i_n\}$ converges strongly to a unique fixed point of T which is p . \square

Theorem 3.2. *Let (G, D) be a complete F -metric space, and $T : X \rightarrow X$ be a Ciric-Kannan $\alpha - \psi$ contraction. Let the sequence $\{i_n\}$ be defined by the Noor iterative scheme. If $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{i_n\}$ converges strongly to the unique fixed point of T .*

Proof. Let p be the fixed point of T using Ciric-Kannan $\alpha - \psi$ contraction and Noor iteration implies that

$$\begin{aligned} D(z_n, p) &= D((1 - \alpha_n)i_n + \alpha_n T i_n, p) \\ &\leq D((1 - \alpha_n)i_n, p) + D(\alpha_n T i_n, p) \\ &= (1 - \alpha_n)D(i_n, p) + \alpha_n D(T i_n, p) \\ &\leq (1 - \alpha_n)D(i_n, p) + \alpha_n k D(i_n, p) \\ &= [(1 - \alpha_n) + \alpha_n k]D(i_n, p) \\ &= [(1 - (1 - k)\alpha_n)]D(i_n, p). \end{aligned}$$

$$\begin{aligned} D(j_n, p) &= D((1 - \beta_n)i_n + \beta_n T z_n, p) \\ &\leq (1 - \beta_n)D(i_n, p) + \beta_n D(T z_n, p) \\ &\leq (1 - \beta_n)D(i_n, p) + \beta_n k D(z_n, p) \\ &\leq (1 - \beta_n)D(i_n, p) + \beta_n k [(1 - (1 - k)\alpha_n)]D(i_n, p) \\ &= [(1 - \beta_n) + \beta_n k [(1 - \alpha_n + \alpha_n k)]]D(i_n, p) \\ &= [(1 - \beta_n) + g_n \alpha_n \beta_n]D(i_n, p) \end{aligned}$$

where

$$g_n = k(1 - (1 - k))$$

$$\begin{aligned}
D(j_n, p) &\leq [(1 - \beta_n) + g_n \alpha_n \beta_n] D(i_n, p) \\
D(x_{n+1}, p) &= D((1 - a_n)i_n + a_n Tj_n, p) \\
&\leq (1 - a_n)D(i_n, p) + a_n D(Tj_n, p) \\
&\leq (1 - a_n)D(i_n, p) + a_n k[(1 - \beta_n) + g_n \alpha_n \beta_n] D(i_n, p) \\
&\leq (1 - a_n)D(i_n, p) + a_n k[(1 - \beta_n) + g_n \alpha_n \beta_n] D(i_n, p) \\
&= [(1 - a_n) + a_n k[(1 - \beta_n) + g_n \alpha_n \beta_n]] D(i_n, p)
\end{aligned}$$

Since $0 < k < 1$ and by assumption of lemma 2.3 $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, it follows that $\sum_{n=0}^{\infty} g_n = \infty$. Hence $\lim_{n \rightarrow \infty} D(i_n, p) = 0$.

Then the sequence $\{i_n\}$ converges strongly to a fixed point p of T . \square

Theorem 3.3. *Let (G, D) be a complete F -metric space, and $T : X \rightarrow X$ be a Ciric-Kannan $\alpha - \psi$ contraction. Let the sequence $\{i_n\}$ be defined by the JK iterative scheme. If $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{i_n\}$ converges strongly to the unique fixed point of T .*

Proof. Let p be the fixed point of T using Ciric-Kannan $\alpha - \psi$ contraction and JK iteration we have

$$\begin{aligned}
D(j_n, p) &= D((1 - b_n)i_n + b_n T i_n, p) \\
&\leq D((1 - b_n)i_n, p) + b_n D(T i_n, p) \\
&\leq (1 - b_n)D(i_n, p) + b_n D(T i_n, p) \\
&\leq (1 - b_n)D(i_n, p) + b_n k D(i_n, p) \\
&\leq [(1 - b_n) + b_n k] D(i_n, p).
\end{aligned}$$

$$\begin{aligned}
D(z_n, p) &= D(T j_n, p) \\
&\leq k D(j_n, p) \\
&\leq k[(1 - b_n) + b_n k] D(i_n, p)
\end{aligned}$$

$$\begin{aligned}
D(x_{n+1}, p) &= D(T((1 - a_n)T j_n + a_n T z_n, p) \\
&\leq D(T(1 - a_n)T j_n, p) + D(a_n T z_n, p) \\
&\leq k D((1 - a_n)T j_n, p) + a_n D(T z_n, p) \\
&\leq k(1 - a_n)D(T j_n, p) + a_n D(T z_n, p) \\
&\leq k^2(1 - a_n)D(j_n, p) + a_n k D(z_n, p) \\
&\leq [k^2(1 - a_n)(1 - b_n + k b_n) + (a_n k^2(1 - b_n + k b_n))] D(i_n, p)
\end{aligned}$$

Since $0 < k < 1$ and by lemma 2.3, we get that $\lim_{n \rightarrow \infty} D(x_n, p) = 0$

Hence the sequence converges strongly to a fixed point p at T .

□

Theorem 3.4. Let (G, D) be a F -metric space and $T : X \rightarrow X$ a mapping satisfying Ciric-Kamma $\alpha - \psi$ contraction. Suppose T has a fixed point p . Let $i_0 \in i$ and the sequence $\{i_n\}$ be Ishikawa iteration. Then, $\{i_n\}$ is stable with respect to T .

Proof. Let the sequece $\{a_n\}_{n=1}^{\infty}$ be arbitrary in G . Define

$$\varepsilon_n = D(a_{n+1}, f(T, a_n))$$

for every $n \in \mathbb{N}$, such that

$$a_{n+1} = (1 - \alpha_n)a_n + Tb_n$$

$$b_n = (1 - \beta_n)a_n + \beta Ta_n.$$

Suppose the $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We show that $\lim_{n \rightarrow \infty} a_n = p$

$$\begin{aligned} D(a_{n+1}, p) &\leq D(a_{n+1}, f(T, a_n)) + D(f(T, a_n), p) \\ &\leq \varepsilon_n + kD((1 - \alpha_n)a_n + Tb_n, p) \\ &\leq \varepsilon_n + D((1 - \alpha_n)a_n, p) + D(Tb_n, p) \\ &\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + kD(b_n, p) \end{aligned}$$

Recall that, we have prove fixed point of contraction as

$$D(Tj_n, p) \leq kD(j_n, p)$$

$$\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + kD(b_n, p) \tag{9}$$

$$\begin{aligned} D(b_n, p) &\leq D((1 - \beta_n)a_n + \beta_n Ta_n, p) \\ &\leq D((1 - \beta_n)a_n, p) + D(\beta_n Ta_n, p) \\ &\leq (1 - \beta_n)D(a_n, p) + \beta_n D(Ta_n, p) \end{aligned}$$

$$D(b_n, p) \leq (1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p) \tag{10}$$

$$\begin{aligned} D(a_{n+1}, p) &\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + k[(1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p)] \\ &\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + k(1 - \beta_n)D(a_n, p) + k^2\beta_n D(a_n, p) \\ &\leq \varepsilon_n + [(1 - \alpha_n) + k(1 - \beta_n) + k^2\beta_n]D(a_n, p) \end{aligned}$$

since $\alpha_n, \beta_n, k \in [0, 1]$, we have $[(1 - \alpha_n) + k(1 - \beta_n) + k^2\beta_n] < 1$

Hence,

$$D(a_{n+1}, p) \leq kD(a_n, p) + \varepsilon_n$$

by lemma 2.3

we have

$$\lim_{n \rightarrow \infty} a_n = p$$

Assuming the $\lim_{n \rightarrow \infty} a_n = p$. Then we prove that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

$$\begin{aligned} \varepsilon_n &= D(a_{n+1}, f(T, a_n)) \\ &\leq D(a_{n+1}, p) + D((1 - \alpha_n)a_n + Tb_n, p) \\ &\leq D(a_{n+1}, p) + D((1 - \alpha_n)a_n, p) + D(Tb_n, p) \\ &\leq D(a_{n+1}, p) + (1 - \alpha_n)D(a_n, p) + kD(b_n, p) \end{aligned} \tag{11}$$

$$\leq D(a_{n+1}, p) + (1 - \alpha_n)D(a_n, p) + k[(1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p)] \tag{12}$$

$$\varepsilon_n \leq D(a_{n+1}, p) + [(1 - \alpha_n) + k(1 - \beta_n) + k\beta_n]D(a_n, p) \tag{13}$$

Taking $\lim_{n \rightarrow \infty}$ of eqn(93), we obtain $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

Hence, the $\{i_n\}$ is stable with respect to T . □

Theorem 3.5. *Let (G, D) be a F -metric space and $T : X \rightarrow X$ a mapping satisfying Ciric-Kamma $\alpha - \psi$ contraction. Suppose T has a fixed point p . Let $i_0 \in G$ and let the sequence $\{i_n\}$ be Noor iteration.*

Then, $\{i_n\}$ is stable with respect to T .

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X . Define

$$a_{n+1} = (1 - \alpha_n)a_n + \alpha_n Tb_n$$

$$b_n = (1 - \beta_n)a_n + \beta_n Tc_n$$

$$c_n = (1 - \gamma_n)a_n + \gamma_n Ta_n$$

In every $n \in \mathbb{N}$,

Suppose that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We show that $\lim_{n \rightarrow \infty} a_n = p$:

$$\begin{aligned} D(a_{n+1}, p) &\leq D(a_{n+1}, f(T, a_n)) + D(f(T, a_n), p) \\ &\leq \varepsilon_n + D((1 - \alpha_n)a_n + Tb_n, p) \\ &\leq \varepsilon_n + D((1 - \alpha_n)a_n, p) + D(\alpha_n Tb_n, p) \\ &\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + \alpha_n D(Tb_n, p) \end{aligned}$$

$$\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + \alpha_n k D(b_n, p) \quad (14)$$

$$\begin{aligned} D(b_n, p) &= D((1 - \beta_n)a_n, \beta_n Tc_n, p) \\ &\leq D((1 - \beta_n)a_n, p) + D(\beta_n T a_n, p) \\ &\leq (1 - \beta_n)D(a_n, p) + \beta_n D(Tc_n, p) \\ &\leq (1 - \beta_n)D(a_n, p) + k\beta_n D(Tc_n, p) \end{aligned} \quad (15)$$

$$\begin{aligned} D(c_n, p) &= D((1 - \gamma_n)a_n + \gamma_n T a_n, p) \\ &\leq D((1 - \gamma_n)a_n, p) + D(\gamma_n T a_n, p) \\ &\leq (1 - \gamma_n)D(a_n, p) + \gamma_n D(T a_n, p) \\ D(c_n, p) &\leq (1 - \gamma_n)D(a_n, p) + \gamma_n k D(a_n, p) \end{aligned} \quad (16)$$

substitute (16) into (15)

$$\begin{aligned} \varepsilon_n &\leq D(a_{n+1}, y) + (1 - \alpha_n)D(a_n, p) + \alpha_n k(1 - \beta_n)D(a_n, p) \\ &\quad + k^2 \alpha_n \beta_n [(1 - \alpha_n)D(a_n, p) + j_n k D(a_n, p)] \\ &\leq D(a_{n+1}, p) + (1 - \alpha_n)D(a_n, p) + \alpha_n k(1 - \beta_n)D(a_n, p) \\ &\quad + k^2 \alpha_n p_n (1 - j_n)D(a_n, p) + k^2 \alpha_n \beta_n j_n D(a_n, p) \\ \varepsilon_n &\leq D(a_n, p) + [(1 - \alpha_n) + \alpha_n k(1 - \beta_n) + k^2 \alpha_n (1 - j_n) + k^3 \alpha_n \beta_n j_n] D(a_n, p) \\ \varepsilon_n &\leq D(a_n, p) + k D(a_n, p) \\ \varepsilon_n &\leq D(a_n, p) \end{aligned}$$

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

Hence, thus $\{i_n\}$ is stable with respect to T . \square

Theorem 3.6. *Let (G, D) be a F -metric space and $T : X \rightarrow X$ a mapping satisfying Ciric-Kannan $\alpha - \psi$ contraction. Suppose T has a fixed point p . Let $i_0 \in G$ and $\{i_n\}$ be JK iteration. Then, $\{i_n\}$ is stable with respect to T .*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence in G . Defined

$$\varepsilon_n = D(a_{n+1}, f(T, a_n))$$

$$a_{n+1} = T((1 - \alpha_n)Tc_n + \alpha_n T b_n)$$

$$b_n = Tc_n$$

$$c_n = (1 - \beta_n)a_n + \beta_n T a_n$$

In every $n \in N$; $\{\alpha_n, \beta_n\} \in \{0, 1\}$ and $k \in [0, 1]$
suppose that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We show that $\lim_{n \rightarrow \infty} a_n = p$.

$$\begin{aligned}
D(a_{n+1}, p) &\leq D(a_{n+1}, f(T, a_n)) + D(f(T, a_n), p) \\
&\leq \varepsilon_n + D(T((1 - \alpha_n)Tc_n + \alpha_n T b_n), p) \\
&\leq \varepsilon_n + kD((1 - \alpha_n)Tc_n + \alpha_n T b_n, p) \\
&\leq \varepsilon_n + k(1 - \alpha_n)D(Tc_n, p) + k\alpha_n D(b_n, p) \\
&\leq \varepsilon_n + k^2(1 - \alpha_n)D(c_n, p) + k^2\alpha_n D(b_n, p)
\end{aligned} \tag{17}$$

$$\begin{aligned}
D(b_n, p) &= D(Tc_n, p) \\
&\leq kD(c_n, p)
\end{aligned} \tag{18}$$

$$\begin{aligned}
D(c_n, p) &= D((1 - \beta_n)a_n + \beta_n T a_n, p) \\
&\leq D((1 - \beta_n)a_n, p) + D(\beta_n T a_n, p) \\
&\leq (1 - \beta_n)D(a_n, p) + \beta_n D(T a_n, p)
\end{aligned}$$

$$D(c_n, p) \leq (1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p) \tag{19}$$

Substitute (18) into (19) yields

$$\begin{aligned}
D(b_n, p) &\leq k[(1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p)] \\
&\leq k(1 - \beta_n)D(a_n, p) + k^2\beta_n D(a_n, p)
\end{aligned} \tag{20}$$

$$\begin{aligned}
D(a_{n+1}, p) &\leq \varepsilon_n + k^2(1 - \alpha_n)[(1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p)] \\
&\quad + k^2\alpha_n[k(1 - \beta_n)D(a_n, p) + k^2\beta_n D(a_n, p)] \\
&\leq \varepsilon_n + [[k^2(1 - \alpha_n)[(1 - \beta_n) + k\beta_n] + [k^3\alpha_n(1 - \beta_n) + k^2\beta_n]]D(a_n, p) \\
&\leq \varepsilon_n + kD(a_n, p)
\end{aligned} \tag{21}$$

since $k \in [0, 1]$ and $\alpha_n, \beta_n \in [0, 1]$, we have

$$k^2(1 - \alpha_n)[(1 - \beta_n) + k\beta_n] + k^3\alpha_n(1 - \beta_n) + k^2\beta_n < 1 \quad (22)$$

Hence, $D(a_{n+1}, p) \leq kD(a_n, p) + \varepsilon_n$, by lemma 2.3, $\lim_{n \rightarrow \infty} a_n = p$

Assuming the $\lim_{n \rightarrow \infty} a_n = p$. It remains to prove that
 $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

$$\begin{aligned} \varepsilon_n &= D(a_{n+1}, f(T, a_n)) \\ &\leq D(a_{n+1}, p) + D(f(T, a_n), p) \\ &\leq D(a_{n+1}, p) + D(f(T((1 - \alpha_n)Tc_n + \alpha_nTb_n), p)) \\ &\leq D(a_{n+1}, p) + kD(f(T((1 - \alpha_n)Tc_n + \alpha_nTb_n), p)) \\ &\leq D(a_{n+1}, p) + k(1 - \alpha_n)D(Tc_n, p) + k\alpha_nD(Tb_n, p) \\ &\leq D(a_{n+1}, p) + k^2(1 - \alpha_n)D(c_n, p) + k^2\alpha_nD(b_n, p) \end{aligned} \quad (23)$$

$$\begin{aligned} \varepsilon_n &\leq D(a_{n+1}, p) + k^2(1 - \alpha_n)[(1 - \beta_n)D(a_n, p) + k\beta_nD(a_n, p)] \\ &\quad + k^2\alpha_n[k(1 - \beta_n)D(a_n, p) + k^2\beta_nD(a_n, p)] \\ &\leq D(a_{n+1}, p) + [k^2(1 - \alpha_n)((1 - \beta_n) + k\beta_n)]D(a_n, p) \\ &\quad + [k^2\alpha_n[k(1 - \beta_n) + k^2\beta_n]]D(a_n, p) \\ &\leq D(a_{n+1}, p) + [[k^2(1 - \alpha_n)((1 - \beta_n) + k\beta_n)] \\ &\quad + [k^2\alpha_n(k(1 - \beta_n) + k^2\beta_n)]]D(a_n, p) \\ &\leq D(a_{n+1}, p) + kD(a_n, p) \end{aligned}$$

$\varepsilon_n \leq D(a_{n+1}, p)$, $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Hence, the sequence $\{i_n\}$ is stable with respect to T . \square

Conclusion

The work established the strong convergence and stability of the sequence with respect to T using Ciric-Kannan contraction in a complete F -metric space.

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