On Some Classical-type Integral Inequalities with Convex Functions

Abstract

The article establish a new Hermite-Hadamard type inequalities with convex functions by employing some known concepts in mathematical analysis. The consequences of our main result extend and generalize many other results in the literature.

Keywords: Measurable function, Convexity, Young's and Holder's Inequalities.
AMS (MOS) Subject Classifications: 15A39, 39B62, 35A23.

1 Introduction

Inequalities are very important concepts in mathematics and many other applied sciences. They offer new insights and effective dimension of study and applications. They are applicable in various fields including economics, computer sciences, and statistics. Due to their applications, there has been a constantly increasing interest of researchers in such an area of research. Understanding the properties and behavior of integral of convex functions plays a crucial role in solving complex problems. One of the most well-known inequalities in mathematics for convex function is called Hermite-Hadamard integral inequality. The Hermite-Hadamard inequality plays an important role in non-linear analysis. This inequality has attracted many researchers with various generalizations, refinements, extensions and variants. The inequality is usually valid for convex functions in line with the growing interest in convexity theory.
A result of Hadamard (1893) states that if $f : I \longleftrightarrow \mathbb{R}$ is a convex function, where $I \subset \mathbb{R}$ is an open interval and for all $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \tag{1}$$

and

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} \tag{2}$$

Moreover, each of the inequalities (1) and (2) provides a characterization of convex functions by Hardy, Littlewood and Polya (1934), Roberts and Varbery (1973). Fejer (1906) established weighted versions of the above inequalities as follows: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex function and p a nonnegative integrable function that is symmetric with respect to $\frac{a+b}{2}$. Then,

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)dt \leq \int_a^b f(t)p(t)dt \tag{3}$$

and

$$\int_a^b f(t)p(t)dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)dt \tag{4}$$

Hermite-Hadamard inequality asserts that the mean value of a continuous convex functions $f : [a, b] \longrightarrow \mathbb{R}$ lies between the value of f at the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of f at the end points of this interval, that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

Since its discovery, inequalities (3) and (4) have been proved to be one of the most useful inequalities in mathematical analysis. It's improvement and applications has been provided over the years. Many researchers focused on ways of obtaining a simplified proof of the inequality. In the early nineties, Pecaric and Dragomir (1991) introduced a new dimension to the research work by generating Hadamard's inequality using isotonic linear funtionals. However, in 1992, Pecaric $et.al.$ improved on the inequality by using partial orderings and statistical applications. Dragomir (1992) gave some results on two mappings in connection to the inequalities. Kirmaci $et.al.$ (2007) extended the result of Pecaric *et. al.* (1992) and obtained a result that gave $s - convex$ function for Hadamard-type inequalities. Fink (1998) established the best possible constant for Hadamard inequality. Bessenyei and Pales (2002) shifted the attention of researchers to higher-order generalizations for a generalized convex functions. Bakula $et.al.$ (2008) extened Hadamard-type inequalities for $m-convex$ and $(\alpha, m)-convex$ functions. Elfarissi $et.al.$ (2009) gave further improvement via twice differentiable funtions. Furthermore, Sarikaya and Ozdemir (2010) improved on the result obtained by Bakula *et. al.* (2008), to established some new inequalities of Hadamard-type involving $h - convex$ functions. However, Set $et.al.$ (2010) established results on the integral inequalities involving two functions. Gurbuz (2013) established results on the integral inequalities on product of different type of convex functions and their applications in his Ph.D Thesis. Hussain $et.al.$ (2016) obtained the results on Hermite-Hadamard-type inequalities for k- Riemann-Liouville fractional integrals via two kinds of convexity. Meftah and Souahi (2018) worked on fractional Hermite-Hadamard type inequalities for co-ordinated $mt - convex$ functions. Set $et.al.$ (2018) provided inequalities for product of different convex functions involving certain fractional integral operators. Gurbaz and Ozdemir (2020) developed some inequalities for product of different kinds of convex functions. A collection of several results related to Hermite-Hadamard and Fejer-type inequalities could be found in [Mitrinovic and Lackovic (1985)], [Budak $et.al.$ (2023)], [Samet (2023)]. Kumar $et.al.$ (2015) studied results on generalized fixed point in 2–Metric Space. Furthermore, in 2022, Duraj and Liftaj obtained a common fixed point in S-metric spaces. One of the question of interest is whether the value of the constant in the Hadamard inequality can exceed the original one by introducing more functions in the earlier result. The main purpose of the study is to derive new integral inequalities of Hadamard-type for convex functions as an extension of Sarikaya and Bingol (2024).

2 Preliminary Results

In this section, some concepts are provided in an attempt to obtain an extension to Hadamard inequality. Some of the results could be sourced from Sarikaya and Ozdemir (2010).
Definition 2.1.5 (Criterion For Convexity) [Niculescu and Persson 2006]
A continuous function f on an interval I is convex if and only if

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{f(a)+f(b)}{2} \tag{5}$$

for all $a, b \in I$. A twice differentiable function f on an interval I is convex if and only if $f''(x) \geq 0$, for all $x \in I$

Lemma 3.1

Let I be an open subset of $\mathbb{R}$. Let $g : I \to \mathbb{R}$. We say g is convex if, for all $x, y \in I$ and for all $\alpha \in [0, 1]$ we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) \tag{6}$$

Concave functions are defined analogously.

Lemma 3.2

Let I be an open subset of $\mathbb{R}$ .Let $g : I \to \mathbb{R}$ .If either
1. g' is non decreasing and continuous on I , or
2. $g'' \geq 0$ on I
Then g is convex.

Lemma 3.3

The following statements are equivalent for a mapping $f : [a, b] \longrightarrow \mathbb{R}$:
i. f is convex on [a,b];
ii. for all x, y in $[a, b]$ the mapping $g : [0, 1]$ into $\mathbb{R}$, defined by
$g(t) = f(tx + (1-t)y)$ is convex on [0,1].

Lemma 3.4

Let f and g be real valued, non-negative and convex functions on [a,b]. Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{7}$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \tag{8}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

REMARK: If $a = 0$, $b = 1$ and the convex function $f(x) = cx$ with $g(x) = d(1 - x)$, where c, d are positive constants, then it is observe that the inequalities obtained in (7) and (8) are sharp in the senses that equalities in (10) and (11) hold.

Proof: Since f and g are convex on [a,b], then for t in [0,1] implies

$$f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b) \tag{9}$$

$$g(ta + (1 - t)b) \leq tg(a) + (1 - t)f(b) \tag{10}$$

From (9) and (10) then

$$f(ta+(1-t)b)g(ta+(1-t)b) \leq t^2 f(a)g(a)+(1-t)^2 f(b)g(b)+t(1-t)[f(a)g(b)+f(b)g(a)] \tag{11}$$

From Lemma 3.3, $f(ta + (1 - t)b)$ and $g(ta + (1 - t)b)$ are convex on [0,1], they are integrable on [0,1] and consequently $f(ta+(1-t)b)g(ta+(1-t)b)$ is also integrable on [0,1]. Similarly, f and g are convex on [0,1], they are integrable on [a,b] and hence fg is also integrable on [a,b]. Integrating both sides of (11) yields

$$\int_0^1 f(ta + (1 - t)b)g(ta + (1 - t)b)dt \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b) \tag{12}$$

By substituting $ta + (1 - t)b = x$, it is observe that

$$\int_0^1 f(x)g(x)dt = \frac{1}{b - a}\int_a^b f(x)g(x)dx \tag{13}$$

By substituting (12) into (13)

$$\frac{1}{b - a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b)$$

Also Since f and g are convex on [a,b], then for $t \in [a, b]$ and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &= f\left(\frac{ta + (1 - t)b}{2} + \frac{(1 - t)a + tb}{2}\right) \\ &.g\left(\frac{ta + (1 - t)b}{2} + \frac{(1 - t)a + tb}{2}\right) \qquad (14) \\ &\leq \frac{1}{4}[f(ta + (1 - t)b) + f(1 - t)a + tb] \\ &.[g(ta + (1 - t)b) + g((1 - t)a + tb)] \\ &\leq \frac{1}{4}[f(ta + (1 - t)b)g(ta + (1 - t)b) \\ &+f((1 - t)a + tb)g((1 - t)a + tb)] \\ &+\frac{1}{4}[tf(a) + (1 - t)f(b)][(1 - t)g(a) + tg(b)] \\ &+[(1 - t)f(a) + tf(b)][tg(a) + (1 - t)g(b)] \end{aligned}$$

$$= \frac{1}{4}[f(ta+(1-t)b)g(ta+(1-t)b)$$
$$+f((1-t)a+tb)g((1-t)a+tb)]$$
$$+\frac{1}{4}[2t(1-t)[f(a)g(a)+f(b)g(b)]$$
$$+[t^2+(1-t)^2][f(a)g(b)+f(b)g(a)]] \tag{15}$$

From (7), we integrate both sides of (15) over [0,1] and obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{4}\int_0^1 [f(ta+(1-t)b)g(ta+(1-t)b)$$
$$+f((1-t)a+tb)g((1-t)a+tb)]dt$$
$$+\frac{1}{12}M(a,b)+\frac{1}{6}N(a,b) \tag{16}$$

From (16) it is observe that

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\int_0^1 [f(ta+(1-t)b)g(ta+(1-t)b)dt$$
$$+\frac{1}{12}M(a,b)+\frac{1}{6}N(a,b) \tag{17}$$

By multiplying both sides of (17) by 2 and using (13), the required inequality is achieved.

3 Main Results

The results in the preliminary section are used to obtain new results in this section.

Theorem 4.1:

Let f, g and h be real-valued, nonnegative and convex functions on $[a,b]$. Then

$$3\int_a^b f(x)g(x)h(x)dx \leq \frac{(b-a)}{4}[3M(a,b)+N(a,b)+K(a,b)+C(a,b)] \tag{18}$$

where
$M(a,b)=f(a)g(a)h(a)+f(b)g(b)h(b)$ and $N(a,b)=f(a)g(b)h(a)+f(b)g(a)h(b)$;
$K(a,b)=f(a)g(a)h(b)+f(b)g(b)h(a)$ and $C(a,b)=f(a)g(b)h(b)+f(b)g(a)h(a)$

Proof: Since f, g, and h are convex on $[a,b]$, then for t in [0,1]

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b), \tag{19}$$

$$g(ta+(1-t)b) \leq tg(a)+(1-t)g(b) \tag{20}$$

and

$$h(ta+(1-t)b) \leq th(a)+(1-t)h(b) \tag{21}$$

From (19), (20) and (21), the inequality
$f(ta+(1-t)b)g(ta+(1-t)b)h(ta+(1-t)b)$

$$\leq (tf(a)+(1-t)f(b))\,(tg(a)+(1-t)g(b))\,(th(a)+(1-t)h(b)) \tag{22}$$

$$= t^3 f(a)g(a)h(a) + t(1-t)^2 f(b)g(b)h(a)$$

$$+t^2(1-t)f(a)g(b)h(a) + t^2(1-t)f(b)g(a)h(a)$$

$$+t^2(1-t)f(a)g(a)h(b) + (1-t)^3 f(b)g(b)h(b)$$

$$+t(1-t)^2 f(a)g(b)h(b) + t(1-t)^2 f(b)g(a)h(b).$$

$$\leq t^3 f(a)g(a)h(a) + (1-t)^3 f(b)g(b)h(b) + t^2(1-t)[f(a)g(b)h(a)$$

$$+f(b)g(a)h(a) + f(a)g(a)h(b)] + t(1-t)^2[f(b)g(b)h(a)$$

$$+f(a)g(b)h(b) + f(b)g(a)h(b)]$$

from Lemma 3.3, $f(ta+(1-t)b)$, $g(ta+(1-t)b)$ and $h(ta+(1-t)b)$ are convex on [0,1], they are integrable on [0,1] and consequently $f(ta+(1-t)b)g(ta+(1-t)b)h(ta+(1-t)b)$ is also integrable on [0,1]. Similarly, f, g, h are convex on $[a,b]$, they are integrable on $[a,b]$ and hence fgh is also integrable on $[a,b]$. Integrating both sides of (22) over [0,1],

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)h(ta+(1-t)b)dx$$

$$\leq \frac{1}{4}M(a,b) + \frac{1}{12}N(a,b) + \frac{1}{12}K(a,b) + \frac{1}{12}C(a,b) \tag{23}$$

By substituting $ta+(1-t)b = x$, it is observe that

$$\int_0^1 f(x)g(x)h(x)dx = \frac{1}{(b-a)}\int_a^b f(x)g(x)h(x)dx \tag{24}$$

Imposing (23) on (24) then

$$\frac{1}{(b-a)}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{4}M(a,b) + \frac{1}{12}N(a,b) + \frac{1}{12}K(a,b) + \frac{1}{12}C(a,b) \tag{25}$$

Now multiplying both sides of (25) by $3(b-a)$ the required inequality is achieved as

$$3\int_a^b f(x)g(x)h(x)dx \leq \frac{(b-a)}{4}[3M(a,b) + N(a,b) + K(a,b) + C(a,b)]$$

Theorem 4.2:

Let f, g and h be real-valued, non-negative and convex functions on $[a, b]$. Then

$$\frac{2}{3}(b-a)^2\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)h(tx+(1-t)y)dtdxdy$$

$$\leq \frac{(b-a)}{3}\int_a^b f(x)g(x)h(x)dx+\frac{1}{6}(b-a)^2[M(a,b)+N(a,b)] \tag{26}$$

where
$M(a,b)=f(x)g(x)h(x)+f(y)g(y)h(y)$ and $N(a,b)=f(x)g(y)h(x)+f(y)g(x)h(y)$.

Proof: Since f, g and h are convex on [a,b], then for x, y in $[a, b]$ and t in $[0, 1]$ then

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y) \tag{27}$$

$$g(tx+(1-t)y)\leq tg(x)+(1-t)g(y) \tag{28}$$

and

$$h(tx+(1-t)y)\leq th(x)+(1-t)h(y) \tag{29}$$

From (27), (28) and (29), the following inequalities are obtained

$$f(tx+(1-t)y)g(tx+(1-t)y)h(tx+(1-t)y)$$

$$\leq (tf(x)+(1-t)f(y))(tg(x)+(1-t)g(y))(th(x)+(1-t)h(y)) \tag{30}$$

$$\leq t^3f(x)g(x)h(x)+t(1-t)^2f(y)g(y)h(x)+t^2(1-t)f(x)g(y)h(x)$$

$$+t^2(1-t)f(y)g(x)h(x)+t^2(1-t)f(x)g(x)h(y)+(1-t)^3f(y)g(y)h(y)$$

$$+t(1-t)^2f(x)g(y)h(y)+t(1-t)^2f(y)g(x)h(y)$$

$$=t^3f(x)g(x)h(x)+(1-t)^3f(y)g(y)h(x)+t^2(1-t)[f(x)g(y)h(x)$$

$$+f(y)g(x)h(x)+f(x)g(x)h(y)]+t(1-t)^2[f(y)g(y)h(y)+f(x)g(y)h(y)$$

$$+f(y)g(x)h(y)]$$

From (30), it is achieved that

$$f(tx+(1-t)y)g(tx+(1-t)y)h(tx+(1-t)y)$$

$$\leq t^3f(x)g(x)h(x)+(1-t)^3f(y)g(y)h(x)+t^2(1-t)[f(x)g(y)h(x)$$

$$+f(y)g(x)h(x)+f(x)g(x)h(y)]+t(1-t)^2[f(y)g(y)h(y)$$

$$+f(x)g(y)h(y)+f(y)g(x)h(y)] \tag{31}$$

By integrating both sides of (31) over $[0,1]$, we obtain

$$\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)h(tx+(1-t)y)$$

$$\leq \frac{1}{4}[f(x)g(x)h(x)+f(y)g(y)h(y)]+\frac{1}{12}[f(x)g(y)h(x)+f(y)g(x)h(x)]$$

$$+\frac{1}{12}[f(x)g(x)h(y)+f(y)g(y)h(x)]+\frac{1}{12}[f(x)g(y)h(y)+f(y)g(x)h(y)] \tag{32}$$

Integrating both sides of (32) over $[a,b]$ x $[a,b]$ x $[a,b]$ then,

$$\int_a^b\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)h(tx+(1-t)y)dtdxdy$$

$$\leq \frac{1}{4}(b-a)\left[\int_a^b f(x)g(x)h(x)+\int_a^b f(y)g(y)h(y)\right]+\frac{1}{4}[M(a,b)+N(a,b)] \tag{33}$$

Dividing both sides of (33) by $\frac{3}{2}(b-a)^2$ yields the desired result as

$$\frac{2}{3}(b-a)^2\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)h(tx+(1-t)y)dtdxdy$$

$$\leq \frac{(b-a)}{3}\int_a^b f(x)g(x)h(x)dx+\frac{1}{6}(b-a)^2[M(a,b)+N(a,b)]$$

Theorem 4.3:

Let f, g, h and k be real-valued, non-negative and convex functions on $[a,b]$. Then

$$\frac{5}{2}(b-a)\int_a^b f(x)g(x)h(x)k(x)dx$$

$$\leq \frac{1}{2}(b-a)^2L(a,b)+\frac{1}{8}(b-a)^2[T(a,b)+G(a,b)+Z(a,b)+U(a,b)]$$

$$+\frac{1}{12}(b-a)^2[W(a,b)+R(a,b)+C(a,b)] \tag{34}$$

where

$$L(a,b)=f(a)g(a)h(a)k(a)+f(b)g(b)h(b)k(b)$$

and

$$T(a,b) = f(a)g(b)h(a)k(b) + f(b)g(a)h(b)k(a),$$

also

$$G(a,b) = f(a)g(a)h(b)k(b) + f(b)g(b)h(a)k(a)$$

and

$$Z(a,b) = f(a)g(b)h(b)k(a) + f(b)g(a)h(a)k(b)$$

furthermore

$$U(a,b) = f(a)g(a)h(a)k(b) + f(b)g(b)h(b)k(a)$$

and

$$W(a,b) = f(a)g(b)h(a)k(a) + f(b)g(a)h(b)k(b),$$

with

$$R(a,b) = f(a)g(a)h(b)k(a) + f(b)g(b)h(a)k(b)$$

and

$$C(a,b) = f(a)g(b)h(b)k(b) + f(b)g(a)h(a)k(b)$$

Proof:

Since f, g, h and k are convex on $[a,b]$, then for t in $[0,1]$,

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \quad (35)$$

$$g(ta + (1-t)b) \leq tg(a) + (1-t)g(b) \quad (36)$$

$$h(ta + (1-t)b) \leq th(a) + (1-t)h(b) \quad (37)$$

and

$$k(ta + (1-t)b) \leq tk(a) + (1-t)k(b) \quad (38)$$

From (35), (36), (37) and (38), then

$$f(ta + (1-t)b)g(ta + (1-t)b)h(ta + (1-t)b)k(ta + (1-t)b)$$

$$\leq (tf(a) + (1-t)f(b))(tg(a) + (1-t)g(b))(th(a) + (1-t)h(b))(tk(a) + (1-t)b) \quad (39)$$

$$= \left(t^2 f(a)g(a) + (1-t)^2 f(b)g(b) + t(1-t)[f(a)g(b) + f(b)g(a)]\right)$$

$$\left(t^2 h(a)k(a) + (1-t)^2 h(b)k(b) + t(1-t)[h(a)k(b) + h(b)k(a)]\right)$$

$$f(ta + (1-t)b)g(ta + (1-t)b)h(ta + (1-t)b)k(ta + (1-t)b)$$

then

$$\leq t^4 f(a)g(a)h(a)k(a) + t^2(1-t)^2 f(a)g(a)h(b)k(b) + t^3(1-t)[f(a)g(a)h(a)k(b)$$

$$+f(a)g(a)h(b)k(a)] + t^2(1-t)^2 f(b)g(b)h(a)k(a) + (1-t)^4 f(b)g(b)h(b)k(b)$$

$$+t(1-t)^3[f(b)g(b)h(a)k(b)+f(b)g(b)h(b)k(a)]+t^3(1-t)[f(a)g(b)h(a)k(a)$$

$$+f(a)g(b)h(a)k(a)]+t(1-t)^3[f(a)g(b)h(b)k(b)+f(b)g(a)h(b)k(a)]$$

$$+t^2(1-t)^2[f(a)g(b)h(a)k(b)+f(a)g(b)h(b)k(b)+f(b)g(a)h(a)k(b)$$

$$+f(b)g(a)h(b)k(a)$$

From (39), it is achieved that

$$f(ta+(1-t)b)g(ta+(1-t)b)h(ta+(1-t)b)k(ta+(1-t)b)$$

$$\leq t^4 f(a)g(a)h(a)k(a)+t^2(1-t)^2 f(a)g(a)h(b)k(b)+t^3(1-t)[f(a)g(a)h(a)k(b)$$

$$+f(a)g(a)h(b)k(a)]+t^2(1-t)^2 f(b)g(b)h(a)k(a)+(1-t)^4 f(b)g(b)h(b)k(b)$$

$$+t(1-t)^3[f(b)g(b)h(a)k(b)+f(b)g(b)h(b)k(a)]+t^3(1-t)[f(a)g(b)h(a)k(a)$$

$$+f(a)g(b)h(a)k(a)]+t(1-t)^3[f(a)g(b)h(b)k(b)+f(b)g(a)h(b)k(a)]$$

$$+t^2(1-t)^2[f(a)g(b)h(a)k(b)+f(a)g(b)h(b)k(b)+f(b)g(a)h(a)k(b)$$

$$+f(b)g(a)h(b)k(a) \tag{40}$$

By Lemma 3.3, $f(ta+(1-t)b)$ and $g(ta+(1-t)b)$ are convex on $[0,1]$ likewise $h(ta+(1-t)b)$ and $k(ta+(1-t)b)$ are also convex on $[0,1]$. They are integrable on [0,1] and consequently $f(ta+(1-t)b)g(ta+(1-t)b)h(ta+(1-t)b)k(ta+(1-t)b)$ is also integrable on $[0,1]$. Similarly, f, g, h and k are convex on $[a,b]$, they are integrable on $[a,b]$ and hence $fghk$ is integrable on $[a,b]$. Integrating both sides of (40) over $[0,1]$ yields

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)h(ta+(1-t)b)k(ta+(1-t)b)dt$$

$$\leq \frac{1}{5}L(a,b)+\frac{1}{20}[T(a,b)+G(a,b)+Z(a,b)+U(a,b)]+\frac{1}{30}[W(a,b)+R(a,b)+C(a,b)] \tag{41}$$

By substituting $ta+(1-t)b=x$, it is observe that

$$\int_0^1 f(x)g(x)h(x)k(x)dt = \frac{1}{b-a}f(x)g(x)h(x)k(x)dx \tag{42}$$

Substitute (42) in (41) then

$$\frac{1}{(b-a)}\int_a^b f(x)g(x)h(x)k(x)dx$$

$$\leq \frac{1}{5}L(a,b)+\frac{1}{20}[T(a,b)+G(a,b)+Z(a,b)+U(a,b)]+\frac{1}{30}[W(a,b)+R(a,b)+C(a,b)] \tag{43}$$

Multiplying both sides of (43) by $\frac{5}{2}(b-a)^2$, the result is achieved as:

$$\frac{5}{2}(b-a)\int_a^b f(x)g(x)h(x)k(x)dx$$

$$\leq \frac{1}{2}(b-a)^2L(a,b)+\frac{1}{8}(b-a)^2[T(a,b)+G(a,b)+Z(a,b)+U(a,b)]$$

$$+\frac{1}{12}(b-a)^2[W(a,b)+R(a,b)+C(a,b)]$$

Conclusion

The concept of Hadamard-type inequalities with convex functions are refined and used as an essential tools in the work of Sarikaya $et.al.$ (2024). Investigation and improvement of several other functions are use.

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