Well-posedness and exponential stability of the dissipative Bresse-Timoshenko system without second spectrum

Abstract

This paper investigates the dissipative Bresse-Timoshenko system without second spectrum. By using the theory of C_0-semigroup, the well-posedness and exponential stability results are got.

Keywords: Bresse-Timoshenko system; Well-posedness; Exponential stability.

MSC: 35B40, 35B35, 81U30, 65H04.

1 Introduction and main results

In 1921, Tymoshenko [14] optimized the Euler-Bernoulli beam model and the Rayleigh beam model and proposed the following hyperbolic system of two coupled wave equations

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \end{cases} \tag{1.1}$$

*

which is called *Timoshenko beam model*, where φ and ψ are the deflection of the beam from its equilibrium position and the rotation of the neutral axis, respectively, $\rho_1 = \rho A, \rho_2 = \rho I, b = EI$ and $k = k'GA$ are positive constants with ρ is the density, A is the cross-sectional area, I is the second moment of area of the cross-sectional area, E is the Young modulus of elasticity, G is the modulus of rigidity, k' is the transverse shear factor. However, it was later discovered that the Timoshenko beam model admits two wave speeds

$$\sqrt{k/\rho_1} \text{ and } \sqrt{b/\rho_2},$$

which contributes to a physical paradox called the *second spectrum* (see, for example, [6, 7, 10]). Based on these reasons, Elishakoff [8] proposed the following truncated version model by combining d'Alembert's principle for dynamic equilibrium from Timoshenko hypothesis,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \end{cases} \tag{1.2}$$

which eliminates the anomaly of the second spectrum since it admits one wave speed

$$\sqrt{b/[\rho_2(1 + \rho_1 b/k\rho_2)]}.$$

The model (1.2) is called *Bresse-Timoshenko system without second spectrum* and has been extensively in recent years (see [1, 2, 5, 9, 13] and references therein).

In this paper, we consider the following dissipative Bresse-Timoshenko system without second spectrum proposed in [4]

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \end{cases} \tag{1.3}$$

where $\mu > 0$ represents the damping coefficient acting on displacement function. Moreover, we consider the boundary conditions of Dirichlet-Neumann type given by

$$\varphi(0, t) = \varphi(L, t) = \psi_x(0, t) = \psi_x(L, t) = 0, \quad t \geq 0. \tag{1.4}$$

and initial conditions given by

$$\varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \quad x \in (0, L). \tag{1.5}$$

When $\rho_2 = 0$, problem (1.3) with boundary conditions of Dirichlet-Neumann type was studied in [3] and the authors showed the exponential decay of the energy. In [4], the authors studied (1.3) with boundary conditions of Dirichlet-Dirichlet or Neumann-Dirichlet type, and the exponential decay of the energy was obtained. However,

1. for boundary conditions of Dirichlet-Neumann type only the case $\rho_2 = 0$ was considered in [3];

2. the well-posedness results were not considered in [4].

Based on the above reasons, we will consider the problem (1.3)-(1.5). The C_0-semigroup theory are applied to study the well-posedness and exponential stability, which is different from [3] and [4], where the multiplying method and energy method were used to study the exponential stability.

From $(1.3)_1$, we get

$$\psi_x = \frac{\rho_1}{k}\varphi_{tt} + \frac{\mu}{k}\varphi_t - \varphi_{xx}.$$

By substituting ψ_x into $(1.3)_2$, we have

$$-\rho_2\varphi_{ttxx} - b\left(\frac{\rho_1}{k}\varphi_{ttxx} + \frac{\mu}{k}\varphi_{txx} - \varphi_{xxxx}\right) + k\varphi_{xx} + k\left(\frac{\rho_1}{k}\varphi_{tt} + \frac{\mu}{k}\varphi_t - \varphi_{xx}\right) = 0,$$

i.e., problem (1.3)-(1.5) can be transformed to

$$\begin{cases} (k\rho_1 I - (b\rho_1 + k\rho_2)\partial_{xx})\,\varphi_{tt} + (k\mu I - b\mu\partial_{xx})\,\varphi_t + bk\varphi_{xxxx} = 0, & \text{in } (0,L)\times(0,\infty), \\ \varphi(0,t) = \varphi(L,t) = \varphi_{xx}(0,t) = \varphi_{xx}(L,t) = 0, & t > 0, \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), & x \in (0,L). \end{cases}$$

Let

$$\mathbf{A} := \partial_{xxxx}, \tag{1.6}$$

$$\mathbf{B} := k\rho_1 I - (b\rho_1 + k\rho_2)\partial_{xx}, \tag{1.7}$$

$$\mathbf{C} := k\mu I - b\mu\partial_{xx}. \tag{1.8}$$

Obviously, $\mathbf{A}$ is a positive self-adjoint operator from $\{\zeta \in H^4(0,L)\cap H_0^1(0,L) : \zeta_{xx} \in H_0^1(0,L)\}$ to $L^2(0,L)$, which can be extended as an isomorphism from $H_*^3(0,L)$ to $H^{-1}(0,L)$; $\mathbf{B}$ and $\mathbf{C}$ are positive self-adjoint operators from $H^2(0,L)\cap H_0^1(0,L)$ to $L^2(0,L)$, which can be extended as an isomorphism from $H_0^1(0,L)$ to $H^{-1}(0,L)$, where

$$H_*^3(0,L) := \left\{\zeta \in H^3(0,L)\cap H_0^1(0,L) : \zeta_{xx} \in H_0^1(0,L)\right\}. \tag{1.9}$$

Then, problem (1.3)-(1.5) can be written as the following abstract form in $H_0^1(0,L)$:

$$\begin{cases} \varphi_{tt} + \mathbf{B}^{-1}\mathbf{C}\varphi_t + bk\mathbf{B}^{-1}\mathbf{A}\varphi = 0, & t > 0, \\ \varphi(0) = \varphi_0 \in H^2(0,L)\cap H_0^1(0,L), \varphi_t(0) = \varphi_1(x) \in H_0^1(0,L). \end{cases} \tag{1.10}$$

Let

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ -bk\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{C} \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} := \begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}, \quad \Phi_0 := \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \tag{1.11}$$

and

$$\mathcal{H} := \left(H^2(0,L) \cap H_0^1(0,L)\right) \times H_0^1(0,L). \tag{1.12}$$

It is obvious that $\mathcal{H}$ is a Hilbert space with scalar product

$$\langle \Phi, \Phi^* \rangle_{\mathcal{H}} = \rho_1 \langle \phi, \phi^* \rangle + \rho_2 \langle \phi_x, \phi_x^* \rangle + b \langle \mathbf{T}\varphi_x, \varphi_x^* \rangle, \quad \forall\, \Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in \mathcal{H},\ \Phi^* = \begin{pmatrix} \varphi^* \\ \phi^* \end{pmatrix} \in \mathcal{H}, \tag{1.13}$$

where

$$\mathbf{T} := -\frac{k}{b\rho_1 + k\rho_2} \left(\rho_2 I + b\rho_1^2 \mathbf{B}^{-1}\right) \circ \partial_{xx} \tag{1.14}$$

and $\langle \cdot, \cdot \rangle$ denotes the standard L^2-scalar product.

With the above preparations, one can see $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, and problem (1.10) can be written as

$$\begin{cases} \dfrac{d}{dt}\Phi = \mathcal{A}\Phi \in \mathcal{H}, \quad t > 0, \\ \Phi(0) = \Phi_0, \end{cases} \tag{1.15}$$

where

$$D(\mathcal{A}) = \left\{ \Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in \mathcal{H} : \varphi \in H_*^3(0,L), \phi \in H^2(0,L) \cap H_0^1(0,L) \right\}. \tag{1.16}$$

Theorem 1.1. *$\mathcal{A}$ generates a C_0-semigroup of contraction $\left(e^{t\mathcal{A}}\right)_{t \geq 0}$ on $\mathcal{H}$, which is exponential stable, i.e., there exist two positive constants M and α such that*

$$\left\| e^{t\mathcal{A}} \right\| \leq M e^{-\alpha t}$$

for any $t \geq 0$.

The rest of this paper is devoted to prove the above theorem.

2 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 by using the following two theorems. Let $\theta \in \mathbb{C}$, A be an operator, and f, g be two quantities, Reθ denotes the real part of θ, $\overline{\theta}$ denotes the conjugate complex of θ, $\rho(A)$ denotes the resolvent set of A, the notation $f \lesssim g$ means there exists a constant such that $f \leq Cg$.

To show $\mathcal{A}$ generates a C_0-semigroup of contraction, we need the following theorem[11, Theoem 1.2.4], which can be seen as a corollary of the Lumer-Phillips theorem.

Theorem 2.1. *Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and A be a linear operator with dense domain $D(A)$ in H. If A is dissipative, i.e.,*

$$\mathrm{Re}\langle \zeta, A\zeta \rangle_H \leq 0$$

for any $\zeta \in H$ and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ of contraction on H.

To prove the exponential stability of a C_0-semigroup we need the following theorem [11, Theoem 1.3.2].

Theorem 2.2. *Let $S(t)$ be a C_0-semigroup of contractions on a Hilbert space with infinitesimal generator A. Then $S(t)$ is exponentially stable if and only if*

$$\rho(A) \supset i\mathbb{R} := \{i\beta : \beta \in \mathbb{R}\}$$

and

$$\limsup_{|\beta|\to\infty} \left\| (i\beta I - A)^{-1} \right\| < \infty.$$

Proof of Theorem 1.1. It is obvious that $D(\mathcal{A})$ is dense in $\mathcal{H}$. We first show that $\mathcal{A}$ is dissipative. For any

$$\Phi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in D(\mathcal{A}),$$

by (1.11) and (1.6),

$$\begin{aligned} \mathcal{A}\Phi &= \begin{pmatrix} 0 & I \\ -bk\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{C} \end{pmatrix} \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \\ &= \begin{pmatrix} \phi \\ -\mathbf{B}^{-1}\mathbf{C}\phi - bk\mathbf{B}^{-1}\mathbf{A}\varphi \end{pmatrix} = \begin{pmatrix} \phi \\ -\mathbf{B}^{-1}\mathbf{C}\phi - bk\mathbf{B}^{-1}\varphi_{xxxx} \end{pmatrix}. \end{aligned}$$

Then it follows from (1.13), (1.14), (1.7), and (1.8) that

$$\begin{aligned} \mathrm{Re}\langle \Phi, \mathcal{A}\Phi \rangle_{\mathcal{H}} = \mathrm{Re} \int_0^L \rho_1 \phi \left[-\mathbf{B}^{-1}\mathbf{C}\overline{\phi} - bk\mathbf{B}^{-1}\overline{\varphi}_{xxxx} \right] dx + \mathrm{Re} \int_0^L \rho_2 \phi_x \left[-\mathbf{B}^{-1}\mathbf{C}\overline{\phi} - bk\mathbf{B}^{-1}\overline{\varphi}_{xxxx} \right]_x dx \\ - \mathrm{Re} \int_0^L \frac{bk}{b\rho_1 + k\rho_2} \left(\rho_2 I + b\rho_1^2 \mathbf{B}^{-1} \right) \varphi_{xxx} \overline{\phi}_x dx \end{aligned}$$

$$
\begin{aligned}
&=\mathrm{Re}\int_0^L \mathbf{B}^{-1}\overline{\left[-\rho_1 bk\varphi_{xx}+\rho_2 bk\varphi_{xxxx}+\frac{bk}{b\rho_1+k\rho_2}\left(\rho_2\mathbf{B}+b\rho_1^2 I\right)\varphi_{xx}\right]}\phi_{xx}dx\\
&\quad-\mathrm{Re}\int_0^L \mathbf{B}^{-1}\left[\rho_1(\mathbf{C}\overline{\phi})\phi+\rho_2(\mathbf{C}\overline{\phi}_x)\phi_x\right]dx\\
&=\mathrm{Re}\int_0^L \mathbf{B}^{-1}\overline{\left[\underbrace{-\rho_1 bk\varphi_{xx}+\rho_2 bk\varphi_{xxxx}+\frac{bk^2\rho_1\rho_2}{b\rho_1+k\rho_2}\varphi_{xx}-\rho_2 bk\varphi_{xxxx}+\frac{b^2k\rho_1^2}{b\rho_1+k\rho_2}\varphi_{xx}}_{=0}\right]}\phi_{xx}dx\\
&\quad-\mathrm{Re}\left(\mu k\rho_1\int_0^L \mathbf{B}^{-1}\overline{\phi}\phi dx+(\mu b\rho_1+\mu k\rho_2)\int_0^L \mathbf{B}^{-1}\overline{\phi}_x\phi_x dx+\mu b\rho_2\int_0^L \mathbf{B}^{-1}\overline{\phi}_{xx}\phi_{xx}dx\right)\\
&=-\left(\mu k\rho_1\left\|\mathbf{B}^{-\frac{1}{2}}\phi\right\|^2_{L^2(0,L)}+(\mu b\rho_1+\mu k\rho_2)\left\|\mathbf{B}^{-\frac{1}{2}}\phi_x\right\|^2_{L^2(0,L)}+\mu b\rho_2\left\|\mathbf{B}^{-\frac{1}{2}}\phi_{xx}\right\|^2_{L^2(0,L)}\right)\\
&\le 0, \qquad (2.1)
\end{aligned}
$$

hence $\mathcal{A}$ is dissipative.

Secondly, we prove that $0\in\rho(\mathcal{A})$. At first, we show $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is surjective, i.e., for given $G=\begin{pmatrix} g_1\\ g_2\end{pmatrix}\in\mathcal{H}$, we need to show there exists $\Phi=\begin{pmatrix}\varphi\\ \phi\end{pmatrix}\in D(\mathcal{A})$ satisfying

$$
-\mathcal{A}\Phi=G, \tag{2.2}
$$

this means

$$
\begin{cases}
-\phi=g_1\in H^2(0,L)\cap H_0^1(0,L),\\
\mathbf{C}\phi+bk\varphi_{xxxx}=\mathbf{B}g_2\in H^{-1}(0,L),\\
\varphi(0)=\varphi(L)=\varphi_{xx}(0)=\varphi_{xx}(L)=0.
\end{cases}
$$

Then we get

$$
\phi=-g_1\in H^2(0,L)\cap H_0^1(0,L),
$$

and φ satisfies

$$
\begin{cases}
bk\varphi_{xxxx}=\mathbf{B}g_2+\mathbf{C}g_1\in H^{-1}(0,L),\\
\varphi(0)=\varphi(L)=\varphi_{xx}(0)=\varphi_{xx}(L)=0.
\end{cases} \tag{2.3}
$$

The standard theory of elliptic equations shows that (2.3) admits a unique solution $\varphi\in H_*^3(0,L)$ and

$$
\|\varphi\|_{H_*^3(0,L)}\lesssim\|\mathbf{B}g_2+\mathbf{C}g_1\|_{H^{-1}(0,L)}\lesssim\|G\|_{\mathcal{H}}.
$$

So the above analysis shows that (2.2) admits a unique solution $\Phi\in D(A)$ and

$$
\|\Phi\|_{H_*^3(0,L)\times\left(H^2(0,L)\cap H_0^1(0,L)\right)}\lesssim\|G\|_{\mathcal{H}}. \tag{2.4}
$$

Following (2.4), we get, $\mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ is injective. So $\mathscr{A}^{-1} : \mathscr{H} \to D(\mathscr{A})$ exists, and by (2.4) again $\mathscr{A}^{-1}$ is a bounded linear operator on $\mathscr{H}$. Therefore, we get $0 \in \rho(\mathscr{A})$.

So by Theorem 2.1, $\mathscr{A}$ generates a C_0-semigroup of contraction $\left(e^{t\mathscr{A}}\right)_{t\geq 0}$ on $\mathscr{H}$.

Next we show $\left(e^{t\mathscr{A}}\right)_{t\geq 0}$ is exponentially stable by using Theorem 2.2. We first show

$$i\mathbb{R} \subset \rho(\mathscr{A}) \tag{2.5}$$

by contradiction argument. If (2.5) in not true, since we have shown $0 \in \rho(\mathscr{A})$, by the proof of [11, Theoem 2.2.1], there is a constant $\omega \in \mathbb{R}$ with $\|\mathscr{A}^{-1}\| \leq |\omega| < \infty$ such that $\{i\beta : |\beta| < |\omega|\} \subset \rho(\mathscr{A})$ and

$$\sup_{|\beta|<|\omega|} \left\|(i\beta - \mathscr{A})^{-1}\right\| = \infty.$$

Then there exists a sequence $\{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$ with $\beta_n \to \omega$ ($n \to \infty$), $|\beta_n| < |\omega|$ and a sequence

$$\{\Phi_n\}_{n=1}^{\infty} = \left\{ \begin{pmatrix} \varphi_n \\ \phi_n \end{pmatrix} \right\}_{n=1}^{\infty} \subset D(\mathscr{A})$$

with

$$\begin{aligned}
\|\Phi_n\|_{\mathscr{H}}^2 &= \rho_1 \|\phi_n\|_{L^2(0,L)}^2 + \rho_2 \|\phi_{nx}\|_{L^2(0,L)}^2 + b\langle \mathbf{T}\varphi_{nx}, \varphi_{nx}\rangle \\
&= \rho_1 \|\phi_n\|_{L^2(0,L)}^2 + \rho_2 \|\phi_{nx}\|_{L^2(0,L)}^2 + \frac{kb}{b\rho_1 + k\rho_2}\left(\rho_2 \|\varphi_{nxx}\|_{L^2(0,L)}^2 + b\rho_1^2 \left\|\mathbf{B}^{-\frac{1}{2}}\varphi_{nxx}\right\|_{L^2(0,L)}^2\right) \\
&= 1
\end{aligned} \tag{2.6}$$

such that

$$\|(i\beta_n - \mathscr{A})\Phi_n\|_{\mathscr{H}} \to 0 \tag{2.7}$$

as $n \to \infty$, i.e.,

$$i\beta_n\varphi_n - \phi_n \to 0 \qquad \text{in } H^2(0,L) \cap H_0^1(0,L), \tag{2.8}$$

$$i\beta_n\phi_n + \mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx} \to 0 \qquad \text{in } H_0^1(0,L). \tag{2.9}$$

Similar to the proof of (2.1), we get

$$\begin{aligned}
&\mathrm{Re}\langle (i\beta_n I - \mathscr{A})\Phi_n, \Phi_n\rangle_{\mathscr{H}} \\
&\quad = \mu k\rho_1 \left\|\mathbf{B}^{-\frac{1}{2}}\phi_n\right\|_{L^2(0,L)}^2 + (\mu b\rho_1 + \mu k\rho_2)\left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nx}\right\|_{L^2(0,L)}^2 + \mu b\rho_2 \left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nxx}\right\|_{L^2(0,L)}^2,
\end{aligned}$$

which, together with (2.6) and (2.7), implies

$$\lim_{n\to\infty} \|\phi_n\|_{L^2(0,L)} = 0 \text{ and } \lim_{n\to\infty} \|\phi_{nx}\|_{L^2(0,L)} = 0, \tag{2.10}$$

where we have used the facts that

$$\|\phi_n\|_{L^2(0,L)} \lesssim \left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nx}\right\|_{L^2(0,L)} \text{ and } \|\phi_{nx}\|_{L^2(0,L)} \leq \left\|\mathbf{B}^{-\frac{1}{2}}\phi_{nxx}\right\|_{L^2(0,L)}.$$

Then, it follows from (2.9) that

$$\begin{aligned}\limsup_{n\to\infty}\|\varphi_{nxx}\|_{L^2(0,L)} &\lesssim \limsup_{n\to\infty}\left\|\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)}\\ &\lesssim \lim_{n\to\infty}\left\|i\beta_n\phi_n + \mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} + \lim_{n\to\infty}\|\phi_n\|_{L^2(0,L)} = 0,\end{aligned} \tag{2.11}$$

where we have used the facts that

$$\|\varphi_{nxx}\|_{L^2(0,L)} \lesssim \left\|\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)},$$

$|\beta_n| \leq \omega + 1 < \infty$ for n large enough since $\beta_n \to \omega$ $(n \to \infty)$ and $|\omega| < \infty$, and

$$\left\|\mathbf{B}^{-1}\mathbf{C}\phi_n\right\|_{L^2(0,L)} \lesssim \|\phi_n\|_{L^2(0,L)}.$$

By (2.8) and (2.10) we get

$$\begin{aligned}\limsup_{n\to\infty}\left\|\mathbf{B}^{-\frac{1}{2}}\varphi_{nxx}\right\|_{L^2(0,L)} &\lesssim \limsup_{n\to\infty}\|\varphi_{nx}\|_{L^2(0,L)} \leq \frac{2}{|\omega|}\limsup_{n\to\infty}\|i\beta_n\varphi_{nx}\|_{L^2(0,L)}\\ &\lesssim \lim_{n\to\infty}\|i\beta_n\varphi_{nx} - \phi_{nx}\| + \lim_{n\to\infty}\|\phi_{nx}\|_{L^2(0,L)} = 0,\end{aligned} \tag{2.12}$$

where we have used the facts that

$$\left\|\mathbf{B}^{-\frac{1}{2}}\varphi_{nxx}\right\|_{L^2(0,L)} \lesssim \|\varphi_{nx}\|_{L^2(0,L)}$$

and $|\beta_n| \geq \frac{|\omega|}{2}$ for n large since $\beta_n \to \omega$ $(n \to \infty)$ and $|\omega| \geq \|\mathcal{A}^{-1}\| > 0$.

By (2.10), (2.11) and (2.12), we get $\lim_{n\to\infty}\|\Phi_n\|_{\mathcal{H}} = 0$, which contradicts $\|\Phi_n\|_{\mathcal{H}} = 1$. So (2.5) holds.

We now prove

$$\limsup_{|\beta|\to\infty}\left\|(i\beta I - A)^{-1}\right\| < \infty \tag{2.13}$$

by a contradiction argument again. Suppose that (2.13) is not true. Then there exists a sequence $\{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}$ with $|\beta_n| \to \infty$ $(n \to \infty)$, and a sequence

$$\{\Phi_n\}_{n=1}^{\infty} = \left\{\begin{pmatrix}\varphi_n\\ \phi_n\end{pmatrix}\right\}_{n=1}^{\infty} \subset D(\mathcal{A})$$

satisfying (2.6) such that (2.7) holds. Again we also have (2.8), (2.9), (2.10) and (2.12) except for (2.11) since in this case $\{\beta_n\}_{n=1}^{\infty}$ is unbounded.

Since

$$\|i\beta_n\phi_n\|_{L^2(0,L)} \lesssim \left\|i\beta_n\phi_n+\mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)}+\left\|\mathbf{B}^{-1}\mathbf{C}\phi_n\right\|_{L^2(0,L)}+\left\|bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)}$$
$$\lesssim \left\|i\beta_n\phi_n + \mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} + \|\phi_n\|_{L^2(0,L)} + \|\varphi_{nxx}\|_{L^2(0,L)}$$

it follows from $\left\{\|\varphi_{nxx}\|_{L^2(0,L)}\right\}_{k=1}^{\infty}$ and $\left\{\|\phi_n\|_{L^2(0,L)}\right\}_{k=1}^{\infty}$ are bounded sequences (see (2.6)), and (2.9) that

$$\left\{\|i\beta_n\phi_n\|_{L^2(0,L)}\right\}_{n=1}^{\infty} \text{ is a bounded sequence.} \tag{2.14}$$

Similar to (2.11), we get

$$\|\varphi_{nxx}\|_{L^2(0,L)} \lesssim \left\|\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} \lesssim \left\|i\beta_n\phi_n + \mathbf{B}^{-1}\mathbf{C}\phi_n + bk\mathbf{B}^{-1}\varphi_{nxxxx}\right\|_{L^2(0,L)} + \|i\beta_n\phi_n\|_{L^2(0,L)}.$$

Then we get from (2.9) and (2.14) that

$$\left\{\|\varphi_{nxx}\|_{L^2(0,L)}\right\}_{n=1}^{\infty} \text{ is a bounded sequence,}$$

which, together with $|\beta_n| \to \infty$ as $n \to \infty$ that

$$\lim_{n\to\infty} \left\|\frac{\varphi_{nxx}}{\beta_n}\right\|_{L^2(0,L)} = 0. \tag{2.15}$$

Dividing (2.8) by β_n, we get

$$i\varphi_n - \frac{\phi_n}{\beta_n} \to 0 \text{ in } H^2(0,L) \cap H_0^1(0,L).$$

Then it follows from (2.15) that

$$\limsup_{n\to\infty} \|\varphi_{nxx}\|_{L^2(0,L)} \le \lim_{n\to\infty} \left\|i\varphi_{nxx} - \frac{\phi_{nxx}}{\beta_n}\right\|_{L^2(0,L)} + \lim_{n\to\infty} \left\|\frac{\varphi_{nxx}}{\beta_n}\right\|_{L^2(0,L)} = 0,$$

i.e., (2.11) also holds. Then we get $\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = 0$, which contradicts $\|\Phi_n\|_{\mathcal{H}} = 1$. So (2.13) holds. The desired result follows from Theorem 2.2, (2.5) and (2.13). □

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