Convergence and Stability Analysis of Ciric-Kannan- α-ψ Contraction on Some Multi-Step Iterative Procedures in F-Metric Space

Abstract

In this work, we study the convergence and stability analysis of Ciric-Kannan α- ψ contraction using some multi-step iterative procedures in F-metric space. Metric fixed point theory has many applications in functional analysis. The contractive conditions with underlying functions play an important role for finding solutions of metric fixed point problems. Some topological properties of such spaces were discussed. By considering this notion, they proved fixed-point theorems for some classes of contractive mappings. The work established the strong convergence and stability of the sequence with respect to operator T using Ciric-Kannan contraction in a complete F-metric space.

Keyword: Convergence, Ciric-Kannan-α-ψ-contraction, Stability and F-metric space.
Mathematical Subject Classification: 32H50.

1. Introduction

Two distance-controlled functions have been used extensively by the researchers working on fixed-point theory for obtaining fixed points of mappings such as contractive or expansive mappings in nature. Metric fixed point theory has many applications in functional analysis. The contractive conditions with underlying functions play an important role for finding solutions of metric fixed point problems. Over the years, Banach contraction principle has been generalized in different directions by several mathematicians, [1] introduced the concept of α-ψ contraction and α-admissible mapping and established various fixed point theorems for such mappings in complete metric spaces. His work was later modified by [2] which are generalizations of the results in [1].
[3] studied the concept of sequential F-metric spaces which is a generalization of the usual metric spaces, b-metric spaces, JS-metric spaces, and mainly F-metric spaces. Some topological properties of such spaces were discussed. By considering this notion, they proved fixed-point theorems for some classes of contractive mappings. Examples were given in order to examine the validity of the underlying space and in support of there fixed-point theorems. They applied the fixed-point theorem to obtain solution of a system of linear algebraic equations.
[4] used the general class of contractive-like operators introduced by [5] to prove strong convergence and stability results for Picard-Mann hybrid iterative schemes considered in a real normed linear space. They established the strong convergence and stability of the Picard iterative scheme as a corollary.
A new concept of Ciric-Kannan-α-ψ-contractions in the setting of F-metric space was introduced in [6]. Using this idea, endowed with suitable hypotheses, some fixed point theorems for such mappings in F-complete metric space were established. As an application, existence conditions for a solution of nonlinear neutral differential equations were investigated to show the usability of the obtained results.
[7] presented an approximation method for finding the fixed point of generalized Suzuki nonexpansive mappings on hyperbolic spaces. Strong and Δ-convergence

theorems are proved using the Noor iterative process for generalizing Suzuki nonexpansive mappings (GSNM) on uniform convex hyperbolic spaces. Due to the richness of uniform convex hyperbolic spaces, the results of there paper can be used as an extension and generalization of many famous results in Banach spaces together with CAT(0) spaces. [8] introduced new hybrid iterative schemes, namely Jungck-Kirk-SP and Jungck-Kirk-CR iterative schemes and proved convergence and stability results for these iterative schemes using certain quasi-contractive operators. Numerical examples showing the comparison of convergence rate and applications of the newly introduced iterative schemes were also provided. The obtained results improve, generalize and extended the works of Olatinwo.

[9] introduced the notion of a $(\phi,\ \psi)$-metric space, which extends the metric space concept. In that space, the symmetry property is preserved. They presented a natural topology $\tau(\phi,\psi)$ in such spaces and discuss their topological properties. They also established the Banach contraction principle in the context of $(\phi,\ \psi)$-metric spaces and illustrated the significance of their main theorems with examples. Ultimately, as a means of application, the existence of a unique solution of Fredholm type integral equations in one and two dimensions were ensured with an example.

[1] proved three fixed point theorems for α-ψ class of operators in complete metric spaces. There results extended some results as well as some known fixed point theorems due to Banach, Kannan, Chatterjea, Zamfirescu, Berinde, Suzuki, Ciric, Nieto, Lopez, and many others. They proved that α-ψ contractions unify large classes of contractive type operators, whose fixed points can be obtained by means of Picard iteration. Finally, they utilize there results to discuss the existence and uniqueness of solutions to a class of quadratic integral equations.

In this work, we study the convergence and stability analysis of Ciric-Kannan α- ψ contraction using some multi-step iterative procedures in F-metric space.

2. Preliminary

This section discussed some concepts that will be needed in the sequel. Throughout this work Ω represents the family of monotone functions ψ on $[0,\infty)$ satisfying $\sum_{j=1}^{\infty}\psi^j(t) < \infty\ \forall\ 0 < t$.

Lemma 2.1. [10]. *For every self map $\psi \in [0,\infty)$, the following holds: if ψ is increasing, $0 < t$, $0 = \lim\limits_{j\to\infty}\psi^j(t)$ yields $t > \psi(t)$.*

Definition 2.2. [10] Let (X, D) be an F-metric space $T : X \to X$ be a map. We call T an $\alpha - \psi$- contraction if $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Omega$ with

$$D(T_x, T_y)\alpha(x, y) \leq \psi(D(x, y)) \tag{1}$$

Definition 2.3. [10]. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be maps. We call T an α-admissible if $1 \leq \alpha(x, y)$ yields $\alpha(T_x, T_y) \geq 1 \; \forall x, y \in X$

Example 2.4. [10] Let $X = (0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by
$T_x = ln(x) \; \forall x \in X$
and
$\alpha(x, y) = \begin{cases} 2 & \text{if} \quad y \leq x \\ 0 & \text{if} \quad y > x \end{cases}$ Then, T is α-admissible.

Definition 2.5. [11]. Let J denote the closed unit interval, T a selfmap of J.

The Ishikawa iterates of T are defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n \tag{2}$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n \; n \geq 0 \tag{3}$$

were $x_0 \in J$ and $\alpha_n, \;\; \beta_n)$ satisfy the conditions
(i) $0 \leq \alpha_n \leq \beta_n \leq 1 \; \forall n$
(ii) $\lim \beta_n = 0$ and
(iii) $\sum \alpha_n \beta_n = \infty$

Definition 2.6. [12] Let X denote the Banach space, T a selfmap of X forall $x_0 \in X$

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n T y_n \\ y_n &= (1 - b_n)x_n + b_n T z_n \\ z_n &= (1 - c_n)x_n + c_n T x_n \; \forall n \geq 0. \end{aligned}$$

where $\{a_n\}, \{b_n\} \; and \; \{c_n\}$ are sequences in $[0, 1]$

Definition 2.7. [13] Let X denote the Banach space, T a selfmap of X forall $x_0 \in X$

$$\begin{aligned} x_{n+1} &= T(1 - a_n)x_n + a_n T y_n \\ y_n &= T z_n \\ z_n &= (1 - b_n)x_n + b_n T x_n \; \forall n \geq 0. \end{aligned}$$

where $\{a_n\}, \; and \; \{b_n\}$ are sequences in $(0, 1)$

Definition 2.8. F-Metric space

[14] made use of a certain class of auxiliary functions to coin the idea of F-metric spaces. We begin with the collection of such functions.

Let $f \in F$ and $f : (0, +\infty) \to \mathbb{R}$ be such that:
$(F1)$ $0 < x < y \implies f(x) \leq f(y)$; and
$(F2)$ for $\{x_n\} \subseteq R^+$, $\lim_{n\to\infty} x_n = 0 \leftrightarrow \lim_{n\to\infty} f\{x_n\} = -\infty$

Example 2.9. [14] The following are some examples of the previously discussed kind of auxiliary functions:

i $-\frac{1}{t}$ where $t \in (0, \infty)$;
ii $-\exp^{\frac{1}{t}}$ for all $t \in (0, \infty)$.

Utilizing such functions, the authors generalized the concept of usual metric spaces and originated the notion of F -metric spaces as follows:

Definition 2.10. [14] Let M be a nonempty set, and let $d_F : M \times M \to [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in F \times [0, +\infty)$ such that:

D1 $(x, y) \in M \times M, D(x, y) = 0 \rightleftarrows x = y$;
D2 $D(x, y) = D(x, y)$ for all $(x, y) \in M \times M$; and
D3 for every $(x, y) \in M \times M, N \in \mathbb{N}, N \geq 2$, and $(x_i)_i^n = 1 \subset M$ with $(x_1, x_N) = (x, y)$, we get

$$D(x, y) > 0 \textit{ implies } f(D(x, y)) \leq f(\sum_{i=1}^{N-1} D(x_i, y_{i+1})) + \alpha$$

Then D is an F-metric on M, and the pair (M, D) is said to be an F-metric space.

It is observed that any metric on X is an F -metric, but the converse is not true.

Lemma 2.11. *Let X be a F-metric space*
i. For $x, y \in X$ and $t \in [0, 1], \exists$ a unique point $z \in [x, y]$ such that

$$D(x, z) = tD(x, y) \textit{ and } D(y, z) = (1 - t)D(x, y)$$

ii. For x,y $\in$ X and t $\in$ [0,1], we have

$$D(1 - t)x \;\oplus\; tyz) \leq (1 - t)D(x, z) + tD(y, z)$$

Lemma 2.12. *Let $\{p_n\}_{n=p}^{\infty}$, $\{q_n\}_{n=p}^{\infty}$, $\{r_n\}_{n=p}^{\infty}$ be sequence of non negative numbers satisfying the following condition:*

$$p_{n+1} \leq (1 - S_n)p_n + q_n + r_n, \;\forall\; x \geq 0$$

where

$$[S_n]_{n=p}^{\infty} \subset [0, 1]. \,if\, \sum_{n=0}^{\infty} S_n = \infty; \lim_{n\to\infty} q_n = 0 \textit{ and } \sum_{n=p}^{\infty} r_n < \infty, \textit{ Then } \lim_{n\to\infty} p_n = 0.$$

Definition 2.13. Let (X, D) be a F- metric space, $T : X \to X$ is a mapping, $x_0 \in X$ and assume that the iteration procedure used is the sequence $\{x_n\}_{n=1}^{\infty}$ provided by the iterative procedure, converges to a fixed point p of T. Let $\{y_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X and $\varepsilon_n = D(y_{n+1}, f(T, y_n))$, in $n = 0, 1, 2, ...$ Then, the fixed point iteration procedure is T-stable or stable with respect to T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \iff \lim_{n\to\infty} y_n = p$$

3. Main Result

Definition 3.1. [15]. Let (X, D) be an F-metric space $T : X \to X$ be a map. T is called a Ciric-Kannan $\alpha - \psi$ contraction, if there exists $l \in [0, \frac{1}{2})$, $\psi \in \Omega$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y)D(T_x, T_y) \leq \psi(M(x, y) + l[D(x, T_x) + D(y, T_y)]$$

where

$$M(x, y) = max[D(x, y), D(x, T_x), D(y, T_y)]$$

if a fixed point $u = T_u = y$ exists for the contraction, then

$$\begin{aligned} M(x, u) &= max[D(x, u), D(x, T_x) + D(u, T_u)] \\ &= max[D(x, u), D(x, T_x)] \\ &= max[D(x, u), D(x, u) + D(u, T_x)] \end{aligned}$$

Hence

$$\begin{aligned} M(x, u) &= D(x, u) + D(u, T_x) \\ &= D(x, T_x) \end{aligned}$$

$$\begin{aligned} D(T_x, u) &\leq \psi D(x, T_x) + L[(D(x, T_x) + D(u, T_u))]\alpha(x, u) \\ &\leq \psi D(x, T_x) + L[D(x, T_x)]\alpha(x, u) \\ &\leq [\psi + L\alpha(x, u)]D(x, T_x) \\ &\leq [\psi + L\alpha(x, u)][D(x, u) + D(u, T_x)] \end{aligned}$$

$$D(T_x, u) - [\psi + L\alpha(x, u)]D(T_x, u) \leq [\psi + L\alpha(x, u)]D(x, u)$$

$$[1 - (\psi + L\alpha(x, u)]D((T_x, u) \leq [\psi + L\alpha(x, u)]D(x, n)$$

$$D(T_x, u) \leq \frac{[\psi + L\alpha(x,u)]}{[1 - (\psi + L\alpha(x,u))]} D(x,u)$$

Let

$$k = \frac{[\psi + L\alpha(x,u)]}{[1 - (\psi + L\alpha(x,u))]} < 1$$

$\therefore$

(4) $$D(T_x, u) \leq kD(x,u)$$

similarly,

(5) $$D(Tx_n, u) \leq kD(x_n, u)$$

(6) $$D(Ty_n, u) \leq kD(y_n, u)$$

(7) $$D(Tz_n, u) \leq kD(z_n, u)$$

Theorem 3.2. *Let (X, D) be a complete F-metric space and $T : X \to X$ be a Ciric-Kannan $\alpha - \psi$ contraction. Let the sequence $\{x_n\}$ be defined by the Ishikawa iterative scheme. If $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. Let p be the fixed point of T using Ciric-Kannan $\alpha - \psi$ contraction and Ishikawa iteration we have

$$\begin{aligned}
D(Tx_{n+1}, Tx_n) &\leq D(Tx_{n+1}, Tx_n)\alpha(x_{n+1}, x_n) \\
D(y_n, p) &\leq D((1-\beta_n)x_n + \beta_n Tx_n, p) \\
&\leq D((1-\beta_n)x_n, p) + D(\beta_n Tx_n, p) \\
&= (1-\beta_n)D(x_n, p) + \beta_n D(Tx_n, p) \\
&\leq (1-\beta_n)D(x_n, p) + \beta_n kD(x_n, p) \\
&= [(1-\beta_n) + \beta_n k]D(x_n, p)
\end{aligned}$$

(8) $$= [(1 - \beta_n + \beta_n k]D(x_n, p)$$

$$\begin{aligned} D(x_{n+1},p) &= D((1-\alpha_n)x_n + \alpha_n T y_n, p) \\ &\le (1-\alpha_n)D(x_n,p) + \alpha_n D(Ty_n,p) \\ &\le (1-\alpha_n)D(x_n,p) + \alpha_n k D(y_n,p) \\ &\le (1-\alpha_n)D(x_n,p) + \alpha_n k[(1-\beta_n+k\beta_n)]D(x_n,p) \\ &= [(1-\alpha_n) + \alpha_n k(1-\beta_n+k\beta_n)]D(x_n,p) \\ &= [(1-\alpha_n+\alpha_n k) - g_n]D(x_n,p) \end{aligned}$$

since $0 < k < 1$ and by the assumption $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, it shows that $\sum_{n=1}^{\infty} g_n = \infty$. Hence, $\lim_{n\to\infty} D(x_n,p) = 0$. Thus $\{x_n\}$ converges strongly to a unique fixed point of T which is p. □

Theorem 3.3. *Let (X,D) be a complete F-metric space, and $T : X \to X$ be a Ciric-Kannan $\alpha - \psi$ contraction. Let the sequence $\{x_n\}$ be defined by the Noor iterative scheme. If $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. Let p be the fixed point of T using Ciric-Kannan $\alpha - \psi$ contraction and Noor iteration implies that

$$\begin{aligned} D(z_n,p) &= D((1-\alpha_n)x_n + \alpha_n T x_n, p) \\ &\le D((1-\alpha_n)x_n, p) + D(\alpha_n T x_n, p) \\ &= (1-\alpha_n)D(x_n,p) + \alpha_n D(Tx_n,p) \\ &\le (1-\alpha_n)D(x_n,p) + \alpha_n k D(x_n,p) \\ &= [(1-\alpha_n) + \alpha_n k]D(x_n,p) \\ &= [(1-(1-k)\alpha_n]D(x_n,p). \end{aligned}$$

$$\begin{aligned} D(y_n,p) &= D((1-\beta_n)x_n + \beta_n T z_n, p) \\ &\le (1-\beta_n)D(x_n,p) + \beta_n D(Tz_n,p) \\ &\le (1-\beta_n)D(x_n,p) + \beta_n k D(z_n,p) \\ &\le (1-\beta_n)D(x_n,p) + \beta_n k[(1-(1-k)\alpha_n]D(x_n,p) \\ &= [(1-\beta_n) + \beta_n k[(1-\alpha_n+\alpha_n k)]]D(x_n,p) \\ &= [(1-\beta_n) + g_n\alpha_n\beta_n]D(x_n,p) \end{aligned}$$

where

$$g_n = k(1-(1-k))$$

$$\begin{aligned}
D(y_n,p) &\leq [(1-\beta_n)+g_n\alpha_n\beta_n]D(x_n,p)\\
D(x_{n+1},p) &= D((1-a_n)x_n+a_nTy_n,p)\\
&\leq (1-a_n)D(x_n,p)+a_nD(Ty_n,p)\\
&\leq (1-a_n)D(x_n,p)+a_nk[(1-\beta_n)+g_n\alpha_n,\beta_n]D(x_n,p)\\
&\leq (1-a_n)D(x_n,p)+a_nk[(1-\beta_n)+g_n\alpha_n,\beta_n]D(x_n,p)\\
&= [(1-a_n)+a_nk[(1-\beta_n)+g_n\alpha_n\beta_n]]D(x_n,p)
\end{aligned}$$

Since $0<k<1$ and by assumption of Lemma 2.3 $\sum_{n=1}^{\infty}\alpha_n\beta_n=\infty$, it follows that $\sum_{n=0}^{\infty}g_n=\infty$. Hence $\lim_{n\to\infty}D(x_n,p)=0$.
Then the sequence $\{x_n\}$ converges strongly to a fixed point p of T. □

Theorem 3.4. *Let (X,D) be a complete F-metric space, and $T:X\to X$ be a Ciric-Kannan $\alpha-\psi$ contraction. Let the sequence $\{x_n\}$ be defined by the JK iterative scheme. If $\sum_{n=1}^{\infty}\alpha_n\beta_n=\infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. Let p be the fixed point of T using Ciric-Kannan $\alpha-\psi$ contraction and JK iteration we have

$$\begin{aligned}
D(y_n,p) &= D((1-b_n)x_n+b_nTx_n,p)\\
&\leq D((1-b_n)x_n,p)+b_nD(Tx_n,p)\\
&\leq (1-b_n)D(x_n,p)+b_nD(Tx_n,p)\\
&\leq (1-b_n)D(x_n,p)+b_nkD(x_n,p)\\
&\leq [(1-b_n)+b_nk]D(x_n,p).
\end{aligned}$$

$$\begin{aligned}
D(z_n,p) &= D(Ty_n,p)\\
&\leq kD(y_n,p)\\
&\leq k[(1-b_n)+b_nk]D(x_n,p)
\end{aligned}$$

$$\begin{aligned}
D(x_{n+1},p) &= D(T((1-a_n))Ty_n+a_nTz_n,p)\\
&\leq D(T(1-a_n)Ty_n,p)+D(a_nTz_n,p)\\
&\leq kD((1-a_n)Ty_n,p)+a_nD(Tz_n,p)\\
&\leq k(1-a_n)D(Ty_n,p)+a_nD(Tz_n,p)\\
&\leq k^2(1-a_n)D(y_n,p)+a_nkD(z_n,p)\\
&\leq [k^2(1-a_n)(1-b_n+kb_n)+(a_nk^2(1-b_n+kb_n)]D(x_n,p)
\end{aligned}$$

Since $0<k<1$ and by Lemma 2.3, we get that $\lim_{n\to\infty}D(a_n,p)=0$
Hence the sequence converges strongly to a fixed point p at T.

□

Theorem 3.5. *Let (X,D) be a F-metric space and $T : X \to X$ a mapping satisfying Ciric-Kamma $\alpha - \psi$ contraction. Suppose T has a fixed point p. Let $x_0 \in X$ and the sequence $\{x_n\}$ be Ishikawa iteration. Then, $\{x_n\}$ is stable with respect to T.*

Proof. Let the sequece $\{a_n\}_{n=1}^{\infty}$ be arbitrary in X. Define

$$\varepsilon_n = D(a_{n+1}, f(T, a_n))$$

for every n $\in \mathbb{N}$, such that

$$a_{n+1} = (1 - \alpha_n)a_n + Tb_n$$

$$b_n = (1 - \beta_n)a_n + \beta Ta_n.$$

Suppose the $\lim_{n\to\infty} \varepsilon_n = 0$. We show that $\lim_{n\to\infty} a_n = p$

$$\begin{aligned} D(a_{n+1}, p) &\leq D(a_{n+1}, f(T, a_n) + D(f(T, a_n), p) \\ &\leq \varepsilon_n + kD((1 - \alpha_n)a_n + Tb_n, p) \\ &\leq \varepsilon_n + D((1 - \alpha_n)a_n, p) + D(Tb_n, p) \\ &\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + kD(b_n, p) \end{aligned}$$

Recall that, we have prove fixed point of contraction as

$$D(Ty_n, p) \leq kD(y_n, p)$$

$$\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + kD(b_n, p) \tag{9}$$

$$\begin{aligned} D(b_n, p) &\leq D((1 - \beta_n)a_n + \beta_n Ta_n, p) \\ &\leq D((1 - \beta_n)a_n, p) + D(\beta_n Ta_n, p) \\ &\leq (1 - \beta_n)D(a_n, p) + \beta_n D(Ta_n, p) \end{aligned}$$

$$D(b_n, p) \leq (1 - \beta_n)D(a_n, p) + k\beta_n D(a_n, p) \tag{10}$$

$$\begin{aligned} D(a_{n+1}, p) &\leq \varepsilon_n + (1 - \alpha_x)D(a_n, p) + k[(1 - p_n)D(a_n, p) + k\beta_n D(a_n, p)] \\ &\leq \varepsilon_n + (1 - \alpha_n)D(a_n, p) + k(1 - p_n)D(a_n, p) + k^2\beta_n D(a_n, p) \\ &\leq \varepsilon_n + [(1 - \alpha_n) + k(1 - \beta_n) + k^2\beta_n]D(a_n, p) \end{aligned}$$

since $\alpha_n, \beta_n, k \in [0, 1]$, we have $[(1 - \alpha_n) + k(1 - p_n) + k^2\beta_n] < 1$
Hence,

$$D(a_{n+1}, p) \leq kD(a_n, p) + \varepsilon_n$$

by Lemma 2.3, we have $\lim_{n\to\infty} a_n = p$

Assuming the $\lim_{n\to\infty} a_n = p$. Then we need to prove that $\lim_{n\to\infty} \varepsilon_n = 0$

$$\begin{aligned}\varepsilon_n &= D(a_{n+1}, f(T, a_n))\\ &\leq D(a_{n+1}, p) + D((1-\alpha_n)a_n + Tb_n, p)\\ &= D(a_{n+1}, p) + D((1-\alpha_n)a_n, p) + D(Tb_n, p)\end{aligned}$$

$$\leq D(a_{n+1}, p) + (1-\alpha_n)D(a_n, p) + kD(b_n, p) \tag{11}$$

$$\leq D(a_{n+1}, p) + (1-\alpha_n)D(a_n, p) + k[(1-\beta_n)D(a_n, p) + k\beta_n D(a_n, p)] \tag{12}$$

$$\varepsilon_n \leq D(a_{n+1}, p) + [(1-\alpha_n) + k(1-\beta_n) + k\beta_n]D(a_n, p) \tag{13}$$

Taking $\lim_{n\to\infty}$ of (13), we obtain $\lim_{n\to\infty} \varepsilon_n = 0$
Hence, the $\{x_n\}$ is stable with respect to T. □

Theorem 3.6. *Let (X,D) be a F-metric space and $T : X \to X$ a mapping satisfying Ciric-Kamma $\alpha - \psi$ contraction. Suppose T has a fixed point p. Let $x_0 \in X$ and let the sequence $\{x_n\}$ be Noor iteration. Then, $\{x_n\}$ is stable with respect to T.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X. Define
$a_{n+1} = (1-\alpha_n)a_n + \alpha_n Tb_n$
$b_n = (1-\beta_n)a_n + \beta_n Tc_n$
$c_n = (1-\gamma_n)a_n + \gamma_n Ta_n$
for every $n \in \mathbb{N}$,
Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. It would be shown that $\lim_{n\to\infty} a_n = p :$

$$\begin{aligned}D(a_{n+1}, p) &\leq D(a_{n+1}, f(T, a_n)) + D(f(T, a_n), p)\\ &\leq \varepsilon_n + D((1-\alpha_n)a_n + Tb_n, p)\\ &\leq \varepsilon_n + D((1-\alpha_n)a_n, p) + D(\alpha_n Tb_n, p)\\ &= \varepsilon_n + (1-\alpha_n)D(a_n, p) + \alpha_n D(Tb_n, p)\end{aligned}$$

$$\leq \varepsilon_n + (1-\alpha_n)D(a_n, p) + \alpha_n kD(b_n, p) \tag{14}$$

$$\begin{aligned}D(b_n, p) &= D((1-\beta_n)a_n\beta_n Tc_n, p)\\ &\leq D((1-\beta_n)a_n, p) + D(\beta_n Ta_n, p)\\ &\leq (1-\beta_n)D(a_n, p) + \beta_n D(Tc_n, p)\end{aligned}$$

(15) $$\leq (1-\beta_n)D(a_n,p) + k\beta_n D(Tc_n,p)$$

$$\begin{aligned} D(c_n,p) &= D((1-\gamma_n)a_n + \gamma_n Ta_n, p) \\ &= D((1-\gamma_n)a_n, p) + D(\gamma_n Ta_n, p) \\ &\leq (1-\gamma_n)D(a_n,p) + \gamma_n kD(a_n,p) \end{aligned}$$

Therefore

(16) $$D(c_n,p) \leq (1-\gamma_n)D(a_n,p) + \gamma_n kD(a_n,p)$$

substitute (16) into (15)

$$\begin{aligned} \varepsilon_n &\leq D(a_{n+1},y) + (1-\alpha_n)D(a_n,p) + \alpha_n k(1-\beta_n)D(a_n,p) \\ &+ k^2\alpha_n\beta_n[(1-\alpha_n)D(a_n,p) + y_n kD(a_n,p)] \\ &\leq D(a_{n+1},p) + (1-\alpha_n)D(a_n,p) + \alpha_n k(1-\beta_n)D(a_n,p) \\ &+ k^2\alpha_n p_n(1-y_n)D(a_n,p) + k^2\alpha_n\beta_n y_n D(a_n,p) \\ &\leq D(a_n,p) + [(1-\alpha_n) + \alpha_n k(1-\beta_n) + k^2\alpha_n(1-y_n) + k^3\alpha_n\beta_n y_n]D(a_n,p) \\ &\leq D(a_n,p) + kD(a_n,p) \\ &\leq D(a_n,p) \end{aligned}$$

Since $\lim_{n\to\infty}\varepsilon_n = 0$. Thus $\{x_n\}$ is stable with respect to T. □

Theorem 3.7. *Let (X,D) be a F-metric space and $T : X \to X$ a mapping satisfying Ciric-Kannan $\alpha - \psi$ contraction. Suppose T has a fixed point p. Let $x_0 \in X$ and $\{x_n\}$ be JK iteration. Then, $\{x_n\}$ is stable with respect to T.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X. Defined
$\varepsilon_n = D(a_{n+1}, f(T,a_n))$
$a_{n+1} = T((1-\alpha_n)Tc_n + \alpha_n Tb_n)$
$b_n = Tc_n$
$c_n = (1-\beta_n)a_n + \beta_n Ta_n$
For every $n \in \mathbb{N}$; $\{\alpha_n, \beta_n\} \in \{0,1\}$ and $k \in [0,1]$
suppose that $\lim_{n\to\infty}\varepsilon_n = 0$. We show that $\lim_{n\to\infty} a_n = p$.

$$\begin{aligned} D(a_{n+1},p) &\leq D(a_{n+1}, f(T,a_n)) + D(f(T,a_n),p) \\ &\leq \varepsilon_n + D(T((1-\alpha_n)Tc_n + \alpha_n Tb_n, p) \\ &\leq \varepsilon_n + kD((1-\alpha_n)Tc_n + \alpha_n Tb_n, p) \\ &\leq \varepsilon_n + k(1-\alpha_n)D(Tc_n,p) + k\alpha_n D(b_n,p) \end{aligned}$$

$$\leq \varepsilon_n + k^2(1-\alpha_n)D(c_n,p) + k\alpha_n D(b_n,p) \tag{17}$$

$$D(b_n,p) = D(Tc_n,p)$$

$$\leq kD(c_n,p) \tag{18}$$

$$\begin{aligned} D(c_n,p) &= D((1-\beta_n)a_n + \beta_n Ta_n, p) \\ &= D((1-\beta_n)a_n, p) + D(\beta_n Ta_n, p) \\ &\leq (1-\beta_n)D(a_n,p) + \beta_n D(Ta_n,p) \end{aligned}$$

Hence

$$D(c_n,p) \leq (1-\beta_n)D(a_n,p) + k\beta_n D(a_n,p) \tag{19}$$

Substitute (19) into (17) yields

$$D(b_n,p) \leq k[(1-\beta_n)D(a_n,p) + k\beta_n D(a_n,p)]$$

$$\leq k(1-\beta_n)D(a_n,p) + k^2\beta_n D(a_n,p) \tag{20}$$

substitute (20) and (19) into (17)

$$\begin{aligned} D(a_{n+1},p) &\leq \varepsilon_n + k^2(1-\alpha_n)[(1-\beta_n)D(a_n,p) + k\beta_n D(a_n,p)] \\ &+ k^2\alpha_n[k(1-\beta_n)D(a_n,p) + k^2\beta_n D(a_n,p)] \\ &= \varepsilon_n + [[k^2(1-\alpha_n)[(1-\beta_n) + k\beta_n] + [k^3\alpha_n(1-\beta_n) + k^2\beta_n]]D(a_n,p) \end{aligned}$$

$$\leq \varepsilon_n + kD(a_n,p) \tag{21}$$

since $k \in [0,1]$ and $\alpha_n, \beta_n \in [0,1]$, we have

$$k^2(1-\alpha_n)[(1-\beta_n) + k\beta_n] + k^3\alpha_n(1-\beta_n) + k^2\beta_n) < 1 \tag{22}$$

Hence, $D(a_{n+1},p) \leq kD(a_n,p) + \varepsilon_n$, by Lemma 2.3, $\lim_{n\to\infty} a_n = p$.

Assuming the $\lim_{n\to\infty} a_n = p$. It remains to prove that $\lim_{n\to\infty} \varepsilon_n = 0$

$$
\begin{aligned}
\varepsilon_n &= D(a_{n+1}, f(T, a_n)) \\
&\leq D(a_{n+1}, p) + D(f(T, a_n), p) \\
&\leq D(a_{n+1}, p) + D(f(T((1-\alpha_n)Tc_n + \alpha_n Tb_n, p) \\
&\leq D(a_{n+1}, p) + kD(f(T((1-\alpha_n)Tc_n + \alpha_n Tb_n, p) \\
&\leq D(a_{n+1}, p) + k(1-\alpha_n)D(Tc_n, p) + k\alpha_n D(Tb_n, p)
\end{aligned}
$$

$$
\leq D(a_{n+1}, p) + k^2(1-\alpha_n)D(c_n, p) + k^2\alpha_n D(b_n, p) \tag{23}
$$

substitute (19) and (20) into (23)

$$
\begin{aligned}
\varepsilon_n &\leq D(a_{n+1}, p) + k^2(1-\alpha_n)[(1-\beta_n)D(a_n, p) + k\beta_n D(a_n, p)] \\
&+ k^2\alpha_n[k(1-\beta_n)D(a_n, p) + k^2\beta_n D(a_n, p)] \\
&= D(a_{n+1}, p) + [k^2(1-\alpha_n)((1-\beta_n) + k\beta_n)]D(a_n, p) \\
&+ [k^2\alpha_n[k(1-\beta_n) + k^2\beta]]D(a_n, p) \\
&= D(a_{n+1}, p) + [[k^2(1-\alpha_n)((1-\beta_n) + k\beta_n)] \\
&+ [k^2\alpha_n(k(1-\beta_n) + k^2\beta_n)]]D(a_n, p) \\
&\leq D(a_{n+1}, p) + kD(a_n, p)
\end{aligned}
$$

$\varepsilon_n \leq D(a_{n+1}, p)$, $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, the sequence $\{x_n\}$ is stable with respect to T. □

Conclusion: The work established the strong convergence and stability of the sequence with respect to operator T using Ciric-Kannan contraction in a complete F-metric space.

References

[1] B. Samet. Fixed points for $\alpha - \psi$ contractive mappings with an application to quadratic integral equations, Electron. J. Differ. Equ. **152**(2014), 2014.

[2] S. A. Alizadeh, F. Moradlou & P. Salimi. Some fixed point results for (α, β)-$(\psi - \varphi)$-contractive mappings, Filomat. **28**(3), (2014) 635-647.

[3] K. Roy, H. Alaeidizaji, M. Saha, B. Mohammadi & V. Parvaneh. Some fixed-point theorems over a generalized f-metric space, Advances in Mathematical Physics. **2021**(1), (2021), 5570653.

[4] H. Akewe & G. A. Okeke. Convergence and stability theorems for the picard-mann hybrid iterative scheme for a general class of contractive-like operators, Fixed Point Theory and Applications. **2015**(2015), 1-8.

[5] A. O. Bosede. Strong convergence results for the Jungck-Ishikawa and Jungck-Mann iteration processes, Bulletin of Mathematical analysis and applications. **2**(3), (2010) 65-73.

[6] H. A. Hammad, H. U. Rehman & M. De la Sen. A new four-step iterative procedure for approximating fixed points with application to 2d volterra integral equations, Mathematics. **10**(22), (2022) 4257.

[7] A. Tassaddiq, S. Kanwal, F. Lakhani & R. Srivastava. Strong and δ-convergence fixed-point theorems using noor iterations, Axioms. **12**(3), (2023), 271.

[8] V. Berinde. Convergence theorems for fixed point iterative methods defined as admissible perturbations of a nonlinear operator, Carpathian Journal of Mathematics. **2013**, (2013), 9-18.

[9] E. Ameer, H. Aydi, H. A. Hammad, W. Shatanawi & N. Mlaiki. On $(\phi - \psi)$-metric spaces with applications, Symmetry. **12**(9), (2020) 1459.

[10] B. Samet, C. Vetro & P. Vetro. Fixed point theorems for $\alpha - \psi$-contractive type mappings, Nonlinear analysis: theory, methods & applications. **75**(4), (2012) 2154-2165.

[11] S. Ishikawa. Fixed points by a new iteration method, Proceedings of the American Mathematical Society. **44**(1), (1974) 147-150.

[12] M. A. Noor. New approximation schemes for general variational inequalities, Journal of Mathematical Analysis and applications. **251**(1) (2000) 217-229.

[13] J. Ahmad, K. Ullah, M. Arshad & Z. Ma. A new iterative method for suzuki mappings in banach spaces, Journal of Mathematics. **2021**(1), (2021) 6622931.

[14] M. Jleli & B. Samet. On a new generalization of metric spaces, Journal of Fixed Point Theory and Applications. **20**(3), (2018) 128.

[15] S. S. Mohammed, Y. Sirajo & I. A. Fulatan. On fixed point results in f-metric space with applications to neutral differential equations, Mathematical Analysis and its Contemporary Applications. **4**(3), (2022) 47-62.