Subsemigroup of signed contraction mapping of full transformation semigroup

Abstract

The full transformation semigroup, T_n on the set $X_n = \{1,2,3,\dots\}$ is the set of all maps $\alpha: X_n \longrightarrow X_n$, under the operation of composition of mapping. The signed contraction mappings of full transformation and its subsemigroup $\alpha \in SCT_n$ is a contraction for all $i, j \in Dom(\alpha)$ if $|i\alpha| - |j\alpha| = |i| - |j|$ and is defined on the set $\alpha: X_n \longrightarrow \mathbb{Z}^*$ where $\mathbb{Z}^* = \mathbb{Z}/\{0\}$ and $\mathbb{Z}^* = \{-n \dots, -3, -2, 1, 2, 3 \dots, n\}$. The cardinality of signed order preserving of full contraction mapping, $SOCT_n$, order decreasing signed full contraction mapping, $SODT_n$ and both order preserving and order decreasing signed full contraction mapping, $SOCCT_n$ were obtained. The results are tabulated based on the number of positive and negative elements in the domain of α $[Dom(\alpha)]$.

Key words: Full transformation semigroup, signed contraction mapping, cardinality, order preserving and decreasing.

1. INTRODUCTION

The study of permutations, symmetric group and group of permutations of a set are the origins of group theory, Howie (2006). Let T_n denote the full transformation semigroup which is the semigroup of all mappings $\alpha: X_n \longrightarrow X_n$ under composition. A map $\alpha: Dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subseteq X_n$ is called full (total) transformation of X_n if $Dom(\alpha) = X_n$. The set T_n, of all full transformation of X_n forms a semigroup under composition of mappings called the full transformation semigroup. The domain of α is denoted by $Dom(\alpha)$ while the image of α is denoted by$Im(\alpha)$. The semigroup of full transformation has been studied, see (Bashar (2010), Tero (1996), Umar (1993), Clifford and Preston (1966) and Laradji (2006).

Howie (1971) studied the semigroup of order- preserving and order- decreasing of a finite set

$X_n = \{1,2,3,\dots,n\}$. A mapping in $T_n$is called order preserving, O_n, for all $i, j \in X_n, i \leq j \Longrightarrow i\alpha \leq j\alpha$. He also showed that $|O_n| = \binom{2n-1}{n-1}$. A map α is order- decreasing, D_n, for all $i \in X_n$, $i\alpha \leq i$ or $i \geq i\alpha$. The semigroup of all order-decreasing maps is of cardinality $n!$. The semigroup of all maps that are both order-preserving and order-decreasing is denoted by $C_n = O_n \cap D_n$ while the cardinality of C_n implies $|C_n| = \frac{1}{n+1}\binom{2n}{n}$ and also known as Catalan number.

2. PRELIMINARIES

Contraction mappings is a new class of transformation semigroups. Kehinde and Umar (2014), Al-kharous *et. Al.* (2014) introduced the concepts of contraction mappings and semigroups of injective contraction mappings of a finite chain. Many authors has worked on

Some classes of semigroups of full transformation semigroups and its sub-semigroups were obtained in 1971, 1995 and 2002 by Howie. Recently, Adeniji (2012), Adesola (2013), Mogbonju (2015), Rauf and Ogunleke (2021) and Adeniji and Obafemi (2022) studied some classes of semigroups of identity difference transformations, full contraction mapping, signed transformation and extension of polarity in signed transformation semigroups.

A general study of D_n was initiated by Umar (1996). Adeniji (2012) study identity difference transformation semigroups and showed that the order of identity of full transformation semigroup $|IDT_n| = n + (n-1)(2^n - 2)$ and order of identity difference of order preserving full transformation semigroup $|OIDT_n| = n^2 - n + 1$. Richard (2008) studied transformation semigroup, he introduced the signed permutation group S_n and signed full transformation semigroups. Mogbonju (2015) studied some combinatorics properties of signed transformation semigroups. He defined signed transformation semigroups as a set of all maps from $X_n \longrightarrow Z^*$ where $Z^* = \{-1,1,2-2,3,-3,\ldots-n,n\}$ and $Im(\alpha) \subset Z^*$. He showed that the order of signed full transformation semigroup, signed order preserving and signed order decreasing full transformation semigroup are $|ST_n| = 2n^2 + n^2(2^n - 2)$, $|SO_n| = 2^n\binom{2n-1}{n-1}$, $|SD_n| = n!\sum_{k=0}^{n}\binom{n}{k}$. Also, signed full transformation for both order-preserving and order decreasing full transformation semigroup $|SC_n| = \frac{1}{n}\binom{2n-1}{n-1}$ and $\sum_{k=0}^{n}\binom{n}{k}$ respectively. Mogbonju (2017, 2019) studied signed singular mapping transformation semigroup and signed full1-1 symmetric group and signed full transformation semigroups. Mogbonju (2018) was first to study the signed semigroups of order – preserving transformation. Also, Mogbonju *et. Al.* (2019) investigated the polarity in signed symmetric group and signed transformation semigroup. Recently Adeniji and Obafemi (2022) studied transformation semigroup of alternating nonnegative Integers, in which the set of integers, Z_n is splits into even and odd parts. The even part is arranged in $\binom{\frac{n}{2}}{n}$ ways, while the odd parts fixes one point at a point to compliment the even part thereby forming the semigroup

$Z_n = \{0,1,2,\dots,n-1\}$. The $Z_{n-even} = \{0,2,4,\dots,2k\}$ and $Z_{n-odd} = \{1,3,\dots,2k+1\}$ respectively. Ogunleke (2021) studied the characterization of subsemigroups of transformation semigroups. The signed (partial) transformation semigroups were defined in the form $\alpha: Dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset Z^*$ where $X_n = \{1,2,3,\dots,n\}$ and $Z^* = \{-1,1,2-2,3,-3,\dots-n,n\}$. The composition of mapping is associative.

A transformation $\alpha \in T_n$ is said to be contraction mapping if $\forall\, i,j \in X_n$ such that

$|i-j| \geq |i\alpha - j\alpha|$ or $|i\alpha - j\alpha| \leq |i-j|$. While $\alpha \in T_n$ is semigroup of signed contraction mapping of full transformation semigroup defined by $|i| - |j| \geq |i\alpha| - |j\alpha|$. Signed contraction mapping as order preserving $i \leq j \Longrightarrow |i\alpha| \leq |j\alpha|$ and order- decreasing as $|i\alpha| \leq i\ or\ i \geq |i\alpha|$.

3. METHODOLOGY.

Let SCT_n be a signed contraction mapping of full transformation semigroup. Let $\alpha \in SCT_n$ and $x, y \in X_n$ be any element, then α is called a transformation. The subsemigroup of SCT_n is defined on $\alpha \in SCT_n$ of subsemigroup and was listed while the table were formed using the number of elements in the $dom(\alpha)$. The number of elements with positive integers, negative integers and both positive and negative integers in the image of α denoted by $|lm(\alpha^+)|, |lm(\alpha^-)|\ and\ |lm(\alpha^*)|$ respectively. The tables are formed and the sequences were generated.

3.1: Element in signed order- preserving full contraction mapping; $SOCT_n$

when $n = 1$

$$|SOCT_1| = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\} = 2$$

$$|lm(\alpha^+)| = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} = 1$$

$$|lm(\alpha^-)| = \left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\} = 1$$

when $n = 2$

$$|SOCT_2| = \left\{\begin{matrix} \begin{pmatrix}1&2\\1&1\end{pmatrix}, \begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}, \begin{pmatrix}1&2\\-1&1\end{pmatrix}, \begin{pmatrix}1&2\\-1&2\end{pmatrix}, \begin{pmatrix}1&2\\-2&2\end{pmatrix}, \\ \begin{pmatrix}1&2\\1&-1\end{pmatrix}, \begin{pmatrix}1&2\\1&-2\end{pmatrix}, \begin{pmatrix}1&2\\2&-2\end{pmatrix}, \begin{pmatrix}1&2\\-1&-1\end{pmatrix}, \begin{pmatrix}1&2\\-1&-2\end{pmatrix}, \begin{pmatrix}1&2\\-2&-2\end{pmatrix} \end{matrix}\right\} = 12$$

$$|lm(\alpha^+)| = \left\{\begin{pmatrix}1&2\\1&1\end{pmatrix}, \begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}\right\} = 3$$

$$|lm(\alpha^-)| = \left\{\begin{pmatrix}1&2\\-1&-1\end{pmatrix}, \begin{pmatrix}1&2\\-1&-2\end{pmatrix}, \begin{pmatrix}1&2\\-2&-2\end{pmatrix}\right\} = 3$$

$$|lm(\alpha^*)| = \left\{\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix}\right\} = 6$$

Table 1: Value of Elements in $SOCT\ _n$

$^n/_{lm(\alpha)}$	$\lvert lm(\alpha^+)\rvert$	$\lvert lm(\alpha^-)\rvert$	$\lvert lm(\alpha^*)\rvert$	$\lvert SOCT_n\rvert$
1	1	1	-	2
2	3	3	6	12
3	8	8	48	64
4	20	20	280	320
5	48	48	1440	1536
6	112	112	6944	7168

3.2: Element in signed order- decreasing full contraction mapping; $SDCT_n$

when $n = 1$

$$|SODT_1| = \left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\} = 2$$

$$|lm(\alpha^+)| = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} = 1$$

$$|lm(\alpha^-)| = \left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\} = 1$$

when $n = 2$

$$|SODT_2| = \left\{\begin{matrix}\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix},\\ \begin{pmatrix}1 & 2\\1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & -2\end{pmatrix}\end{matrix}\right\} = 8$$

$$|lm(\alpha^+)| = \left\{\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\right\} = 2$$

$$|lm(\alpha^-)| = \left\{\begin{pmatrix}1 & 2\\-1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix}\right\} = 2$$

$$|lm(\alpha^*)| = \left\{\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix}\right\} = 4$$

Table 2: Value of Elements in $SDCT_n$

$^{n}/_{lm(\alpha)}$	$\lvert lm(\alpha^{+})\rvert$	$\lvert lm(\alpha^{-})\rvert$	$\lvert lm(\alpha^{*})\rvert$	$\lvert SDCT_n\rvert$
1	1	1	-	2
2	2	2	4	8
3	4	4	24	32
4	8	8	112	128
5	16	16	480	512
6	32	32	1984	2048

3.3: Element in signed order- preserving and signed order decreasing full contraction mapping; $\boldsymbol{SOPDT_n}$

when $n = 1$

$$|SOPDT_1| = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\} = 2$$

$$|lm(\alpha^{+})| = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} = 1$$

$$|lm(\alpha^{-})| = \left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\} = 1$$

when $n = 2$

$$|SOPDT_2| = \left\{\begin{matrix} \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix}, \\ \begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\-2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & -2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & -2\end{pmatrix}, \\ \begin{pmatrix}1 & 2\\-1 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix}, \begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\-2 & -2\end{pmatrix} \end{matrix}\right\} = 16$$

$$|lm(\alpha^{+})| = \left\{\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 2\end{pmatrix}\right\} = 4$$

$$|lm(\alpha^{-})| = \left\{\begin{pmatrix}1 & 2\\-1 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix}, \begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\-2 & -2\end{pmatrix}\right\} = 4$$

$$|lm(\alpha^{*})| = \left\{\begin{matrix} \begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & -2\end{pmatrix}, \\ \begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\-2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \end{matrix}\right\} = 8$$

Table 3: Value of Elements in $SOPDT_n$

$^{n}/_{lm(\alpha)}$	$\lvert lm(\alpha^{+})\rvert$	$\lvert lm(\alpha^{-})\rvert$	$\lvert lm(\alpha^{*})\rvert$	$\lvert SOPDT_n\rvert$
1	1	1	-	2
2	4	4	8	16
3	13	13	78	104
4	13	13	78	104
5	91	91	2730	2912
6	218	218	13516	13952

4.0: RESULTS:

The results of the subsemigroups of $\alpha \in SCT_n$ are investigated from the sequences and the pattern of the sequences. The following formulae were formed and theorems were stated and proved.

Theorem 4.1: Let $S = SOCT_n$, then $|SOCT_n| = (n+1)2^{2n-2}$

Proof: Let $\propto \in S$ and $X_n = \{1, 2, 3, \dots \dots \dots \dots . n\}$ such that $\propto : dom(\propto) \subseteq X_n;$ for each n, the $|lm(\alpha^{+})| = lm(\alpha^{-})|$. By contraction and order preserving properties $|x - y| \geq |x \propto - y \propto| \Longrightarrow x \propto \leq y \propto$ where each $i's$ maps all $j's$. Since the element of $dom(\propto)$ can occur in $(n+1)(2^{n-2})$ satisfying $S = (n+1)(2^{n-2})$. Since the semigroup is a full transformation, hence the result.

Theorem 4.2: Let $S = SOCT_n$, then $|SDCT_n| = 2^{2n-1}$

Proof: Let $S = SOCT_n$ and $\propto : dom(\propto) \subseteq X_n \rightarrow Im(\propto) \subseteq Z^{*}$, $Im(\propto) = \{i, -i\}$ where $i = 1, 2, 3, \dots . n$. Since the mapping is one–to–one and by chosen some elements from image $Im(\propto) \subseteq Z^{*}$, we have 2^{2n-1} choice with 2–degree each which occurs in 2^{2n-1} ways.

Hence, the result.

Theorem 4.3: Let $S = SOPDT_n$, then $|S| = \frac{2^n}{2}[2^n(n-1) - 2n]$

Proof: Let $\propto$ be transformation in S, $X_n = \{1,2,3, \dots, n\}$ and $X_n \rightarrow Z^{*}$. The , $Im(\propto) = \{i, -i\}$ where where $i = 1, 2, 3, \dots . n$. By the property of order reversing in which $i \leq j \Rightarrow i \propto \geq j \propto$ for each $i \in dom(\propto)$ and $j \in Im(\propto)$. But $|lm(\alpha^{+})| = lm(\alpha^{-})|$. By Binomial theorem, for a positive n, we have $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. Taking the different of 2^n to satisfying $|S| = \frac{2^n}{2}[2^n(n-1) - 2n]$.

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