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## 2x2 Matrix: Calculation of the n-th Power

**Original Research  
Article**

### Abstract

This work explores a robust method for calculating the n-th power of a 2x2 matrix  $A^n$  without using diagonalization techniques. This approach significantly simplifies complex matrix operations. Firstly, we give here, a single formula for the  $n$ th power for any type of 2x2 matrix, that is to say whether it is diagonalizable or not, whether it has two distinct eigenvalues or not and whether these eigenvalues are real or not. We also add a second method.

*Keywords: Eigenvalue, The nth power of matrices, The second-order linear recurrent sequence, The characteristic polynomial*

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## 1 Introduction

In 1992 *Kenneth S. Williams* (see (1)) gives us a general formula working with the Matrices  $X = \frac{A-\beta I}{\alpha-\beta}$ ,  $Y = \frac{A-\alpha I}{\beta-\alpha}$  et  $Z = A - \alpha I$ , which is

$$A^n = \begin{cases} \alpha^n \left( \frac{A-\beta I}{\alpha-\beta} \right) + \beta^n \left( \frac{A-\alpha I}{\beta-\alpha} \right) & \text{if } \alpha \neq \beta \\ \alpha^{n-1} (nA - (n-1)\alpha I) & \text{if } \alpha = \beta \end{cases}$$

In the article (2), they study and determine a general formula for the nth power of 2x2 matrices using the eigenvalues of A, and give some Excel programming examples.

On the other hand, in paper (6) the authors seek to determine other combinatorial identities deriving from the power n of a 2x2 matrix.

But the real problem for mathematicians and even computer scientists is determining the exponential of a matrix (see (3), (4), (5), (10)) since this problem is often found especially in systems of differential equations of the type  $X'(t) = AX(t)$  (see (7), (8), (9))

Here we give, and this is new, a single formula for the nth power for any type of 2x2 matrix, that is to say whether it is diagonalizable or not, whether it has two distinct eigenvalues or not and whether these eigenvalues are real or not. We also add a second method.

The results can be applied to areas such as recurrence relations, differential equations, dynamical systems, and computer algorithms where powers of small matrices appear frequently.

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## 2 The power of any (diagonalizable or not) matrix 2x2

**Theorem 2.1.** Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  a  $2 \times 2$  matrix, so  $\forall n \geq 2$  ( $n \in \mathbb{N}$ )

$$A^n = \begin{pmatrix} a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix}$$

with  $\alpha_1$  and  $\alpha_2$  solutions of  $|A - \alpha I| = 0$ .

*Proof.* We will demonstrate our formula using recurrence.

(i) For  $n = 2$ , we will show that we actually have

$$A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

$$A^2 = \begin{pmatrix} a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} - \alpha_1 \alpha_2 & b \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \\ c \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} & \sum_{k=0}^2 \alpha_1^k \alpha_2^{2-k} - a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \end{pmatrix}$$

we must therefore show the following 4 equalities

$$\begin{cases} a^2 + bc = a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} - \alpha_1 \alpha_2 \\ d^2 + bc = \sum_{k=0}^2 \alpha_1^k \alpha_2^{2-k} - a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \\ b(a+d) = b \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \\ c(a+d) = c \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \end{cases}$$

what does it mean

$$\begin{cases} a^2 + bc = a(\alpha_2 + \alpha_1) - \alpha_1 \alpha_2 \\ d^2 + bc = \alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) \\ b(a+d) = b(\alpha_2 + \alpha_1) \\ c(a+d) = c(\alpha_2 + \alpha_1) \end{cases}$$

$\alpha_1$  and  $\alpha_2$  solutions of  $|A - \alpha I| = 0$ , so

$$\begin{cases} \alpha_1 + \alpha_2 = \text{Tr}(A) = a + d \\ \alpha_1 \alpha_2 = |A| = ad - bc \end{cases}$$

So

$$a(\alpha_2 + \alpha_1) - \alpha_1 \alpha_2 = a(a+d) - (ad - bc) = a^2 + ad - ad + bc = a^2 + bc$$

$$\alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) = (\alpha_1 + \alpha_2)^2 - \alpha_1 \alpha_2 - a(\alpha_2 + \alpha_1) \implies$$

$$\alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) = (a+d)^2 - (ad - bc) - a(a+d) = d^2 + bc$$

$$b(\alpha_2 + \alpha_1) = b(a+d)$$

$$c(\alpha_2 + \alpha_1) = c(a+d)$$

hence the result is true for  $n = 2$ .

(ii) Let's suppose

$$A^n = \begin{pmatrix} a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix}$$

(iii) Let us show that

$$A^{n+1} = A^n \cdot A$$

$$A^{n+1} = \begin{pmatrix} a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - \sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} & b \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\ c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} & \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k} - a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \end{pmatrix}$$

We have  $A^{n+1} = A^n \cdot A =$

$$= \begin{pmatrix} a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix} A$$

$$A^{n+1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$x = a^2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} + cb \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$y = ba \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - b \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} + db \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$z = ac \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - ca \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$t = bc \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - da \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$\Rightarrow \begin{cases} x = (a^2 + cb) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \\ y = b \left[ (a + d) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right] \\ z = c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\ t = (bc - da) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \end{cases}$$

$\Rightarrow$

$$x = [a(\alpha_1 + \alpha_2) - \alpha_1 \alpha_2] \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$y = b \left[ (\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right]$$

$$z = c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k}$$

$$t = -\alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k}$$

because  $\alpha_1 + \alpha_2 = a + d$ ,  $\alpha_1 \alpha_2 = ad - bc$ ,  $a(\alpha_1 + \alpha_2) - \alpha_1 \alpha_2 = a^2 + cb$

$$A^{n+1} = \begin{pmatrix} a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & b \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\ c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} & \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k} - a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \end{pmatrix}$$

because

$$x = [a(\alpha_1 + \alpha_2) - \alpha_1 \alpha_2] \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$= a(\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$= a\alpha_1 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + a\alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$= a\alpha_1^n + a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$x = a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} ;$$

$$\begin{aligned}
y &= b \left[ (\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right] \\
y &= b \left[ \sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-1-k} + \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right] \\
y &= b \left[ \alpha_1^n + \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} \right] = b \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\
\text{and} \\
t &= -\alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\
t &= -\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} + (\alpha_1 + \alpha_2 - a) \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \text{ (as } \alpha_1 + \alpha_2 = a + d) \\
t &= -\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} - a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} + \sum_{k=0}^n \alpha_1^{k+1} \alpha_2^{n-k} + \sum_{k=0}^n \alpha_1^k \alpha_2^{n+1-k} \\
t &= -a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} + \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k}
\end{aligned}$$

hence the result.  $\square$

**Remark 2.1.** Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  a  $2 \times 2$  matrix and  $\alpha_1$  and  $\alpha_2$  solutions of  $|A - \alpha I| = 0$ .

1) If  $\alpha_1 = \alpha_2 = \alpha$ , so

$$A^n = \left( \frac{a+d}{2} \right)^{n-1} \begin{pmatrix} \frac{(n+1)a - (n-1)d}{2} & nb \\ nc & \frac{(n+1)d - (n-1)a}{2} \end{pmatrix}, \forall n \geq 2 \ (n \in \mathbb{N})$$

with  $\alpha = \frac{a+d}{2}$  and  $(a+d)^2 = 4(ad-bc)$  ( $(a-d)^2 = -4bc$ )

And

$$A^n = n\alpha^{n-1}A - (n-1)\alpha^n I$$

2) If  $\alpha_1 \neq \alpha_2$  so  $\forall n \geq 2 \ (n \in \mathbb{N})$

$$\begin{aligned}
A^n &= \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} (a - \alpha_2)\alpha_1^n - (a - \alpha_1)\alpha_2^n & b(\alpha_1^n - \alpha_2^n) \\ c(\alpha_1^n - \alpha_2^n) & (a - \alpha_2)\alpha_2^n - (a - \alpha_1)\alpha_1^n \end{pmatrix} \\
A^n - \alpha_2^n I &= \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} (A - \alpha_2 I) \text{ and } A^n - \alpha_1^n I = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} (A - \alpha_1 I) \\
A^n &= \left( \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right) A - \alpha_1 \alpha_2 \left( \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} \right) I
\end{aligned}$$

**Proof.** 1)  $A^n = \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & (n+1)\alpha - na \end{pmatrix}$

And

$$\begin{aligned}
A^n &= \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (n+1)\alpha - n(a+d) \end{pmatrix} \\
A^n &= \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (n+1)\alpha - 2n\alpha \end{pmatrix} \\
A^n &= \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (1-n)\alpha \end{pmatrix} \\
A^n &= \alpha^{n-1} \begin{pmatrix} na & nb \\ nc & nd \end{pmatrix} + \alpha^{n-1} \begin{pmatrix} -(n-1)\alpha & 0 \\ 0 & -(n-1)\alpha \end{pmatrix}
\end{aligned}$$

$$A^n = n\alpha^{n-1}A - (n-1)\alpha^n I$$

2)

$$A^n = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} a(\alpha_1^n - \alpha_2^n) - \alpha_1\alpha_2(\alpha_1^{n-1} - \alpha_2^{n-1}) & b(\alpha_1^n - \alpha_2^n) \\ c(\alpha_1^n - \alpha_2^n) & (\alpha_1^{n+1} - \alpha_2^{n+1}) - a(\alpha_1^n - \alpha_2^n) \end{pmatrix}$$

□

## 2.1 Second method:

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$  a 2X2 matrix, so  $|A - xI| = x^2 - \text{Tr}(A)x + |A|$

with  $\text{Tr}(A) = a + d \in \mathbb{R}$  and  $|A| = ad - bc \in \mathbb{R}$

$$\implies A^2 - \text{Tr}(A)A + |A|I = O$$

$$\implies A^2 = \text{Tr}(A)A - |A|I$$

$$\forall A \in M_2(\mathbb{R}), \exists (r, s) \in \mathbb{R}^2 / A^2 = rA + sI$$

$\implies$

$$A^n = r_n A + s_n I, \forall n \in \mathbb{N}$$

so  $A^{n+1} = r_{n+1}A + s_{n+1}I = r_n A^2 + s_n A$

$$\implies r_{n+1}A + s_{n+1}I = r_n(rA + sI) + s_n A = (rr_n + s_n)A + sr_n I$$

$\implies$

$$\begin{cases} r_{n+1} = rr_n + s_n \\ s_{n+1} = sr_n \end{cases}$$

$\implies$

$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} r_n \\ s_n \end{pmatrix}$$

$\implies$

$$\begin{cases} r_{n+1} = rr_n + sr_{n-1} \\ s_{n+1} = sr_n \end{cases}$$

with  $r_1 = 1, r_2 = r, s_1 = 0$  and  $s_2 = s$ .

**Remark 2.2.**

$$(r_n, s_n) \in \mathbb{R}^2, \forall n \in \mathbb{N}^*.$$

*Proof.* By recurrence:

We have  $(r_2, s_2) = (r, s) \in \mathbb{R}^2$

$$\text{and } \begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} r_n \\ s_n \end{pmatrix}.$$

□

First we will solve the second-order linear recurrent sequence with constant coefficients:  $r_{n+1} = rr_n + sr_{n-1}$

the characteristic equation is

$$x^2 = rx + s$$

$$P(x) = x^2 - rx - s = 0$$

which  $(P)$  is the same as the characteristic polynomial of  $A$

$$x = \frac{r \pm (r^2 + 4s)^{\frac{1}{2}}}{2} = \begin{cases} \alpha_1 \\ \alpha_2 \end{cases}$$

$$(\alpha_1, \alpha_2) \in \mathbb{C}^2$$

$\implies$

$$\begin{cases} r_n = c_1 \alpha_1^n + c_2 \alpha_2^n & \text{if } \alpha_1 \neq \alpha_2 \\ \text{and} \\ r_n = (c_1 n + c_2) \alpha^n & \text{if } \alpha_1 = \alpha_2 = \alpha \end{cases}$$

### 2.1.1 If $\alpha_1 \neq \alpha_2$ :

We have

$$r_1 = c_1 \alpha_1 + c_2 \alpha_2 = 1, r_2 = c_1 \alpha_1^2 + c_2 \alpha_2^2 = r$$

so,

$$\begin{cases} c_1 = \frac{r - \alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} \\ c_2 = \frac{r - \alpha_1}{\alpha_2(\alpha_2 - \alpha_1)} \end{cases} \quad \text{if } \alpha_1 \neq \alpha_2$$

$$r_n = \frac{1}{\alpha_1 - \alpha_2} [(r - \alpha_2) \alpha_1^{n-1} - (r - \alpha_1) \alpha_2^{n-1}] \quad \text{if } \alpha_1 \neq \alpha_2$$

$$r_n = \frac{1}{\alpha_1 - \alpha_2} [r(\alpha_1^{n-1} - \alpha_2^{n-1}) - \alpha_1 \alpha_2 (\alpha_1^{n-2} - \alpha_2^{n-2})]$$

and

$$\begin{aligned} s_n &= sr_{n-1} = \frac{s}{\alpha_1 - \alpha_2} [(r - \alpha_2) \alpha_1^{n-2} - (r - \alpha_1) \alpha_2^{n-2}] \\ &= \frac{s}{\alpha_1 - \alpha_2} [r(\alpha_1^{n-2} - \alpha_2^{n-2}) - \alpha_1 \alpha_2 (\alpha_1^{n-3} - \alpha_2^{n-3})] \end{aligned}$$

So

$$(\alpha_1 - \alpha_2) A^n = [(r - \alpha_2) \alpha_1^{n-1} - (r - \alpha_1) \alpha_2^{n-1}] A + s [(r - \alpha_2) \alpha_1^{n-2} - (r - \alpha_1) \alpha_2^{n-2}] I$$

Since  $\alpha_1$  and  $\alpha_2$  are solutions of the equation  $x^2 = rx + s$

$$\implies \alpha_1^2 = r\alpha_1 + s, \alpha_2^2 = r\alpha_2 + s \text{ and } s = -\alpha_1 \alpha_2$$

$$\implies \alpha_1^n = r\alpha_1^{n-1} + s\alpha_1^{n-2}, \alpha_2^n = r\alpha_2^{n-1} + s\alpha_2^{n-2}$$

$$\implies \alpha_1^n - \alpha_2^n = r(\alpha_1^{n-1} - \alpha_2^{n-1}) - \alpha_1 \alpha_2 (\alpha_1^{n-2} - \alpha_2^{n-2})$$

$$\implies \alpha_1^n - \alpha_2^n = (r - \alpha_2) \alpha_1^{n-1} - (r - \alpha_1) \alpha_2^{n-1}$$

$$\text{and } \alpha_1^{n-1} - \alpha_2^{n-1} = (r - \alpha_2) \alpha_1^{n-2} - (r - \alpha_1) \alpha_2^{n-2}$$

$\implies$

$$A^n = \left( \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right) A - \alpha_1 \alpha_2 \left( \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} \right) I$$

$$A^n = \left( \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right) A + s \left( \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} \right) I$$

### 2.1.2 If $\alpha_1 = \alpha_2 = \alpha \neq 0$ :

so  $\alpha_1 = \alpha_2 = \alpha = \frac{r}{2}$  (since  $\Delta = 0$ )

$$r_n = (c_1 n + c_2) \alpha^n$$

$$r_1 = (c_1 + c_2) \alpha = 1, r_2 = (2c_1 + c_2) \alpha^2 = r$$

$\implies$

$$\begin{cases} c_1 = \frac{r - \alpha}{2\alpha^2} \\ c_2 = \frac{2\alpha - r}{\alpha^2} \end{cases} \quad \text{if } \alpha_1 = \alpha_2$$

⇒

$$r_n = [(r - \alpha)n + r - 2\alpha] \alpha^{n-2}$$

And  $s_{n+1} = sr_n \Rightarrow s_n = s [(r - \alpha)(n - 1) + r - 2\alpha] \alpha^{n-3}$   
 $\Rightarrow s_n = s [(r - \alpha)n - \alpha] \alpha^{n-3}$   
 $\Rightarrow A^n = [(r - \alpha)n + r - 2\alpha] \alpha^{n-2} A + s [(r - \alpha)n - \alpha] \alpha^{n-3} I$   
 $\alpha = \frac{r}{2}$  and  $\Rightarrow \alpha^2 = -s$

$$A^n = n \left(\frac{r}{2}\right)^{n-1} A + s(n-1) \left(\frac{r}{2}\right)^{n-2} I$$

$$A^n = n\alpha^{n-1} A - (n-1)\alpha^n I$$

If  $\alpha_1 = \alpha_2 = 0$

so  $A^2 = O$  and  $A^n = O, \forall n \geq 2$ .

**Remark 2.3.** 1) If  $\alpha_1 = \alpha e^{i\theta} \in \mathbb{C} \setminus \mathbb{R}$ , so  $\alpha_2 = \bar{\alpha}_1 = \alpha e^{-i\theta} \in \mathbb{C} \setminus \mathbb{R}$  and

$$A^n = \frac{\alpha^{n-2}}{\sin(\theta)} \{ \alpha \sin(n\theta) A + s \sin[(n-1)\theta] I \}$$

2) We can do the same if  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ .

**Conclusion 2.2.** We conclude that the final formulas are the same as those given in (1) and (2) if we distinguish the cases according to whether the eigenvalues are equal or not, except that in our article we were able to combine all the cases in a single formula, but for the second method it was done only in the case where the matrix has real coefficients, but this still gives us the credibility that our formulas and calculations are indeed true.

### Competing Interests

As the sole author of this work, I declare that I have no conflict of interest with anyone.

### Authors' Contributions

I am the sole author and contributor to the writing of this article. I have read and approved the final manuscript.

### The type of article

Original Research Article.

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